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ON n-FOLD STRONG IDEALS OF BH-ALGEBRAS

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Abstract. The notion of n-fold strong ideal in BH-algebra is introduced and some related properties of it are investigated. The role of initial segments in BH-algebras is described.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([2,3]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. BCK-algebras have some connections with other areas: D. Mundici [5] proved MV-algebras are categorically equivalent to bounded commutative algebra, and J. Meng [6] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. Y. B. Jun, E. H. Roh, and H. S. Kim [4] introduced the notion of a BH-algebra, which is a generalization of BCK/BCI-algebras. They defined the notions of ideal, maximal ideal and translation ideal and investigated some properties. E. H. Roh and S. Y. Kim [8] estimated the number of BH^*-subalgebras of order i in a transitive BH^*-algebras by using Hao's method. S. S. Ahn and J. H. Lee ([1]) introduced the notion of strong ideals in BH-algebra and investigated some properties of it.

In this paper, we introduce the notion of n-fold strong ideal in BH-algebra and investigated some related properties of it. We also describe the role of initial segments in BH-algebras.

2. Preliminaries

By a *BH-algebra* ([4]), we mean an algebra $(X;*,0)$ of type (2,0) satisfying the following conditions:

(I) $x * x = 0$,
(II) $x * 0 = x$,

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(III) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y \in X$.

For brevity, we also call X a BH-algebra. In X we can define a binary operation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. A non-empty subset S of a BH-algebra X is called a *subalgebra* of X if, for any $x, y \in S$, $x * y \in S$, i.e., S is a closed under binary operation.

Definition 2.1. ([4]) A non-empty subset A of a BH-algebra X is called an *ideal* of X if it satisfies:

(I1) $0 \in A$,
(I2) $x * y \in A$ and $y \in A$ imply $x \in A$, $\forall x, y \in X$.

An ideal A of a BH-algebra X is said to be a *translation ideal* of X if it satisfies:

(I3) $x*y \in A$ and $y*x \in A$ imply $(x*z)*(y*z) \in A$ and $(z*x)*(z*y) \in A$, $\forall x, y, z \in X$.

Obviously, $\{0\}$ and X are ideals of X. A mapping $f : X \to Y$ of BH-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. For a homomorphism $f : X \to Y$ of BH-algebras, the *kernel* of f, denoted by $\mathrm{Ker}(f)$, defined to be the set

$$\mathrm{Ker}(f) = \{x \in X | f(x) = 0\}.$$

Definition 2.2. ([8]) A BH-algebra X is called a BH^*-*algebra* if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Example 2.3. ([4]) Let $X := \{0, 1, 2, 3\}$ be a BH-algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	1	0	0
1	1	0	0	0
2	2	2	0	3
3	3	3	3	0

Then $A := \{0, 1\}$ is a translation ideal of X.

Lemma 2.4. *Let X be a BH^*-algebra. Then the following identity holds:*

$$0 * x = 0, \ \forall x \in X.$$

Definition 2.5. A BH-algebra $(X; *, 0)$ is said to be *transitive* if $x * y = 0$ and $y * z = 0$ imply $x * z = 0$.

Definition 2.6. ([1]) A non-empty subset A of a BH-algebra X is called a *strong ideal* of X if it satisfies (I1) and

(I4) $(x * y) * z, y \in A$ imply $x * z \in A$.

Lemma 2.7. ([1]) *In a BH-algebra, any strong ideal is an ideal.*

Lemma 2.8. ([1]) *In a BH^*-algebra X, any ideal is a subalgebra.*

Corollary 2.9. ([1]) *Any strong ideal of BH^*-algebra is a subalgebra.*

Definition 2.10. Let X be a BH-algebra. X is said to be *positive implicative* if it satisfies the following identity:

$$(x * y) * z = (x * z) * (y * z), \forall x, y, z \in X.$$

Lemma 2.11. *Let X be a positive implicative BH^*-algebra. If $x \leq y$, then $x * z \leq y * z$ for any $x, y, z \in X$.*

Proposition 2.12. *If X is a positive implicative BH^*-algebra, then X is a transitive BH^*-algebra.*

Definition 2.13. ([1]) A non-zero element $a \in X$ is called an *atom* of a BH-algebra X if $x \leq a$ implies $x = 0$ or $x = a$. Let n be a positive integer. A non-zero element a of a BH-algebra X is called an *n-atom* of X if $x * a^n = 0$ implies $x = 0$ or $x = a$.

3. n-fold strong ideals

For any elements x and y of a BH-algebra X, $x * y^n$ denotes $(\cdots((x * y) * y) * \cdots) * y$ in which y occurs n times.

Definition 3.1. A non-empty subset A of a BH-algebra X is called an *n-fold strong ideal* of X if it satisfies (I1) and

(I5) for every $x, y, z \in X$ there exists a natural number n such that $x * z^n \in A$ whenever $(x * y) * z^n$ and $y \in A$.

For a BH-algebra X, obviously $\{0\}$ and X itself are n-fold strong ideals of X for every positively integer n.

Example 3.2. Let $X := \{0, 1, 2, 3\}$ be a set with the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	2	2	0

Then $(X;*,0)$ is a BH-algebra. It is easy to check that $A := \{0,1,2\}$ is an n-fold strong ideal of X for every positive integer n.

The 1-fold strong ideal is precisely a strong ideal. Taking $z := 0$ in (I5) and using (II), then $x*y = (x*y)*0^n \in A$ and $y \in A$ which imply that $x = x*0^n \in A$. Hence we have the following theorem.

Theorem 3.3. *In a BH-algebra, every n-fold strong ideal is an ideal.*

Combining Lemma 2.8 and Theorem 3.3, we have the following corollary.

Corollary 3.4. *In a BH^*-algebra, every n-fold strong ideal is a subalgebra.*

The converse of Corollary 3.4 may not be true as seen in the following example.

Example 3.5. Let $X = \{0,1,2,3\}$ be a BH-algebra as in Example 3.2. Then $(X;*,0)$ is a BH^*-algebra. The subset $C = \{0,2\}$ of X is a subalgebra, but not an n-fold strong ideal for every positive integer n, since $(3*2)*0^n = 2*0^n = 2 \in C$ and $3*0^n = 3 \notin C$.

Now we give a condition for a BH-algebra to be an n-fold strong ideal.

Theorem 3.6. *Let A be a subalgebra a BH^*-algebra X. Then A is an n-fold strong ideal of X if and only if $(y*x)*z^n \notin A$ whenever $y*z^n \notin A$ and $x \in A$.*

Proof. Let A be an n-fold strong ideal of X and let $x,y,z \in X$ be such that $y*z^n \notin A$ and $x \in A$. If $(y*x)*z^n \in A$, then $y*z^n \in A$ by (I5). This is a contradiction. Hence $(y*x)*z^n \notin A$.

Conversely, let A be a subalgebra X in which $y*z^n \notin A$ and $x \in A$ imply $(y*x)*z^n \notin A$. Obviously, $0 \in A$. Let $x,y,z \in X$ be such that $(x*y)*z^n \in A$ and $y \in A$, If $x*z^n \notin A$, then $(x*y)*z^n \notin A$ by assumption. This is impossible. Hence A is an n-fold strong ideal of X. □

Theorem 3.7. *Let $f: X \to Y$ be a homomorphism of a BH-algebra X. If B is an n-fold strong ideal of Y, then $f^{-1}(B)$ is an n-fold strong ideal of X for every positive integer n.*

Proof. Since $f(0) = 0 \in B$, we have $0 \in f^{-1}(B)$. Let $x,y,z \in X$ be such that $(x*y)*z^n \in f^{-1}(B)$ and $y \in f^{-1}(B)$. Then $(f(x)*f(y))*f(z)^n = f((x*y)*z^n) \in B$ and $f(y) \in B$. Since B is an n-fold strong ideal of Y, it follows from (I5) that $f(x * z^n) = f(x) * f(z)^n \in B$ so that $x * z^n \in f^{-1}(B)$. Hence $f^{-1}(B)$ is an n-fold strong ideal of X. □

Corollary 3.8. *Let $f : X \to Y$ be a homomorphism of BH-algebras. Then $Kerf := \{x \in X | f(x) = 0\}$ is an n-fold strong ideal of X for every positive integer n.*

Theorem 3.9. *Let $f : X \to Y$ be an isomorphism of BH-algebras. If a non-zero element a is an n-atom of X, then $f(a)$ is an n-atom of Y where n is a positive integer.*

Proof. Let y be a non-zero element of Y such that $y * f(a)^n = 0$. Then there exists a non-zero element $x \in X$ such that $f(x) = y$. Thus

$$\begin{aligned} f(0) = 0 &= y * f(a)^n \\ &= f(x) * f(a)^n = f(x) * f(a^n) \\ &= f(x * a^n). \end{aligned}$$

Since f is 1-1, it follows that $x * a^n = 0$ so that $x = a$ because a is an n-atom if X. Hence $y = f(x) = f(a)$, and $f(a)$ is an n-atom of Y. □

Let $A(\leq)$ be a partially ordered set with the least element 0. If we define a binary operation $*$ on A as follows:

$$x * y := \begin{cases} 0 & \text{if } x \leq y \\ x & \text{otherwise} \end{cases}$$

then the algebraic structure $(A; *, 0)$ is a BH-algebra. Hence any partially ordered set is regarded as a BH-algebra. We say that a BH-algebra with such defined a multiplication has the *trivial structure*.

For any fixed element $a \leq b$ of a BH-algebra X, the set

$$[a, b] = \{x \in X | a \leq x \leq b\} = \{x \in X | a * x = x * b = 0\}$$

is called the *segment* of X. Note that the segment

$$[0, b] = \{x \in X | x \leq b\} = \{x \in X | x * b = 0\}$$

is called *initial*, is the left annihilator of b. Since $[0, b]$ has two elements only in the case when $b \in X$ is an atom of X, a BH-algebra in which all initial segments have at most two elements has the trivial structure.

Proposition 3.10. *Every initial segment of a positive implicative BH^*-algebra is a subalgebra of X.*

Proof. Obviously, $0 \in [0, c]$. If $x, y \in [0, c]$, then $x \leq c$ and $y \leq c$. By Definition 2.2 and Lemma 2.11, we have $x * y \leq c * y \leq c$. Thus $x * y \in [0, c]$, which proves that $[0, c]$ is a subalgebra of X. □

Proposition 3.11. *The set-theoretic union of any two initial segments of a given positive implicative BH^*-algebra is a subalgebra.*

Proof. Straightforward. □

In general, initial segments of a BH^*-algebra X may not be an n-fold strong ideal of X, where n is a positive integer, as seen in the following example.

Example 3.12. Let $X := \{0, a, b, c\}$ be a set with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	b	a	0

It is easy to check that $(X; *, 0)$ is a BH^*-algebra. Then $[0, b] = \{0, a, b\}$ is not an ideal of X, since $c * b = a \in [0, b]$ but $c \notin [0, b]$. Also, it is not both a strong ideal and an n-fold strong ideal of X, because $(c * b) * 0^n = a * 0^n = a \in [0, b]$, but $c * 0^n = c \notin [0, b]$.

Proposition 3.13. *Let c be a fixed element of a BH-algebra X and let n be a positive integer. If the initial segment $[0, c]$ is an n-fold strong ideal of X, then for all $x, z \in X$,*

$$(x * c) * z^n \leq c \Rightarrow x * z^n \leq c.$$

Proof. Assume that for all $x, z \in X$, $(x*c)*z^n \leq c$. Hence $(x*c)*z^n \in [0, c]$. Since $c \in [0, c]$ and $[0, c]$ is an n-fold strong ideal of X, we have $x * z^n \in [0, c]$. Hence $x * z^n \leq c$. □

Corollary 3.14. *Let c be a fixed element of a BH-algebra X. If the initial segment $[0, c]$ is a strong ideal of X, then for all $x \in X$,*

$$x * c \leq c \Rightarrow x \leq c.$$

Corollary 3.15. *If a non-trivial segment $[0, c]$ is an ideal or a strong ideal of a BH-algebra, then $x * c \neq c$ for all non-zero $x \in X$.*

Proof. Let $[0, c]$, where $c \neq 0$, be an ideal. If $x*c = c$ for some $x \in X$, then $x * c \in [0, c]$. Hence $x * c \leq c$. By Corollary 3.14, we have $x \leq c$, which is a contradiction since in this case we obtain $0 = x * c = c$. □

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