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$\mathcal{M}$-SCOTT CONVERGENCE AND $\mathcal{M}$-SCOTT TOPOLOGY ON POSETS

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Abstract. For a subset system $\mathcal{M}$ on any poset, $\mathcal{M}$-Scott notions, such as $\mathcal{M}$-way below relation, $\mathcal{M}$-continuity, $\mathcal{M}$-Scott convergence (of nets and filters respectively) and $\mathcal{M}$-Scott topology are proposed. Any approximating auxiliary relation on a poset can be represented by an $\mathcal{M}$-way below relation such that this poset is $\mathcal{M}$-continuous. It is shown that a poset is $\mathcal{M}$-continuous iff the $\mathcal{M}$-Scott topology is completely distributive. The topology induced by the $\mathcal{M}$-Scott convergence coincides with the $\mathcal{M}$-Scott topology. If the $\mathcal{M}$-way below relation satisfies the property of interpolation, then a poset is $\mathcal{M}$-continuous if and only if the $\mathcal{M}$-Scott convergence coincides with the $\mathcal{M}$-Scott topological convergence. Also, $\mathcal{M}$-continuity is characterized by a certain Galois connection.

1. Introduction

In the past forty years, the concept of a continuous lattice and its generalizations have attracted more and more attention. It was the pioneering work of D. Scott [22, 23] which lead to the discovery that algebraic lattices and their generalizations, continuous lattices, could be used to assign meanings to programs written in high-level programming languages. In pure mathematical aspect, many researchers try to establish the results in continuous lattice to some general poset, such as dcpos or arbitrary posets. In order to do so, generalized Scott notions should be given firstly. Different researchers focus on different aspects.

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(1) Generalization of way below relation, continuity and Scott topology. The way below relation and continuity were firstly defined on/for complete lattice and then on/for dcpos. In [8], related to three families of subset systems $I_m(L)$ of different ideals, three types of generalized way below relation (resp., generalized continuity, generalized Scott topology) were respectively defined and studied on arbitrary posets, namely s_m-way below relation (resp., s_m-continuity, s_m-topology) (m=1,2,3). It is shown in [8], a poset is s_m-continuous iff the s_m-topology is completely distributive (m=1,2,3). Also, 's_m-continuous' was characterized by certain Galois connections. The way below relation and the Scott topology on posets in [12, 32] just are the s_3-way below and s_3-topology respectively. In [32], it is also shown that a poset is continuous iff the Scott topology is completely distributive via sobrification.

(2) Generalization of Scott convergence. Since on any complete lattice or dcpos, the Scott topology can be characterized by Scott convergence of both nets and filters, the Scott convergence has been generalized by many authors. For the filter-theoretical one, three types of Scott convergence, namely s_m-convergence was defined and studied in [8] (m=1,2,3). A kind of Scott convergence, called s-convergence(=s_1-convergence in [8]), is also defined and studied in [30]. For the net-theoretical one, Scott convergence on arbitrary posets is defined and studied in [35] and then $\mathcal{M}$-Scott convergence in [36] related to a subset system $\mathcal{M}$.

(3) Generalization of distributivity. Since a continuous lattice (resp., a completely distributive lattice) is a complete lattice that satisfies the directed distributive law (resp., completely distributive law). Many authors focus on the distributive law and generalized continuous lattices as well as completely distributive lattice to $\mathcal{M}$-distributive lattices or $\mathcal{M}$-continuous lattices for $\mathcal{M}$ a subset system [2, 4, 7, 27, 33].

(4) $\mathcal{Z}$-continuous posets. The background for the categorical equivalence between completely distributive lattices and domains due to Hoffmann [14] and Lawson [17]. It is therefore natural to give a presentation of these matters in a more general framework: the category of $\mathcal{Z}$-continuous posets. The study of $\mathcal{Z}$-continuity was suggested by Wright and Wagner [31], and then widely studied in [3, 5, 9, 19, 24, 26, 31].

The aim of this paper is to study $\mathcal{M}$-continuity, $\mathcal{M}$-Scott topology and $\mathcal{M}$-Scott convergence (w.r.t. both nets and filters) on arbitrary poset for a subset system $\mathcal{M}$.

This paper is organized as follows. In Section 2, we make a preparation for the whole paper. In Section 3, we define $\mathcal{M}$-way below on any poset and then introduce a concept of an $\mathcal{M}$-continuous poset. We show that any approximating auxiliary relation on a poset can be represented by an $\mathcal{M}$-way below relation such that this poset is $\mathcal{M}$-continuous. In Section 4, we propose a definition of $\mathcal{M}$-Scott topology and show that a poset is $\mathcal{M}$-continuous iff the $\mathcal{M}$-Scott topology is completely distributive. In Section 5, we study $\mathcal{M}$-Scott convergence of nets and filters. We show that the topology induced by the $\mathcal{M}$-Scott convergence coincides with the $\mathcal{M}$-Scott topology. If the $\mathcal{M}$-way below relation satisfies the property of interpolation, then a poset is $\mathcal{M}$-continuous if and only if the $\mathcal{M}$-Scott convergence coincides with the $\mathcal{M}$-Scott topological convergence. In Section 6, $\mathcal{M}$-continuity is characterized by a certain Galois connections. In Section 7, for a complete lattice L, the $\mathcal{M}$-continuity is characterized by a kind of distributivity.

2. Preliminaries

In this paper, L always denotes a poset unless otherwise stated.

By an ideal of L, we mean a nonempty directed lower subset of L. By a subset system of a poset L, we mean a family $\mathcal{M}$ of subsets of L such that for all $x \in L,\ x = \bigvee M$ for some $M \in \mathcal{M}$.

Three basic subset systems are family of finite subsets $\mathcal{F}(L)$, family of directed sets $\mathcal{D}(L)$ and family of all subsets $\mathcal{P}(L)$, respectively.

In [8], the following three subset systems are considered:

$I_1(L) = \{I \subseteq L |\ I \text{ is a Frink ideal}\}$,

$I_2(L) = \{I \subseteq L |\ I \text{ is an ideal}\}$,

$I_3(L) = \{I \subseteq I_2(L) |\ I \text{ has a join}\}$,

where a Frink ideal [10] of L is a subset $I \subseteq L$ such that for any finite subset $Z \subseteq I$, the cut Z^δ (i.e., the intersection of all principle ideals containing Z) is also contained in I. By definition, $I_1(L) \supseteq I_2(L) \supseteq I_3(L)$. In $\vee$-semilattices with 0, $I_1(L)$ coincides with $I_2(L)$, while in dcpos, $I_2(L)$ is identical with $I_3(L)$. Hence in a complete lattice L, all three systems agree with the usual ideal lattice

$$I(L) = \{I \subseteq L |\ I \neq \emptyset,\ x \vee y \in I \Leftrightarrow x \in I \text{ and } y \in I\}.$$

In the following, we will fix some notations related to a subset A and a subfamily $\mathcal{S}$ of 2^L.

$A^u = \{x \in L |\ \forall a \in A,\ a \leq x\}$;

$A^l = \{x \in L |\ \forall a \in A,\ a \geq x\}$;

$A^{ul} = (A^u)^l,\ A^{lu} = (A^l)^u$;

$\mathcal{S}^l = \bigcup\{S^l |\ S \in \mathcal{S}\}$ and $\mathcal{S}^u = \bigcup\{S^u |\ S \in \mathcal{S}\}$;

$\mathcal{I}_A = \{I \subseteq L \mid I$ is an ideal containing $A\}$ and $I_A = \bigcap \mathcal{I}_A$ in $(2^L, \subseteq)$ (Notice that if $\mathcal{I}_A = \emptyset$, then $I_A = L$).

Remark 2.1. (1) $\emptyset^l = \emptyset^u = L$.

(2) Both $({}^u, {}^l)$ and $({}^l, {}^u)$ form Galois connections [13] (p. 129) on 2^L in sense of Ore [21], i.e., antitone Galois connections. Thus $(\bigcup_i A_i)^u = \bigcap_i A_i^u$, $(\bigcup_i A_i)^l = \bigcap_i A_i^l$ for all $\{A_i \mid i \in I\} \subseteq 2^L$.

(3) $A^{ulu} = A^u$, $A^{lul} = A^l$ and $(A^{ul})^{ul} = A^{ul}$, $(A^{lu})^{lu} = A^{lu}$.

(4) $A^l = \bigcap\{\downarrow x \mid x \in A\}$, $A^u = \bigcap\{\uparrow x \mid x \in A\}$.

(5) $A^{ul} = \bigcap\{\downarrow x \mid x \in A^u\}$ (which is exactly equal to A^δ, the cut [10] of A) and $A^{lu} = \bigcap\{\downarrow x \mid x \in A^l\}$ (Notice that $\bigcap \emptyset = L$ in $(2^L, \subseteq)$). If A has a join (resp., a meet) a, then $A^{ul} = \downarrow a$ (resp., $A^{lu} = \uparrow a$.)

Lemma 2.2. (1) For any $A \subseteq L$, $I_A \subseteq A^{ul}$.

(2) If A is finite (and nonempty), then $I_A = A^{ul}$.

(3) If $A \subseteq B \subseteq I_A$, then $A^{ul} = B^{ul}$ and $I_A = I_B$.

Proof. (1) If $A^u = \emptyset$, then $A^{ul} = L \supseteq I_A$. If $A^u \neq \emptyset$, then $\mathcal{I}_A \neq \emptyset$. By Remark 2.1(4), $I_A \subseteq A^{ul}$.

(2) We only need to show that $A^{ul} \subseteq I_A$ for $\mathcal{I}_A \neq \emptyset$. Suppose that $a \in A^{ul}$ and $I \subseteq L$ is an ideal with $A \subseteq I$. Since A is finite, there exists an upper bound x of A in I. Then $x \in A^u$ and then $a \leq x$, which implies that $a \in I$ since I is a lower set.

(3) Suppose that $A \subseteq B \subseteq I_A$. By (1) and Remark 2.1(2)(3), $A^{ul} \subseteq B^{ul} \subseteq I_A^{ul} \subseteq (A^{ul})^{ul} = A^{ul}$. Thus $A^{ul} = B^{ul}$. If $\mathcal{I}_A = \emptyset$, then $\mathcal{I}_B = \emptyset$ and in this case $I_A = I_B = L$. Otherwise, we have $I_A \subseteq I_B$. For any $I \in \mathcal{I}_A$, we have $B \subseteq I_A \subseteq I$ and then $I_B \subseteq I$. By the arbitrariness of I, we have $I_B \subseteq I_A$. Hence $I_A = I_B$. $\square$

The above $I_m(L)(m = 1, 2, 3)$ lead to three different generalizations of Scott convergence and Scott topology in arbitrary poset, and were called s_m-convergence and s_m-topology ($m = 1, 2, 3$), respectively. Likewise, the concept of continuous lattices extended in a threefold manner to that of an s_m-continuous poset: using the way below ideals

$$\Downarrow_m y = \bigcap\{I \in I_m(L) \mid y \in I^\delta\}.$$

A poset L is called s_m-continuous if each of the set $\Downarrow_m y$ is directed and has join y.

In this paper, $\mathcal{M}$ always denotes a subset system of a poset L. For a subset system $\mathcal{M}$ of L, we define

$$I_{\mathcal{M}} = \{A \subseteq L \mid M \subseteq A \subseteq I_M \text{ for some } M \in \mathcal{M}\}.$$

Proposition 2.3. The assignment $\mathcal{M} \mapsto I_{\mathcal{M}}$ is a closure operator on $(2^{(2^L)}, \subseteq)$.

Proof. Obviously, the assignment is increasing and order-preserving. In what follows, we need to show that $I_{\mathcal{M}} \supseteq I_{I_{\mathcal{M}}}$ for any subset system $\mathcal{M}$ of L. In fact, $\forall A \in I_{I_{\mathcal{M}}}$, there exists $B \in I_{\mathcal{M}}$ such that $B \subseteq A \subseteq I_B$ and for $B \in I_{\mathcal{M}}$, there exists $M \in \mathcal{M}$ such that $M \subseteq B \subseteq I_M$. By Lemma 2.2, it follows that $M \subseteq B \subseteq A \subseteq I_B = I_M$. Hence $A \in I_{\mathcal{M}}$. $\square$

3. $\mathcal{M}$-way below relations and $\mathcal{M}$-continuity

3.1. $\mathcal{M}$-way below relation and $\mathcal{M}$-continuous posets

Definition 3.1. For $x,\ y \in L$, we call x is $\mathcal{M}$-way below y, in symbols $x \ll_{\mathcal{M}} y$, if $y \in M^{ul}$ implies $x \in I_M$ for any $M \in \mathcal{M}$.

For $x \in L$, put $\twoheaddownarrow_{\mathcal{M}} x = \{y \in L |\ y \ll_{\mathcal{M}} x\}$. Then $\twoheaddownarrow_{\mathcal{M}} x = \bigcap\{I_M |\ M \in \mathcal{M},\ x \in M^{ul}\}$.

Remark 3.2. (1) In [8], $x \ll_1 y$ implies $x \ll_{I_1(L)} y$ and for $m = 2, 3$, $\ll_{I_m(L)} = \ll_m$.

(2) $\ll_{\mathcal{M}} = \ll_{I_{\mathcal{M}}}$.

Proposition 3.3. (1) If L has 0, then $0 \ll_{\mathcal{M}} x$ for all $x \in X$;

(2) $x \ll_2 y \Longrightarrow x \ll_{\mathcal{M}} y$;

(3) $x \ll_{\mathcal{M}} y \Longrightarrow x \leq y$;

(4) $u \leq x \ll_{\mathcal{M}} y \leq v \Longrightarrow u \ll_{\mathcal{M}} v$;

(5) In a $\vee$-semilattice, $x \ll_{\mathcal{M}} z,\ y \ll_{\mathcal{M}} z$ imply $x \vee y \ll_{\mathcal{M}} z$.

Proof. (1) Trivial since each ideal contains 0.

(2) Suppose that $x \ll y$ and $y \in M^{ul}$, where $M \in \mathcal{M}$. For each ideal $I \supseteq M$, we have $y \in I^{ul}$ and then $x \in I$. Hence $x \ll_{\mathcal{M}} y$.

(3) Suppose that $x \ll_{\mathcal{M}} y$. Clearly, $y = \bigvee M$ for some $M \in \mathcal{M}$. Then $y \in M^{ul}$ and $x \in I_M \subseteq \downarrow y$, $x \leq y$.

(4) and (5) are trivial. $\square$

Remark 3.4. (1) For two subset systems $\mathcal{M}_1,\ \mathcal{M}_2$, if $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $\ll_{\mathcal{M}_2} \subseteq \ll_{\mathcal{M}_1}$. The least (resp., the largest) $\mathcal{M}$-way below relation on L is $\ll_2$ (resp., $\leq$), one of the corresponding subset systems is $I_2(L)$ (resp., 2^L).

(2) For a family of subset systems $\{\mathcal{M}_i |\ i \in I\}$, if they induce a same $\mathcal{M}$-way below relation $\prec$, then $\ll_{\bigcup_i \mathcal{M}_i} = \prec$. Thus in all subset systems which induce a same $\mathcal{M}$-way below relation, there is a largest one.

When we define some types of continuity, the following four conditions are always considered:

(M1) $\forall x \in L,\ \Downarrow_{\mathcal{M}} x \in I_{\mathcal{M}}$;

(M2) $\forall x \in L,\ \Downarrow_{\mathcal{M}} x$ is an ideal;

(APP) $x = \bigvee \Downarrow_{\mathcal{M}} x$ for each $x \in X$;

(INT) $\ll_{\mathcal{M}}$ has the property of interpolation, that is $\forall x,\ y \in L$, if $x \ll_{\mathcal{M}} y$, then $x \ll_{\mathcal{M}} z \ll_{\mathcal{M}} y$ for some $z \in L$.

Remark 3.5. (1) The condition (APP) is equivalent to

(APP)′ $x \in (\Downarrow_{\mathcal{M}} x)^{ul}$ for each $x \in X$.

(2) In a $\vee$-semilattice with 0, the condition (M2) always holds.

(3) If $\mathcal{M} = I_1(L)$ or $\mathcal{M} = I_2(L)$, then (M1) can be implied by (M2). If $\mathcal{M} = I_3(L)$, then (M1) can be implied by (M2) and (APP).

(4) For $\mathcal{M} = I_m(L)$ ($m = 1, 2, 3$), (M1) and (INT) can be implied by (M2) and (APP) by Lemma 1.8(2) in [8].

Definition 3.6. A poset L is called $\mathcal{M}$-continuous if it satisfies the conditions (M1), (M2) and (APP). An '$\mathcal{M}$-continuous' is called strongly $\mathcal{M}$-continuous if it satisfies the condition (INT) additionally.

At the end of this subsection, we will compare our $\mathcal{M}$-way below with the $\mathcal{Z}$-($\mathcal{M}$-)notions in [2, 3, 4, 5, 7, 9, 19, 24, 26, 27, 31, 33] and many other literatures.

Remark 3.7. (1) x is said to be $\mathcal{Z}$-way below (or $\mathcal{Z}$-below, $\mathcal{M}$-way below) y if $x \in M^{ul}$ (which is equivalent to $x \leq \bigvee M$ if $\bigvee M$ exists) implies $x \in \downarrow M$ for any $M \in \mathcal{M}$. These generalized way below relations are generalizations of both the classical way below relation and the wedge below relation.

(2) The $\mathcal{Z}$-way below (or $\mathcal{Z}$-below, $\mathcal{M}$-way below) generally does not has the property (5) in Proposition 3.3. For example, let L be the diamond lattice $\{0, a, b, 1\}$ and $\mathcal{Z} = \mathcal{P}(L)$. It is easy to show that $a,\ b \ll_{\mathcal{Z}} 1$ and $1 = a \vee b \not\ll_{\mathcal{Z}} 1$, here $\ll_{\mathcal{Z}}$ is the $\mathcal{Z}$-way below relation in (1).

(3) In the theory of $\mathcal{Z}$-($\mathcal{M}$-)continuous poset, the corresponding condition of (M1) and (M2) is that

(Z1) $\Downarrow_{\mathcal{Z}} x = \downarrow M$ for some $M \in \mathcal{M}$.

(4) The (strongly) $\mathcal{Z}$-($\mathcal{M}$-)continuity is also a generalization of both of the usual continuity and completely distributivity. Sometimes, the poset is assumed to be $\mathcal{Z}$-($\mathcal{M}$-)complete and equipped with a more strongly subset system. The $\mathcal{Z}$-($\mathcal{M}$-)continuity is characterized by $\mathcal{Z}$-($\mathcal{M}$-)distributivity and certain Galois connections.

3.2. $\mathcal{M}$-minimal set

In a complete lattice (resp., a dcpo) L, $B \subseteq L$ is called a minimal set (resp., directed minimal set) of x if $\bigvee B = x$, and for any $A \in \mathcal{P}(L)$ (resp., $\forall A \in \mathcal{D}(L)$), $x \leq \bigvee A$ implies for any $b \in B$, $b \leq a$ for some $a \in A$. A dcpo (resp., complete lattice) is continuous (resp., completely distributive) iff each element has a directed minimal set (resp., minimal set) [28, 29]. An equivalent statement of a directed minimal set is that, $B \subseteq L$ is a directed minimal set of x if and only if $\bigvee B = x$ and for any $D \in \mathcal{D}(L)$, $x \leq \bigvee D$ implies $B \subseteq I_D$. For this reason, we introduce the following concept.

Definition 3.8. $\forall x \in L,\ B \subseteq L$ is called an $\mathcal{M}$-minimal set of x if
(1) $x = \bigvee B$;
(2) $\forall M \in \mathcal{M},\ x \in M^{ul}$ implies $B \subseteq I_M$.

Remark 3.9. (1) In a dcpo L, if $\mathcal{M} = \mathcal{D}(L)$, then an $\mathcal{M}$-minimal set is a directed minimal set.

(2) In a complete lattice L for $\mathcal{M} = \mathcal{P}(L)$, then for any co-prime element x of L, a $\mathcal{P}(L)$-minimal set of x is a minimal set of x in the sense of Wang [29].

(3) If $x \in L$ has an $\mathcal{M}$-minimal set, then the usual union of some $\mathcal{M}$-minimal sets of x is also an $\mathcal{M}$-minimal set of x. That is to say, if there exists an $\mathcal{M}$-minimal set of x, then there is a largest one, in symbols $\beta_{\mathcal{M}}(x)$.

Proposition 3.10. The following two are equivalent.

(1) L satisfies the condition (M2), i.e., $\ll_{\mathcal{M}}$ is approximating;

(2) for any $x \in L$, there exists an $\mathcal{M}$-minimal set of x, thus $\beta_{\mathcal{M}}(x)$ exists.

Proof. (1)$\Longrightarrow$ (2). We only need to show that $\forall x \in L$, $\twoheaddownarrow_{\mathcal{M}} x$ is an $\mathcal{M}$-minimal set of x. In fact, $\forall M \in \mathcal{M}$ with $x \in M^{ul}$. Then for any $y \in \twoheaddownarrow_{\mathcal{M}} x$, we have $y \in I_M$. Hence $\twoheaddownarrow_{\mathcal{M}} x \subseteq I_M$.

(2)$\Longrightarrow$ (1). We only need to show that $\beta_{\mathcal{M}}(x) = \twoheaddownarrow_{\mathcal{M}} x$. On the one hand, $\forall a \in \twoheaddownarrow_{\mathcal{M}} x$, it is easy to verify that $\beta_{\mathcal{M}}(x) \cup \{a\}$ also is an $\mathcal{M}$-minimal set of x, which implies $a \in \beta_{\mathcal{M}}(x)$ and $\beta_{\mathcal{M}}(x) \supseteq \twoheaddownarrow_{\mathcal{M}} x$ by the maximality of $\beta_{\mathcal{M}}(x)$. On the other hand, $\forall M \in \mathcal{M}$ with $x \in M^{ul}$, we have $\beta_{\mathcal{M}}(x) \subseteq I_M$ and then $\beta_{\mathcal{M}}(x) \subseteq \bigcap\{I_M \mid M \in \mathcal{M},\ x \in I_M\} = \twoheaddownarrow_{\mathcal{M}} x$.
□

3.3. Representation of approximating auxiliary relations by $\mathcal{M}$-way below relations

In [20], it is shown that an auxiliary relation on a poset L is approximating iff there exists a subset system of L such that L is $\mathcal{M}$-continuous (in sense of [20]) and the $\mathcal{M}$-way below relation (also in sense of [20]) coincides with the given auxiliary relation. In other words, each approximating auxiliary relation on a poset can be represented by an $\mathcal{M}$-way below relation on an $\mathcal{M}$-continuous poset in sense of [20]. In this section, we also will show that each approximating auxiliary relation on a poset can be represented by an $\mathcal{M}$-way below relation on an $\mathcal{M}$-continuous poset in our sense.

Definition 3.11. [12] A binary relation $\prec$ on a poset L is called an auxiliary relation, or an auxiliary order, if it satisfies the following conditions for all $u,\ x,\ y,\ v \in L$:
(Au1) $x \prec y$ implies $x \leq y$;
(Au2) $u \leq x \prec y \leq v$ implies $u \prec v$.
(Au3) if 0 exists, then $0 \prec x$ for any $x \in L$.

Put $\downarrow_{\prec} x = \{y \in L |\ y \prec x\}$ for each $x \in L$.

Definition 3.12. (See in [12] for L a dcpo) We call an auxiliary relation $\prec$ on L approximating if
(App) $\forall x \in L$, $\downarrow_{\prec} x$ is directed (thus is an ideal) and $x \in (\downarrow_{\prec} x)^{ul}$.

The set of all auxiliary relations and all approximating auxiliary relations on L are denoted by $Aux(L)$ and $App(L)$ respectively.

Clearly, each $\mathcal{M}$-way below relation is an auxiliary relation. Furthermore, if (M2) and (APP) hold, then $\ll_{\mathcal{M}}$ is also an approximating auxiliary relation.

Proposition 3.13. ([12]) Let M be the set of monotone mappings $s : L \longrightarrow \theta(L)$ satisfying $s(x) \subseteq \downarrow x$ for all $x \in L$— considered as a poset with respect to the pointwise order. Then the assignment

$$\prec \longmapsto s_{\prec} = (x \longmapsto \{y |\ y \prec x\})$$

is a well defined isomorphism from $Aux(L)$ to M, whose inverse associates to each mapping $s \in M$, the relation $\prec_s$ given by

$$x \prec_s y \text{ iff } x \in s(y).$$

Proposition 3.14. For any approximating auxiliary relation $\prec$ on L, there is a subset system $\mathcal{M}$ of L such that $\prec = \ll_{\mathcal{M}}$ and L is $\mathcal{M}$-continuous.

Proof. Let $\mathcal{M} = \{M \subseteq L | \ M \subseteq \downarrow_{\prec} a \subseteq I_M$ for some $a \in L\}$. For $x,\ y \in L$, suppose that $x \ll_{\mathcal{M}} y$. Put $M = \downarrow_{\prec} y$. Then $I_M = M$ since M is an ideal and then $M \in \mathcal{M}$. It follows that $y \in (\downarrow_{\prec} y)^{ul} = M^{ul}$. Then $x \in I_M = M = \downarrow_{\prec} y$. Thus $x \prec y$. Conversely, suppose that $x \prec y$. For $M \in \mathcal{M}$ with $y \in M^{ul}$, we can find an $a \in L$ such that $M \subseteq \downarrow_{\prec} a \subseteq I_M$. Then $y \in M^{ul} \subseteq (\downarrow_{\prec} a)^{ul}$, which implies that $y \leq a$ and $x \in \downarrow_{\prec} a \subseteq I_M$ since $x \prec y$. Hence $x \ll_{\mathcal{M}} y$. Hence $\prec = \ll_{\mathcal{M}}$ and L is $\mathcal{M}$-continuous. □

Theorem 3.15. *An auxiliary relation on a poset L is approximating if and only if it is the $\mathcal{M}$-way below relation for some subset system $\mathcal{M}$ such that L is $\mathcal{M}$-continuous.*

4. The $\mathcal{M}$-Scott topologies on posets

Definition 4.1. We call a subset U of L an $\mathcal{M}$-Scott open set if U is an upper set and for any $M \in \mathcal{M}$, $M^{ul} \cap U \neq \emptyset$ implies $I \cap U \neq \emptyset$ for each $I \in \mathcal{I}_M$. We call a subset of L $\mathcal{M}$-Scott closed if its complement is $\mathcal{M}$-Scott open. An $\mathcal{M}$-Scott open set F is called an open filter if it is a filter, that is for any $a, b \in F$, there exists $c \in F$ as a lower bound of a, b. Denote by $OFil(L)$ the family of open filters of L.

Denote $\sigma_{\mathcal{M}}(L)$ the family of all $\mathcal{M}$-Scott open subsets of L. It is easy to verify that $\sigma_{\mathcal{M}}(L)$ is a topology on L, called the $\mathcal{M}$-Scott topology on L. $\Sigma_{\mathcal{M}}(L)$ will denotes the corresponding topological space.

Remark 4.2. (1) By Corollary 2.2 in [8], $\sigma_1(L) \subseteq \sigma_{I_1(L)}$ and for $m = 2, 3$, $\sigma_{I_m(L)} = \sigma_m(L)$.

(2) Let $\mathcal{M}_1,\ \mathcal{M}_2$ be two subset systems of L. If $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $\sigma_{\mathcal{M}_1}(L) \supseteq \sigma_{\mathcal{M}_2}(L)$. The finest $\mathcal{M}$-Scott topology is the Alexandrov topology (one corresponding subset system is $\{\{x\} | \ x \in X\}$) and the coarsest one is $\sigma_2(L)$ in [8] (one corresponding subset system is $I_2(L)$).

(3) Let $\{\mathcal{M}_j | \ j \in J\}$ be a family of subset systems of a poset L. If for each $j \in J$, $\sigma_{\mathcal{M}_j}(L)$ is a same topology on L, then $\sigma_{\bigcup_j \mathcal{M}_j}(L)$ is also equal to this topology.

Proposition 4.3. The following statements hold.

(1) $\sigma_2(L) \subseteq \sigma_{\mathcal{M}}(L) \subseteq \gamma(L)$;

(2) $\{x\}^- = \downarrow x$;

(3) $\sigma_{\mathcal{M}}(L)$ is a T_0 topology;

(4) if $y \in (\uparrow x)^\circ$, then $x \ll_{\mathcal{M}} y$;

(5) $OFil(L)$— the set of all open filter—just is the set of all nonzero co-prime elements of $\sigma_{\mathcal{M}}(L)$.
If L is $\mathcal{M}$-continuous, then

(6) U is $\mathcal{M}$-Scott open if and only if U is an upper set and for any $u \in U$, there exists $v \in U$ such that $v \ll_{\mathcal{M}} u$;

(7) $OFil(L)$ is a basis of $\sigma_{\mathcal{M}}(L)$.
If L satisfies (INT), then

(8) $\forall x \in L,\ \Uparrow_{\mathcal{M}} x$ is $\mathcal{M}$-Scott open;

(9) $y \in (\uparrow x)^{\circ}$ if and only if $x \ll_{\mathcal{M}} y$, thus $\forall x \in L,\ (\uparrow x)^{\circ} = \Uparrow_{\mathcal{M}} x$.
If L is strongly $\mathcal{M}$-continuous, then

(10) $\{ \Uparrow_{\mathcal{M}} x |\ x \in L\}$ is a basis of $\sigma_{\mathcal{M}}(L)$;

Proof. (1) $\forall U \in \sigma_2(L)$, U is an upper set. Suppose that $M \in \mathcal{M}$ and $M^{ul} \cap U \neq \emptyset$. If $M \subseteq I$ and I is an ideal, then $I^{ul} \cap U \neq \emptyset$ and then $I \cap U \neq \emptyset$. Thus $U \in \sigma_{\mathcal{M}}(L)$.

(2) Clearly, $\{x\}^- \supseteq \downarrow x$ since every $\mathcal{M}$-Scott closed set is a lower set. By (1), $\{x\}^- \subseteq \{x\}^-_{\sigma_2(L)} = \downarrow x$, where $\{x\}^-_{\sigma_2(L)}$ denotes the closure of $\{x\}$ in $\sigma(L)$. Hence $\{x\}^- = \downarrow x$.

(3) Trivial since $\sigma_2(L)$ is a T_0 topology.

(4) Suppose that $y \in (\uparrow x)^{\circ}$. Then $\forall M \in \mathcal{M}$ with $y \in M^{ul}$, we have $M^{ul} \cap (\uparrow x)^{\circ} \neq \emptyset$, it follows that $I \cap (\uparrow x)^{\circ} \neq \emptyset$ for any ideal $I \supseteq M$. Then $I \cap \uparrow x \neq \emptyset$ and $x \in I$. Then $x \in I_M$ and hence $x \ll_{\mathcal{M}} y$.

(5) The proof is the same as that of Proposition II-1.11(i) in [12]. Firstly suppose that $U \in \sigma_{\mathcal{M}}(L)$ is a filter and that U is not a co-prime in $\sigma_{\mathcal{M}}(L)$. Then there are $V,\ W \in \sigma_{\mathcal{M}}(L)$ such that $U \subseteq V \cup W$ and elements $v \in U \backslash V$ and $w \in U \backslash W$. Let $z \in U$ satisfy $z \leq v$ and $z \leq w$. Since V and W are upper sets we have $z \notin V \cup W$, a contradiction to $z \in U$. Secondly, suppose that U is a co-prime in $\sigma_{\mathcal{M}}(L)$. To show that U is a filter, note first that it is an upper set. Suppose $v,\ w \in U$. Then $U \nsubseteq L \backslash \downarrow v$ and $U \nsubseteq L \backslash \downarrow w$. By (1), the sets $L \backslash \downarrow v$ and $L \backslash \downarrow w$ are Scott open. Thus, since U is co-prime, $U \nsubseteq (L \backslash \downarrow v) \cup (L \backslash \downarrow w) = L \backslash (\downarrow v \cap \downarrow w)$. Thus there is a $u \in U$ such that $u \leq v$ and $u \leq w$.
If L is $\mathcal{M}$-continuous,

(6) Suppose that $U \in \sigma_{\mathcal{M}}(L)$ and $u \in U$. Then $u \in (\Downarrow_{\mathcal{M}} u)^{ul} \cap U$, $\Downarrow_{\mathcal{M}} u$ is an ideal and $M \subseteq \Downarrow_{\mathcal{M}} u \subseteq I_M$ for some $M \in \mathcal{M}$. Then $(\Downarrow_{\mathcal{M}} u)^{ul} = M^{ul}$ and $M^{ul} \cap U \neq \emptyset$. Since $\Downarrow_{\mathcal{M}} u$ is an ideal containing M, we have $\Downarrow_{\mathcal{M}} u \cap U \neq \emptyset$, i.e., there exists $v \in U$ such that $v \ll_{\mathcal{M}} u$. Conversely, suppose that $M \in \mathcal{M}$ such that $M^{ul} \cap U \neq \emptyset$, then there exists $u \in M^{ul} \cap U$ and there exists a further $v \in U$ such that $v \ll_{\mathcal{M}} u$.

Then $v \in I_M$. For all ideal I with $M \subseteq I$, $v \in I_M \subseteq I$. Thus $v \in I \cap U \neq \emptyset$. Hence U is $\mathcal{M}$-Scott open.

(7) For any $x \in U \in \sigma_{\mathcal{M}}(L)$, by (6), we can find a sequence of $\{x_n\}_{n\in N} \subseteq U$ such that $\cdots \ll_{\mathcal{M}} x_n \ll_{\mathcal{M}} \cdots \ll_{\mathcal{M}} x_1 \ll_{\mathcal{M}} x$. Put $F = \bigcup_{n=1}^{\infty} \uparrow x_n$, then F is a filter and $x \in F \subseteq U$. We only need to prove that F is $\mathcal{M}$-Scott open in the following. In fact, if $M \in \mathcal{M}$ and $M^{ul} \cap F \neq \emptyset$, then there exists $n \in N$ such that $M^{ul} \cap \uparrow x_n \neq \emptyset$, i.e., there exists $a \geq x_n$ such that $a \in M^{ul}$. Thus $x_{n+1} \in I_M$. For each ideal $I \supseteq M$, we have $x_{n+1} \in I \cap F \neq \emptyset$. Hence $F \in \sigma_{\mathcal{M}}(L)$.

If $\ll_{\mathcal{M}}$ has the property of interpolation, i.e., (APP) holds,

(8) Obviously, $\Uparrow_{\mathcal{M}} x$ is an upper set. $\forall M \in \mathcal{M}$ with $M^{ul} \cap \Uparrow_{\mathcal{M}} x \neq \emptyset$, there exists $a \in M^{ul}$ such that $x \ll_{\mathcal{M}} a$ and there further exists $y \in L$ such that $x \ll_{\mathcal{M}} y \ll_{\mathcal{M}} a$. Suppose that I is an ideal with $M \subseteq I$. Then by $y \ll_{\mathcal{M}} a$, we have $y \in I_M \subseteq I$, which implies that $y \in \Uparrow_{\mathcal{M}} x \cap I$ and $\Uparrow_{\mathcal{M}} x \cap I \neq \emptyset$.

(9) If $x \ll_{\mathcal{M}} y$, then $y \in \Uparrow_{\mathcal{M}} x \in \sigma_{\mathcal{M}}(L)$ and $y \in (\uparrow x)^{\circ}$. Then $\Uparrow_{\mathcal{M}} x \subseteq (\uparrow x)^{\circ}$ and by (4), $\Uparrow_{\mathcal{M}} x \supseteq (\uparrow x)^{\circ}$.

If L is strongly $\mathcal{M}$-continuous,

(10) we only need to prove that $U \subseteq \bigcup_{x\in U} \Uparrow_{\mathcal{M}} x$. In fact, $\forall u \in U$, there exists $x \in U$ such that $x \ll_{\mathcal{M}} u$ and $u \in \Uparrow_{\mathcal{M}} x$. $\square$

Lemma 4.4. ([29]) A complete lattice is completely distributive iff each element has a minimal set.

Theorem 4.5. *If L is a strongly $\mathcal{M}$-continuous poset, then $\sigma_{\mathcal{M}}(L)$ is completely distributive.*

Proof. For any $U \in \sigma_{\mathcal{M}}(L)$, we only need to show that $\{\Uparrow_{\mathcal{M}} x \mid x \in U\}$ is a minimal set of U. Clearly, $\bigcup_{x\in U} \Uparrow_{\mathcal{M}} x = U$ by Theorem 4.3(10). Let $\{V_i \mid i \in I\} \subseteq \sigma_{\mathcal{M}}(L)$ with $U \subseteq \bigcup_i V_i$. $\forall x \in U$, $x \in V_i$ for some $i \in I$ and then $\Uparrow_{\mathcal{M}} x \subseteq V_i$. This completes the proof. $\square$

Lemma 4.6. If $OFil(L)$ is a basis of $\sigma_{\mathcal{M}}(L)$ and $\sigma_{\mathcal{M}}(L)$ is a continuous lattice, then for any $x \in U \in \sigma_{\mathcal{M}}(L)$, there exists $y \in U$ such that $y \ll_{\mathcal{M}} x$.

Proof. $\forall x \in U \in \sigma_{\mathcal{M}}(L)$, there exist $V \in \sigma_{\mathcal{M}}(L)$ and $F \in OFil(L)$ such that $x \in F \subseteq V \ll U$. We only need to prove that there exists $y \in U$ such that $F \subseteq \uparrow y$ by Proposition 4.3(4). In fact, if $F \not\subseteq \uparrow y$ for any $y \in U$, then there exists $z_y \in F$ such that $z_y \not\geq y$ and $y \in L \setminus \downarrow z_y$, which implies that there exists $F_y \in OFil(L)$ such that $y \in F_y \subseteq L \setminus \downarrow z_y$. Thus $\{F_y \mid y \in U\}$ is a cover of U in $\sigma_{\mathcal{M}}(L)$. It follows that $F \subseteq V \subseteq$

$F_{y_1} \cup \cdots \cup F_{y_n}$ for $y_1, \cdots, y_n \in U$. Since $z_{y_1}, \cdots, z_{y_n} \in F$ and F is a filter, there exists $z \in F$ such that $z \leq z_{y_1}, \cdots, z_{y_n}$. For $z \in F$, there exists $i \in \{1, 2, \cdots, n\}$ such that $z \in F_{y_i}$. Since $z \leq z_{y_i}$, we have $z_{y_i} \in F_{y_i} \subseteq L \setminus \downarrow z_{y_i}$, which is a contradiction. □

Theorem 4.7. *Consider the following conditions:*
(1) $\{ \Uparrow_{\mathcal{M}} x \mid x \in L\}$ *is a basis of* $\sigma_{\mathcal{M}}(L)$;
(2) $OFil(L)$ *is a basis of* $\sigma_{\mathcal{M}}(L)$ *and* $\sigma_{\mathcal{M}}(L)$ *is a continuous lattice;*
(3) $\sigma_{\mathcal{M}}(L)$ *has enough co-prime elements and is continuous;*
(4) $\sigma_{\mathcal{M}}(L)$ *is a completely distributive lattice;*
(5) L *satisfies* $(M2)$,
we have (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) $\Longrightarrow$ (5). *If* $(M1)$ *and* (APP) *hold, then the five conditions are equivalent to each other.*

Proof. (2) $\Longleftrightarrow$ (3) by Proposition 4.3(5) and (3) $\Longleftrightarrow$ (4) see [11, 12].

(1) $\Longrightarrow$ (2): We only need to prove that $\sigma_{\mathcal{M}}(L)$ is a continuous lattice. It is sufficient to show that $\Uparrow_{\mathcal{M}} x \ll U$ for all $x \in U \in \sigma_{\mathcal{M}}(L)$. In fact, for all directed cover of U, there exists a member of this cover which contains x and $\Uparrow_{\mathcal{M}} x$.

(2) $\Longrightarrow$ (5): Suppose that $x \notin (\Downarrow_{\mathcal{M}} x)^{ul}$. Then there exists $y \in (\Downarrow_{\mathcal{M}} x)^{u}$ such that $x \not\leq y$. Then $L \setminus \downarrow y$ is an open neighborhood of x. By Lemma 4.5, there exists $a \in L \setminus \downarrow y$ such that $a \ll_{\mathcal{M}} x$ and then $a \leq y$, a contradiction.

If L satisfies (M1) and (INT), then by Proposition 4.3(10), (5) $\Longrightarrow$ (1) holds. □

5. $\mathcal{M}$-Scott convergence

In this section, we shall study the net-theoretical and filter-theoretical Scott convergence with respect to a subset system $\mathcal{M}$.

Let X be a nonempty set. A net ξ of X is a mapping $\xi : \Delta \longrightarrow X$, where Δ is a directed set, denoted by $\xi = (x_\delta)_{\delta \in \Delta}$ or just by $(x_\delta)_{\delta \in \Delta}$. A (set-theoretical) filter $\mathcal{F}$ on X is a proper lattice filter of 2^X.

For a net $\xi = (x_\delta)_{\delta \in \Delta}$ of X, ξ is said to be in $A \subseteq L$ eventually if there exists $\delta_0 \in \Delta$ such that $x_\delta \in A$ for any $\delta \geq \delta_0$. The family

$$\mathcal{F}_\xi = \{F \subseteq X \mid \xi \text{ is in } F \text{ eventually}\}$$

is a filter on X, called the induced filter or the associated filter of ξ. For a filter $\mathcal{F}$ on X, let $\Delta_{\mathcal{F}} = \{(x, F) \mid x \in F \in \mathcal{F}\}$ ordered by $(x, U) \leq (y, V)$ iff $V \subseteq U$. Then $\Delta_{\mathcal{F}}$ $(\forall (x, U),\ (y, V) \in \Delta_{\mathcal{F}},\ )$ is a directed set. Define

$\xi_{\mathcal{F}} : \Delta_{\mathcal{F}} \longrightarrow X$ by $(x, U) \longmapsto x$. Then $\xi_{\mathcal{F}}$ is a net, called the induced net or the associated net of $\mathcal{F}$.

Let X be a nonempty set and $\mathbb{F}(X)$ the collection of all filters on X, a subset F of $\mathbb{F}(X) \times X$ is called a filter-theoretical convergence relation [16] if

(1) $([x], x) \in F$ for all $x \in X$, where $[x]$ is the principle ultrafilter generated by the singleton $\{x\}$;

(2) If $(\mathcal{F}, x) \in F$ and $\mathcal{F} \subseteq \mathcal{G}$, then $(\mathcal{G}, x) \in F$.

The family

$$\mathcal{O}_F(X) = \{U \subseteq X |\ (\mathcal{F}, x) \in F \text{ and } x \in U \text{ imply } U \in \mathcal{F}\}$$

is the finest topology on X satisfying $(\mathcal{F}, x) \in F$ implies $\mathcal{F}$ is topologically convergent to x.

Analogously, let $\Xi(X)$ the collection of all nets on X, a subset N of $\Xi(X) \times X$ is called a net-theoretical convergence relation if

(1) $(\overline{x}, x) \in N$ for all $x \in X$, where $\overline{x}$ is the constant net valued at x;

(2) If $(\xi, x) \in N$ and η is a subnet of ξ, then $(\eta, x) \in N$.

The family

$$\mathcal{O}_N(X) = \{U \subseteq X |\ (\xi, x) \in N \text{ and } x \in U \text{ implies } \xi \in U \text{ eventually}\}$$

is the finest topology on X satisfying $(\xi, x) \in N$ implies ξ is topologically convergent to x. A subset C is closed in $\mathcal{O}_N(X)$ iff for any net ξ in C, $(\xi, x) \in N$ implies $x \in C$.

Let F (resp., N) be a filter-theoretical (resp., net-theoretical) convergence relation on X. We call N finer than F, if $(\mathcal{F}, x) \in F$ implies $(\xi_{\mathcal{F}}, x) \in N$ for all $\mathcal{F} \in \mathbb{F}(X)$ and, F finer than N, if $(\xi, x) \in N$ implies $(\xi_{\mathcal{F}}, x) \in F$ for all $\xi \in \Xi(X)$. F and N are called compatible if both F and N are finer than each other.

Lemma 5.1. Let F (resp., N) be a filter-theoretical (resp., net-theoretical) convergence relation on X.

(1) If F is finer than N, then $\mathcal{O}_F(X) \subseteq \mathcal{O}_N(X)$;

(2) If N is finer than F, then $\mathcal{O}_N(X) \subseteq \mathcal{O}_F(X)$;

(3) If F and N are compatible, then $\mathcal{O}_F(X) = \mathcal{O}_N(X)$.

Proof. (1) Suppose that $U \in \mathcal{O}_F(X)$ and $(\xi, x) \in N,\ x \in U$. Then $(\mathcal{F}_\xi, x) \in F$, which implies that $U \in \mathcal{F}_\xi$. Thus $\xi \in U$ eventually and $U \in \mathcal{O}_N(X)$.

(2) Suppose that $U \in \mathcal{O}_N(X)$ and $(\mathcal{F}, x) \in F,\ x \in U$. Then $(\xi_{\mathcal{F}}, x) \in N$, which implies that $\xi_{\mathcal{F}} \in U$. Thus there exists $(a, F) \in \Delta_{\mathcal{F}}$ such that $\forall (b, F_1),\ (b, F_1) \geq (a, F)$ implies $\xi_{\mathcal{F}}(b, F_1) = b \in U$. Let $F_1 = F,\ \forall c \in F$,

we have $(c, F) \geq (a, F)$ and then $F \subseteq U$ and $U \in \mathcal{F}$. Hence $U \in \mathcal{O}_F(X)$. $\square$

In [35], the concept of lim-inf$_{\mathcal{M}}$-convergence of nets on posets is defined, which is a generalization of lim-inf-convergence or Scott convergence (or, S-convergence for short) in dcpos by changing collection of directed subsets to a subset system $\mathcal{M}$. Here we would like to called it $\mathcal{M}$-Scott convergence. Also in [8, 30], the Scott convergence of filters on any posets are defined with respect to some certain subset systems. Likewise, we can generalize it to $\mathcal{M}$-setting.

Definition 5.2. (1) A net $(x_\delta)_{\delta\in\Delta}$ in L is said to be $\mathcal{M}$-Scott convergent to $x \in L$, in symbols $x_\delta \longrightarrow_{\mathcal{M}} x$, if there exists $M \in \mathcal{M}$ such that $x \in M^{ul}$ and $\forall m \in M$, $x_\delta \geq m$ eventually.

(2) A filter $\mathcal{F}$ on L is said to be $\mathcal{M}$-Scott convergent to $x \in L$, in symbols $\mathcal{F} \longrightarrow_{\mathcal{M}} x$, if there exists $M \in \mathcal{M}$ such that $M \subseteq \mathcal{F}^l$ and $x \in M^{ul}$.

Easily, we can see that both of the net-theoretical and filter-theoretical $\mathcal{M}$-Scott convergence are convergence relations since $\mathcal{M}$ is a subset system (for each $x \in X$, $x = \bigvee M$ for some $M \in \mathcal{M}$) and the induced topologies are

$$\mathcal{O}^N_{\mathcal{M}}(L) = \{U \subseteq L|\ x_\delta \longrightarrow_{\mathcal{M}} x \in U \text{ implies } x_\delta \in U \text{ evetually}\}$$

and

$$\mathcal{O}^F_{\mathcal{M}}(L) = \{U \subseteq L|\ \mathcal{F} \longrightarrow_{\mathcal{M}} x \in U \text{ implies } U \in \mathcal{F}\}$$

respectively.

Remark 5.3. (1) C is closed in $\mathcal{O}^N_{\mathcal{M}}(L)$ iff for each net $\xi \subseteq C$, $\xi \to_{\mathcal{M}} x$ implies $x \in C$.

(2) $\mathcal{F} \longrightarrow_{\mathcal{M}} x \geq y$ implies $\mathcal{F} \longrightarrow y$ and $\xi \longrightarrow_{\mathcal{M}} x \geq y$ implies $\xi \longrightarrow y$. Thus if $x \leq y$, then the constant net $\overline{y}$ is $\mathcal{M}$-Scott convergent to x.

Proposition 5.4. (1) $\xi \longrightarrow_{\mathcal{M}} x$ iff $\mathcal{F}_\xi \longrightarrow_{\mathcal{M}} x$.

(2) $\mathcal{F} \longrightarrow_{\mathcal{M}} x$ iff $\xi_{\mathcal{F}} \longrightarrow_{\mathcal{M}} x$.

Proof. (1) Suppose that $\xi \in \Xi(L)$ and $\xi \longrightarrow_{\mathcal{M}} x$. Then there exists $M \in \mathcal{M}$ such that $x \in M^{ul}$ and $\forall m \in M$, $\xi \geq m$ eventually. That is to say, $\forall m \in M$, there exists $\delta_0 \in \Delta$ such that $m \in \{\xi(\delta)|\ \delta \geq \delta_0\}^l \subseteq \mathcal{F}^l_\xi$. Hence $M \subseteq \mathcal{F}^l_\xi$ and $\mathcal{F}_\xi \longrightarrow_{\mathcal{M}} x$. Conversely, suppose that $\mathcal{F}_\xi \longrightarrow_{\mathcal{M}} x$. Then there exists $M \in \mathcal{M}$ such that $x \in M^{ul}$ and $M \subseteq \mathcal{F}^l_\xi$. $\forall m \in M$, there exists $F \in \mathcal{F}_\xi^l$ such that $m \in F^l$ and for this F, there exists $\delta_0 \in \Delta$ such that $F \supseteq \{\xi(\delta) | \ \delta \geq \delta_0\}$, which implies $\xi \geq m$ eventually. Hence $\xi \longrightarrow_{\mathcal{M}} x$.

(2) Suppose that $\mathcal{M}$ is a subset system of L and $\mathcal{F} \longrightarrow_{\mathcal{M}} x$. Then there exists $M \in \mathcal{M}$ such that $M \subseteq \mathcal{F}^l$ and $x \in M^{ul}$. For any $m \in M$, there exists $F \in \mathcal{F}^l$ such that $m \in F^l$ and $F \subseteq \uparrow m$. Choose $a \in F$, we have $(a, F) \in \Delta_{\mathcal{F}}$. $\forall (b, F_1) \geq (a, F)$, i.e., $b \in F_1 \subseteq F$, $\xi_{\mathcal{F}}(b, F_1) = b \in F_1 \subseteq F$. Thus $\xi_{\mathcal{F}} \in F \subseteq \uparrow m$ eventually and $\xi_{\mathcal{F}} \longrightarrow_{\mathcal{M}} x$. $\xi_{\mathcal{F}} \longrightarrow_{\mathcal{M}} x \Rightarrow \mathcal{F} \longrightarrow_{\mathcal{M}} x$ can be induced by (1) since $\mathcal{F}_{\xi_{\mathcal{F}}} = \mathcal{F}$. $\square$

By Lemma 5.1 and Theorem 5.4, the net-theoretical and filter-theoretical $\mathcal{M}$-Scott convergence are compatible and thus they induce a same topology, i.e., $\mathcal{O}_{\mathcal{M}}^F(L) = \mathcal{O}_{\mathcal{M}}^N(L)$, denoted by $\mathcal{O}_{\mathcal{M}}(L)$.

For a dcpo, the Scott topology is equal to the topology induced by the Scott-convergence. Similarly, we have the following:

Theorem 5.5. $\sigma_{\mathcal{M}}(L) = \mathcal{O}_{\mathcal{M}}(L)$.

Proof. Suppose that U is an $\mathcal{M}$-Scott open set of L and $(x_\delta)_{\delta \in \Delta}$ is a net in L which is $\mathcal{M}$-Scott convergent to $x \in U$. For the net $(x_\delta)_{\delta \in \Delta}$, there exists $M \in \mathcal{M}$ such that $x \in M^{ul}$ and $\forall m \in M,\ x_\delta \geq m$ eventually. Since $x \in M^{ul} \cap U \neq \emptyset$, we have $I \cap U \neq \emptyset$ for each ideal $I \supseteq M$. If x_δ is not in U eventually, then for any $\delta \in \Delta$, we can find a $\delta_1 \geq \delta$ such that $x_{\delta_1} \notin U$. These x_{δ_1} forms a subnet of x_δ and $x_{\delta_1} \subseteq U'$. Therefore, $x_{\delta_1} \rightarrow_{\mathcal{M}} x$ by Remark 5.3, we have $x \in U'$, which is a contradiction.

Conversely, suppose that $U \in \mathcal{O}_{\mathcal{M}}(L)$. $\forall x \in U,\ y \in L$ with $x \leq y$, the constant net $\overline{y}$ is $\mathcal{M}$-Scott convergent to x. Then $y \in U$ and U is an upper set. $\forall M \in \mathcal{M}$ with $M^{ul} \cap U \neq \emptyset$ and I is an ideal with $M \subseteq I$. Choose $x \in M^{ul} \cap U$. The ideal I as a net of L is $\mathcal{M}$-Scott convergent to x, which implies that $I \cap U \neq \emptyset$. Hence $U \in \sigma_{\mathcal{M}}(L)$. $\square$

Proposition 5.6. (1) $y \ll_{\mathcal{M}} x$ if and only if $x_\delta \longrightarrow_{\mathcal{M}} x$ implies $x_\delta \geq y$ eventually for any net $(x_\delta)_{\delta \in \Delta}$;

(2) In any $\mathcal{M}$-continuous lattice L, $x_\delta \longrightarrow_{\mathcal{M}} x$ if and only if $y \ll_{\mathcal{M}} x$ implies $x_\delta \geq y$ eventually for any $y \in L$.

Proof. Trivial. $\square$

Again for a dcpo, it is continuous or a domain if and only if the Scott-convergence coincides with the Scott topological convergence. Similarly, we have,

Lemma 5.7. (1) Let $x,\ y \in L$. If U is an open neighborhood of x such that $y \in U^l$, then $y \ll_{\mathcal{M}} x$. Thus $\mathcal{U}(x)^l \subseteq \ \twoheaddownarrow_{\mathcal{M}} x$.

(2) If (APP) holds, then $\mathcal{U}(x)^l = \ \twoheaddownarrow_{\mathcal{M}} x$.

Proof. (1) For any $M \in \mathcal{M}$ such that $x \in M^{ul}$. Let I be an ideal with $I \supseteq M$. Then $I \cap U \neq \emptyset$ and $y \in I$ since $y \in U^l$ and I is a lower set. Hence $y \in I_M$ and $y \ll_{\mathcal{M}} x$.

(2) $\forall m \in \Downarrow_{\mathcal{M}} x$, we have $m \ll_{\mathcal{M}} x$ and $\Uparrow_{\mathcal{M}} m$ is an open neighborhood of x. Then $m \in \downarrow m = (\uparrow m)^l \subseteq \mathcal{U}(x)^l$. Hence $\Downarrow_{\mathcal{M}} x \subseteq \mathcal{U}(x)^l$. □

Theorem 5.8. *For a subset system $\mathcal{M}$ of a poset L, if L satisfies the condition (INT), i.e., the $\mathcal{M}$-way below relation $\ll_{\mathcal{M}}$ satisfies the properties of interpolation, then (M1) and (APP) hold if and only if $\mathcal{M}$-Scott convergence coincides with topological convergence w.r.t. $\sigma_{\mathcal{M}}(L)$.*

Proof. Suppose that L is strongly $\mathcal{M}$-continuous and the net $(x_\delta)_{\delta \in \Delta}$ is topologically convergent to x with respect to $\sigma_{\mathcal{M}}(L)$. We only need to show that $x_\delta \longrightarrow_{\mathcal{M}} x$. In fact, since L is $\mathcal{M}$-continuous, we have $x \in (\Downarrow_{\mathcal{M}} x)^{ul}$ and $M \subseteq \Downarrow_{\mathcal{M}} x \subseteq I_M$ for some $M \in \mathcal{M}$. Then $M^{ul} = (\Downarrow_{\mathcal{M}} x)^{ul} \ni x$. $\forall m \in M \subseteq \Downarrow_{\mathcal{M}} x$, we have $m \ll_{\mathcal{M}} x$ and $\Uparrow_{\mathcal{M}} m$ is an $\mathcal{M}$-Scott open neighborhood of x and $x_\delta \in \Uparrow_{\mathcal{M}} m \subseteq \uparrow m$ eventually. Hence $x_\delta \longrightarrow_{\mathcal{M}} x$.

Conversely, suppose that for any net, $\mathcal{M}$-Scott convergence coincides with Scott topological convergence. Let $\mathcal{U}(x)$ be the $\mathcal{M}$-Scott open neighborhood of x and $\xi = \xi_{\mathcal{U}(x)}$. It is easy to verify that ξ is topological convergent to x, which implies that ξ is $\mathcal{M}$-Scott convergent to x. Then there exists $M \in \mathcal{M}$ such that $x \in M^{ul}$ and $\forall m \in M$, $\xi \geq m$ eventually. We need to show that $M \subseteq \Downarrow_{\mathcal{M}} x \subseteq I_M$ and $x \in (\Downarrow_{\mathcal{M}} x)^{ul}$. In fact, $\forall m \in M$, there exists $U \in \mathcal{U}(x)$ such that $m \in U^l$. By Lemma 5.7, $m \ll_{\mathcal{M}} x$. Thus $M \subseteq \Downarrow_{\mathcal{M}} x$. $\forall m \in \Downarrow_{\mathcal{M}} x$, since $x \in M^{ul}$, we have $m \in I_M$ and then $\Downarrow_{\mathcal{M}} x \subseteq I_M$. $x \in M^{ul} = (\Downarrow_{\mathcal{M}} x)^{ul}$. □

Corollary 5.9. *L is an $\mathcal{M}$-continuous poset iff $\mathcal{U}(x) \to_{\mathcal{M}} x$ for any $x \in L$.*

The conclusions in Theorem 5.5, Proposition 5.6 and Theorem 5.8 indicate that the concepts of $\mathcal{M}$-way below relation and $\mathcal{M}$-Scott topology are defined reasonably.

6. Characterize $\mathcal{M}$-continuity by Galois connections

The $\mathcal{Z}$-($\mathcal{M}$-)continuity is characterized by certain Galois connections in many literatures. In this section, we will characterize our $\mathcal{M}$-continuity by Galois connections.

Let $\theta(L)$ denote the family of all lower set of L. Then $\theta(L)$ is a completely distributive lattice under set inclusion [6]. Define $\Delta_{\mathcal{M}} : \theta(L) \longrightarrow \theta(L)$ by $\forall A \in \theta(L)$,

$$\Delta_{\mathcal{M}}(A) = \{a \in L| \text{ there is a filter } \mathcal{F} \text{ with } \mathcal{F} \to_{\mathcal{M}} a,\ \mathcal{F}^l \text{ is an ideal and } A \in \mathcal{F}\}.$$

and $\nabla_{\mathcal{M}} : \theta(L) \longrightarrow \theta(L)$ by $\forall B \in \theta(L)$,

$$\nabla_{\mathcal{M}}(B) = \ \Downarrow_{\mathcal{M}} B = \bigcup\{\ \Downarrow_{\mathcal{M}} b|\ b \in B\}.$$

Then both $\Delta_{\mathcal{M}}$ and $\nabla_{\mathcal{M}}$ are order-preserving. The definition of $\Delta_{\mathcal{M}}$ is somewhat different from that in [8].

Proposition 6.1. (1) $\nabla_{\mathcal{M}}(A) \subseteq A \subseteq \Delta_{\mathcal{M}}(A)$ for all $A \in \theta(L)$.
(2) If (APP) holds, then $\nabla_{\mathcal{M}}$ is an interior operator on $\theta(L)$.

Proof. (1) For any $x \in \nabla_{\mathcal{M}}(A)$, we have $x \ll_{\mathcal{M}} a$ for some $a \in A$. Since A is a lower set, we have $x \leq a$ and $x \in A$. Thus $\nabla_{\mathcal{M}}(A) \subseteq A$. $\forall x \in A$, we have $x = \bigvee M$ for some $M \in \mathcal{M}$. Then $x \in M^{ul}$. Put $\mathcal{F} = [x]$, we have $\mathcal{F}^l = \downarrow x$ is an ideal and $M \subseteq \downarrow x = \mathcal{F}^l$. Then $x \in \Delta_{\mathcal{M}}(A)$. Thus $A \subseteq \Delta_{\mathcal{M}}(A)$.

(2) We only need to show that $\nabla_{\mathcal{M}}(A) \subseteq \nabla_{\mathcal{M}}(\nabla_{\mathcal{M}}(A))$. $\forall x \in \nabla_{\mathcal{M}}(A)$, we have $x \ll_{\mathcal{M}} a$ for an $a \in A$. By (INT), $x \ll_{\mathcal{M}} a_1 \ll_{\mathcal{M}} a$ for some $a_1 \in L$. Clearly, $a_1 \in \nabla_{\mathcal{M}}(A)$ and then $x \in \nabla_{\mathcal{M}}(\nabla_{\mathcal{M}}(A))$. $\square$

Lemma 6.2. If (M2) holds, then
(1) $\{\uparrow y \cap \nabla(A)|\ y \ll_{\mathcal{M}} x\}$ is a filter base of a filter, denoted by $\mathcal{F}_{A,x}$.
(2) $\mathcal{F}^l_{A,x} = \ \Downarrow_{\mathcal{M}} x$.

Proof. (1) $\{\uparrow y \cap \nabla(A)|\ y \ll_{\mathcal{M}} x\}$ is a filter base. Firstly, for all $y \ll_{\mathcal{M}} x$, $y \in \uparrow y \cap \Downarrow_{\mathcal{M}} x$. Thus each $\uparrow y \cap \nabla(A)$ is nonempty. Secondly, for all $y_1,\ y_2 \ll_{\mathcal{M}} x$, there is a $y_3 \ll_{\mathcal{M}} x$ such that $y_1,\ y_2 \leq y_3$. Then $(\uparrow y_1 \cap \nabla(A)) \cap (\uparrow y_2 \cap \nabla(A)) = (\uparrow y_1 \cap \uparrow y_2) \cap \nabla(A) \supseteq (\uparrow y_3 \cap \nabla(A))$.

(2) It is easy to check that $(\uparrow y \cap \nabla(A))^l = \downarrow y$. Then $\mathcal{F}^l_{A,x} = \bigcup\limits_{y \ll_{\mathcal{M}} x} (\uparrow y \cap \nabla(A))^l = \bigcup\limits_{y \ll_{\mathcal{M}} x} \downarrow y = \ \Downarrow_{\mathcal{M}} x$. $\square$

Proposition 6.3. (1) If L is $\mathcal{M}$-continuous, then $(\Delta_{\mathcal{M}}, \nabla_{\mathcal{M}})$ forms a Galois connection on $\theta(L)$.

(2) If $(\Delta_{\mathcal{M}}, \nabla_{\mathcal{M}})$ is a Galois connection on $\theta(L)$, then (M2) and (APP) hold.

Proof. (1) On one hand, $\forall x \in \nabla_{\mathcal{M}}(\Delta_{\mathcal{M}}(A))$, there exists $y \in \Delta_{\mathcal{M}}(A)$ such that $x \ll_{\mathcal{M}} y$. For $y \in \Delta_{\mathcal{M}}(A)$, there exists a filter $\mathcal{F} \to_{\mathcal{M}} y$, $\mathcal{F}^l$ is an ideal and $A \in \mathcal{F}$. For $\mathcal{F} \to_{\mathcal{M}} y$, there exists

$M \in \mathcal{M}$ such that $M \subseteq \mathcal{F}^l$ and $y \in M^{ul}$. Since $x \ll_{\mathcal{M}} y$, we have $x \in I_M \subseteq \mathcal{F}^l$, which implies $x \in F^l$ for some $F \in \mathcal{F}$. Then $F \cap A \in \mathcal{F}$ and $x \in (F \cap A)^l$. Choose $a \in F \cap A \subseteq A$, we have $x \leq a$. Hence $x \in A$ since A is a lower set. On the other hand, $\forall x \in A$, there exists $M \in \mathcal{M}$ such that $M \subseteq \Downarrow_{\mathcal{M}} x \subseteq I_M$. Since $M^{ul} = (\Downarrow_{\mathcal{M}} x)^{ul} \ni x$, we have $\mathcal{F}_{A,x} \to_{\mathcal{M}} x$. Clearly, $\mathcal{F}_{A,x}^l = \Downarrow_{\mathcal{M}} x$ is an ideal, $M \subseteq \mathcal{F}_{A,x}^l$ and $\nabla(A) \in \mathcal{F}$. Hence $x \in \Delta_{\mathcal{M}}(\nabla_{\mathcal{M}}(A))$.

(2) For any $x \in X$, $x \in \downarrow x \subseteq \Delta_{\mathcal{M}}(\nabla_{\mathcal{M}}(\downarrow x)) = \Delta_{\mathcal{M}}(\Downarrow_{\mathcal{M}} x)$. Then there exists a filter $\mathcal{F} \to_{\mathcal{M}} x$ such that $\mathcal{F}^l$ is an ideal and $\Downarrow_{\mathcal{M}} x \in \mathcal{F}$. There exists $M \in \mathcal{M}$ such that $M \subseteq \mathcal{F}^l$ and $x \in M^{ul}$. Suppose that Z is a finite subset of $\Downarrow_{\mathcal{M}} x$. Since $x \in M^{ul}$, we have $Z \subseteq I_M \subseteq \mathcal{F}^l$ and since Z is finite, there exists $F \in \mathcal{F}$ such that $Z \subseteq F^l$ and then $F \subseteq Z^u$, which implies $Z^u \in \mathcal{F}$ and $Z^u \cap \Downarrow_{\mathcal{M}} x \in \mathcal{F}$. Thus $Z^u \cap \Downarrow_{\mathcal{M}} x \neq \emptyset$ and $\Downarrow_{\mathcal{M}} x$ is directed (hence an ideal), i.e., (M2) holds. Also, $\downarrow x \subseteq \Delta_{\mathcal{M}}(\Downarrow_{\mathcal{M}} x) \subseteq (\Downarrow_{\mathcal{M}} x)^{ul} \subseteq (\downarrow x)^{ul} = \downarrow x$. Hence $x \in (\Downarrow_{\mathcal{M}} x)^{ul}$. That is, (APP) holds. □

By Proposition 6.3, we have

Theorem 6.4. *If (M1) holds, then L is $\mathcal{M}$-continuous iff $(\Delta_{\mathcal{M}}, \nabla_{\mathcal{M}})$ is a Galois connection on $\theta(L)$.*

Lemma 6.5. ([25, 34]) Let f, g be two maps on a poset. If (f, g) is a Galois connection, then f is an interior operator iff g is a closure operator.

Corollary 6.6. *If L is $\mathcal{M}$-continuous, then $\Delta_{\mathcal{M}}$ is a closure operator on $\theta(L)$.*

7. $\mathcal{J}_{\mathcal{M}}$-distributivity

The $\mathcal{Z}$-($\mathcal{M}$-)continuity is characterized by $\mathcal{Z}$-($\mathcal{M}$-)distributivity in many literatures.

In this section, we suppose that L is a complete lattice. Thus I_M just is the ideal generated by M and the condition (M2) automatically holds.

Definition 7.1. Let $\mathcal{S}$ be a collection of subsets of L. L is called an $\mathcal{S}$-distributive lattice or be $\mathcal{S}$-distributive if it satisfies the $\mathcal{S}$-distributive law, that is for any $\{S_i | \ i \in I\} \subseteq \mathcal{S}$ and any $S_i = \{a_{i,j} | \ j \in J_i\}$, the following equation holds:

$$\bigwedge_{i\in I}\bigvee_{j\in J_i} a_{i,j} = \bigvee_{f\in\prod_i J_i}\bigwedge_{i\in I} a_{i,f(i)}.$$

For a subset system $\mathcal{M}$, let $\mathcal{J}_{\mathcal{M}} = \{I_M|\ M \in \mathcal{M}\}$. Then $\mathcal{J}_{\mathcal{M}} \subseteq I_{\mathcal{M}}$.

Proposition 7.2. (1) $\mathcal{M}$-distributivity implies $\mathcal{J}_{\mathcal{M}}$-distributivity.

(2) $\mathcal{M}$-distributivity implies $I_{\mathcal{M}}$-distributivity.

(3) If L is distributive, then $\mathcal{J}_{\mathcal{M}}$-distributivity always implies $\mathcal{M}$-distributivity.

Proof. (1) Trivial since $\bigvee M = \bigvee I_M$.

(2) Trivial by (1) and the definition of $I_{\mathcal{M}}$.

(3) Let $\{M_i|\ i \in I\} \subseteq \mathcal{M}$. Suppose that $M_i = \{a_{i,j}|\ j \in J_i\}$ and $I_{M_i} = \{b_{i,k}|\ k \in K_i\}$ $(\forall i \in I)$. We only need to show that

$$\bigvee_{f\in\prod_i J_i}\bigwedge_{i\in I} a_{i,f(i)} \geq \bigvee_{g\in\prod_i K_i}\bigwedge_{i\in I} b_{i,g(i)}.$$

In fact, $\forall g \in \prod_i K_i$, $\forall i \in I$, $b_{i,g(i)} \leq \bigvee_{l=1}^{n_i} m_{i,l}$ for some $\{m_{i,l}|\ l = 1, 2, \cdots, l\} \subseteq M_i$. Put $B_{i,g(i)} = \{m_{i,l}|\ l = 1, 2, \cdots, n_i\}$, we have $B_{i,g(i)}$ is a finite subset of M_i for all $i \in I$. Then $\bigwedge_{i\in I} b_{i,g(i)} = \bigvee_{h\in\prod_i B_{i,g(i)}}\bigwedge_{i\in I} m_{i,h(i)} \leq \bigvee_{f\in\prod_i J_i}\bigwedge_{i\in I} a_{i,f(i)}$. This completes the proof. □

Proposition 7.3. $(M2)$ holds if and only if L is $\mathcal{J}_{\mathcal{M}}$-distributive.

Proof. Suppose that L satisfies the condition $(M2)$. $\forall\{I_{M_i}|\ i \in I\} \subseteq \mathcal{J}_{\mathcal{M}}$ and $I_{M_i} = \{a_{i,j}|\ j \in J_i\}(\forall i \in I)$. We only need to prove that

$$\bigwedge_{i\in I}\bigvee I_{M_i} \leq \bigvee_{f\in\prod_i J_i}\bigwedge_{i\in I} a_{i,f(i)}.$$

In fact, $\forall x \ll_{\mathcal{M}} \bigwedge_{i\in I}\bigvee I_{M_i} = \bigwedge_{i\in I}\bigvee M_i$, we have $x \ll_{\mathcal{M}} \bigvee M_i(\forall i \in I)$. Then $\forall i \in I$, $x \in I_{M_i}$ and $x = a_{i,g(i)}$, which implies that $x = \bigwedge_{i\in I} a_{i,g(i)} \leq \bigvee_{f\in\prod_i J_i}\bigwedge_{i\in I} a_{i,f(i)}$.

Conversely, suppose that L is $\mathcal{J}_{\mathcal{M}}$-distributive. We only need to prove that for any $x \in L$, there exists an $\mathcal{M}$-minimal set of x. Let $\mathcal{B} = \{I_M|\ x \leq \bigvee M\} = \{B_i|\ i \in I\}$ and $\forall i \in I$, $B_i = \{a_{i,j}|\ j \in J_i\}$. Put

$B = \{\bigwedge_{i\in I} a_{i,f(i)} \mid f \in \prod_i J_i\}$. We will show that B is an $\mathcal{M}$-minimal set of x in the following. In fact,

$$\bigvee B = \bigvee_{f\in\prod_i J_i} \bigwedge_{i\in I} a_{i,f(i)} = \bigwedge_{i\in I} \bigvee_{j\in J_i} a_{i,j} = \bigwedge_{x\leq\bigvee M} \bigvee I_M = \bigwedge_{x\leq\bigvee M} \bigvee M = x,$$

since for each x, there exists $M \in \mathcal{M}$ such that $x = \bigvee M$. For any $M \in \mathcal{M}$ with $x \leq \bigvee M$, we have $I_M \in \mathcal{B}$ and $I_M = B_{i_0}$ for some $i_0 \in I$. $\forall a = \bigwedge_{i\in I} a_{i,f(i)} \in B$, we have $a \leq a_{i_0,f(i_0)} \in B_{i_0} = I_M$ and $B \subseteq I_M$. □

Remark 7.4. (1) In Proposition 7.3, if $\mathcal{M} = \mathcal{D}(L)$, then L is continuous iff L is $\mathcal{J}(L)$-distributive, where $\mathcal{J}(L)$ is the set of all ideals of L. Since for a complete lattice (even for a dcpo) L, $\mathcal{J}(L)$-distributivity is equivalent to $\mathcal{D}(L)$-distributivity=(directed distributivity). Thus Proposition 7.3 can be view as a generalization of distributivity-like characterization of continuous lattices.

(2) In Proposition 7.3, if $(M1)$ holds, then the two conditions are equivalent to that $\downarrow_{\mathcal{M}} : L \longrightarrow I_{\mathcal{M}}$ is the left adjoint of $\bigvee : I_{\mathcal{M}} \longrightarrow L$.

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