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NEW EXACT TRAVELLING WAVE SOLUTIONS OF SOME NONLINEAR EVOLUTION EQUATIONS BY THE $\left(\frac{G'}{G}\right)$-EXPANSION METHOD

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Abstract. In this paper, the $\left(\frac{G'}{G}\right)$-expansion method is used to construct new exact travelling wave solutions of some nonlinear evolution equations. The travelling wave solutions in general form are expressed by the hyperbolic functions, the trigonometric functions and the rational functions, as a result many previously known solitary waves are recovered as special cases. The $\left(\frac{G'}{G}\right)$-expansion method is direct, concise, and effective, and can be applied to many other nonlinear evolution equations arising in mathematical physics.

1. Introduction

Nonlinear evolution equations are widely used as models to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma wave and chemical physics. One of the basic physical problems for nonlinear PDEs is to obtain their travelling wave solutions. During the past decades, searching for explicit solutions of nonlinear evolution equations by using various different methods has been the main goal for many researchers, and many powerful methods to construct exact solutions of those nonlinear equations have been developed such as the inverse scattering transform [1], the Backlund/Darboux transform [2, 3], Hirota's bilinear method [4], the truncated Painleve expansion [5], the tanh-sech method [6], the extended tanh method [7, 8], the Jacobi elliptic function expansion [9],

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the F-expansion [11], the sub-ODE method [12, 13], the homogeneous balance method [14], the sine-cosine method [15, 16], the rank analysis method [17], the ansatz method [18, 19], the Exp-function expansion method [20], and so on.

In this paper, the $\left(\frac{G'}{G}\right)$-expansion method, which is recently developed by Wang et al [21], is used to search for new travelling wave solutions of some nonlinear evolution equations such as the Benjamin, Bona and Mahony equation (BBM equation) [22], a nonlinear model introduced in [23] and a coupled Higgs equation [24]. These equations play an important role in applied scientific fields such as plasma, nonlinear optical fiber and statistical physics. In the following section, the $\left(\frac{G'}{G}\right)$-expansion method is described in great detail.

2. Description of the $\left(\frac{G'}{G}\right)$-expansion method

Suppose that a nonlinear equation, say in two independent variables x and t, is given by

$$P(u, u_t, u_x, u_{xx}, u_{tt}, ...) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. To find solution u explicitly, we take the following steps:

Step 1. Combining the independent variables x and t into one variable $\xi = x - \omega t$, we suppose that

$$u(x,t) = u(\xi), \xi = x - \omega t, \tag{2}$$

the travelling wave variable (2) permits us reducing (1) to an ODE for $u = u(\xi)$

$$Q(u, u', u'', u''', ...) = 0. \tag{3}$$

Step 2. Suppose that the solution of ODE (3) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ ad follows:

$$u(\xi) = a_m \left(\frac{G'}{G}\right)^m + a_{m-1}\left(\frac{G'}{G}\right)^{m-1} + \cdots, \tag{4}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{5}$$

$a_m, a_{m-1}, \cdots, \lambda$ and μ are constants to be determined later, $a_m \neq 0$, the unwritten part in (4) is also a polynomial in $\left(\frac{G'}{G}\right)$, but the degree of which is generally equal to or less than $m-1$, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (3).

Step 3. By substituting (4) into (3) and using second order LODE (5), collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand side of (3) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_m, a_{m-1}, \cdots, \lambda$ and μ.

Step 4. Assuming that the constants $a_m, \cdots, \omega, \lambda$ and μ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LODE (5) have been well known for us, then substituting $a_m, \cdots, \omega$ and the general solutions of (5) into (4) we have more travelling wave solutions of the nonlinear evolution equation (1).

3. Application to the BBM equation

Benjamin, Bona and Mahony [22] established the model

$$u_t + \alpha u_x - \beta u_{xxt} + \kappa(u^2)_x = 0, \tag{6}$$

which is called BBM equation.

Using the travelling wave $u(x,t) = u(\xi), \xi = x - \omega t$ we have from (6) the nonlinear ordinary differential equation

$$-\omega u' + \alpha u' + \beta\omega u''' + 2\kappa u u' = 0. \tag{7}$$

Integrating (7) with respect to ξ we obtain the second-order equation:

$$C + (-\omega + \alpha)u + \beta\omega u'' + \kappa u^2 = 0, \tag{8}$$

where C is a constant of integration that is to be determined later.

Suppose that the solution of ODE (8) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = a_m \left(\frac{G'}{G}\right)^m + \cdots, \tag{9}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0. \tag{10}$$

By using (9) and (10) it is easily derived that

$$u^2 = a_m^2 \left(\frac{G'}{G}\right)^{2m} + \cdots , \tag{11}$$

$$u' = -m a_m \left(\frac{G'}{G}\right)^{m+1} + \cdots , \tag{12}$$

$$u'' = m(m+1) a_m \left(\frac{G'}{G}\right)^{m+2} + \cdots . \tag{13}$$

Considering the homogeneous balance between u'' and u^2 in (8), based on (11) and (13) we require that $2m = m + 2 \Rightarrow m = 2$, so we can write (9) as

$$u(\xi) = a_2 \left(\frac{G'}{G}\right)^2 + a_1 \left(\frac{G'}{G}\right) + a_0, a_2 \neq 0, \tag{14}$$

and therefore

$$\begin{aligned} u^2(\xi) = a_2^2 \left(\frac{G'}{G}\right)^4 + 2a_2 a_1 \left(\frac{G'}{G}\right)^3 + (a_1^2 + 2a_2 a_0) \left(\frac{G'}{G}\right)^2 + \\ 2a_1 a_0 \left(\frac{G'}{G}\right) + a_0^2. \end{aligned} \tag{15}$$

By using (10) and (14) it is derived that

$$\begin{aligned} u'' = 6a_2 \left(\frac{G'}{G}\right)^4 + (2a_1 + 10a_2\lambda) \left(\frac{G'}{G}\right)^3 + (8a_2\mu + 3a_1\lambda + 4a_2\lambda^2) \left(\frac{G'}{G}\right)^2 \\ +(6a_2\lambda b + 2a_1\mu + a_1\lambda^2) \left(\frac{G'}{G}\right) + 2a_2\mu^2 + a_1\lambda\mu. \end{aligned} \tag{16}$$

By substituting (14), (15) and (16) into (8) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left-hand side of (8) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for $a_2, a_1, a_0, \omega, a, b$ and C. Solving the resulting algebraic equations, yields

$$a_2 = \frac{6\beta(\alpha + 2\kappa a_0)}{(\beta\lambda^2 + 8\beta\mu - 1)\kappa}, a_1 = \frac{6\beta(\alpha + 2\kappa a_0)\lambda}{(\beta\lambda^2 + 8\beta\mu - 1)\kappa}, \omega = -\frac{\alpha + 2\kappa a_0}{\beta\lambda^2 + 8\beta\mu - 1},$$
$$C = \frac{K_1}{(\beta\lambda^2 + 8\beta\mu - 1)^2\kappa}, \tag{17}$$

where $\lambda, \mu, \alpha, \beta, \kappa$ and a_0 are arbitrary constants and $K_1 = \kappa a_0 \alpha\beta\lambda^2 + 8\kappa a_0\alpha\beta\mu + \kappa^2 a_0^2 - \kappa a_0\alpha\beta^2\lambda^4 + 8\kappa a_0\alpha\beta^2\lambda^2\mu - 16\kappa a_0\alpha\beta^2\mu^2 + 6\beta^2\lambda^2\mu\alpha^2 + 8\beta^2\lambda^2\mu\kappa^2 a_0^2 + 12\beta^2\mu^2\alpha^2 - 16\beta^2\mu^2\kappa^2 a_0^2 - \kappa^2 a_0^2\beta^2\lambda^4$.

By using (17), expression (14) can be written as

$$u(\xi) = \frac{6\beta(\alpha + 2\kappa a_0)}{(\beta\lambda^2 + 8\beta\mu - 1)\kappa}\left(\frac{G'}{G}\right)^2 + \frac{6\beta(\alpha + 2\kappa a_0)\lambda}{(\beta\lambda^2 + 8\beta\mu - 1)\kappa}\left(\frac{G'}{G}\right) + a_0, \tag{18}$$

where $\xi = x + \frac{\alpha+2\kappa a_0}{\beta\lambda^2+8\beta\mu-1}t$. (18) is the formula of a solution of (8), provided that the integration constant C in (8) is taken as that in (17).

Substituting the general solutions of (10) into (18) we have three types of travelling wave solutions of the bbm equation (6) as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_1(\xi) = \frac{3\beta(\alpha + 2\kappa a_0)(\lambda^2 - 4\mu)}{2(\beta\lambda^2 + 8\beta\mu - 1)\kappa}\left(\frac{C_1 \sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right)^2 - \frac{3\beta(\alpha + 2\kappa a_0)\lambda^2}{2(\beta\lambda^2 + 8\beta\mu - 1)\kappa} + a_0, \tag{19}$$

where $\xi = x + \frac{\alpha+2\kappa a_0}{\beta\lambda^2+8\beta\mu-1}t$, C_1 and C_2 are arbitrary constants.

If C_1 and C_2 are taken as special values, the various known results in the literature can be rediscovered, for instance, if $C_1 > 0, C_1^2 > C_2^2$, then $u_1 = u_1(\xi)$ can be written as

$$u_1(\xi) = -\frac{3\beta(\alpha + 2\kappa a_0)(\lambda^2 - 4\mu)}{2(\beta\lambda^2 + 8\beta\mu - 1)\kappa}\operatorname{sech}^2\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + \xi_0\right) - \frac{6\beta(\alpha + 2\kappa a_0)\mu}{(\beta\lambda^2 + 8\beta\mu - 1)\kappa} + a_0, \tag{20}$$

which is the known solitary wave solution of the BBM equation (6) (see [22]), where $\xi_0 = \tanh^{-1}\frac{C_2}{C_1}, \xi = x + \frac{\alpha+2\kappa a_0}{\beta\lambda^2+8\beta\mu-1}t$.

When $\lambda^2 - 4\mu < 0$,

$$u_2(\xi) = \frac{3\beta(\alpha + 2\kappa a_0)(4\mu - \lambda^2)}{2(\beta\lambda^2 + 8\beta\mu - 1)\kappa} \left(\frac{-C_1 \sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2 \cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1 \cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2 \sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi} \right)^2 - \frac{3\beta(\alpha + 2\kappa a_0)\lambda^2}{2(\beta\lambda^2 + 8\beta\mu - 1)\kappa} + a_0, \tag{21}$$

where $\xi = x + \frac{\alpha+2\kappa a_0}{\beta\lambda^2+8\beta\mu-1}t$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$,

$$u_3(\xi) = \frac{6\beta(\alpha + 2\kappa a_0)}{(\beta\lambda^2 + 8\beta\mu - 1)\kappa} \left(\frac{C_2}{C_1 + C_2\xi} \right)^2 - \frac{3\beta(\alpha + 2\kappa a_0)\lambda^2}{2(\beta\lambda^2 + 8\beta\mu - 1)\kappa} + a_0, \tag{22}$$

where $\xi = x + \frac{\alpha+2\kappa a_0}{12\beta\mu-1}t$, C_1 and C_2 are arbitrary constants.

Remark 1. It is used as an alternative the KdV equation which describes unidirectional propagation of weakly long dispersive waves [8]. As a model that characterizes long waves in nonlinear dispersive media, the BBM equation, like KdV equation, was formally derived to describe an approximation for surface water waves in a uniform channel. the equation covers not only the surface waves of long wavelength in liquids, but also hydromagnetic waves clod plasma, acoustic waves in anharmonic crystals, and acoustic gravity waves in compressible fluids. Many researchers are attracted by the wide applicability of the BBM equation [8].

4. Application to a nonlinear model equation for weak symmetries

Consider the model equation introduced in [23]

$$u_{xxx} + u(u_t + \delta u_x) = 0, \tag{23}$$

where δ is a constant and the subscripts denote differentiation with respect to the variable indicated.

Using the travelling wave $u(x,t) = u(\xi), \xi = x - \omega t$ we have from (23) the nonlinear ordinary differential equation

$$u''' + u(-\omega u' + \delta u') = 0. \tag{24}$$

Integrating (24) with respect to ξ we obtain the second-order equation:

$$C + u'' + \frac{1}{2}(-\omega + \delta)u^2 = 0, \tag{25}$$

where C is a constant of integration that is to be determined later.

Considering the homogeneous balance between u'' and u^2 in (25), we require that $2m = m + 2 \Rightarrow m = 2$, so (4) can be written as

$$u(\xi) = a_2 \left(\frac{G'}{G}\right)^2 + a_1 \left(\frac{G'}{G}\right) + a_0, a_2 \neq 0, \tag{26}$$

and therefore

$$u^2(\xi) = a_2^2 \left(\frac{G'}{G}\right)^4 + 2a_2a_1 \left(\frac{G'}{G}\right)^3 + (a_1^2 + 2a_2a_0) \left(\frac{G'}{G}\right)^2 + 2a_1a_0 \left(\frac{G'}{G}\right) + a_0^2. \tag{27}$$

By using (5) and (26) it is derived that

$$u'' = 6a_2 \left(\frac{G'}{G}\right)^4 + (2a_1 + 10a_2\lambda) \left(\frac{G'}{G}\right)^3 + (8a_2\mu + 3a_1\lambda + 4a_2\lambda^2) \left(\frac{G'}{G}\right)^2 + (6a_2\lambda b + 2a_1\mu + a_1\lambda^2) \left(\frac{G'}{G}\right) + 2a_2\mu^2 + a_1\lambda\mu. \tag{28}$$

By substituting (26), (27) and (28) into (25) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left-hand side of (25) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for $a_2, a_1, a_0, \omega, a, b$ and C. Solving the resulting algebraic equations, yields

$$a_2 = \frac{12a_0}{\lambda^2 + 8\mu}, a_1 = \frac{12a_0\lambda}{\lambda^2 + 8\mu}, \omega = \frac{\lambda^2 + 8\mu + \delta a_0}{a_0}, \quad C = \frac{a_0(-8\lambda^2\mu + 16\mu^2 + \lambda^2)}{2(\lambda^2 + 8\mu)}, \tag{29}$$

where λ, μ, δ and a_0 are arbitrary constants.

By using (29), expression (26) can be written as

$$u(\xi) = \frac{12a_0}{\lambda^2 + 8\mu} \left(\frac{G'}{G}\right)^2 + \frac{12a_0\lambda}{\lambda^2 + 8\mu} \left(\frac{G'}{G}\right) + a_0, \tag{30}$$

where $\xi = x - \frac{\lambda^2+8\mu+\delta a_0}{a_0}t$. (30) is the formula of a solution of (25), provided that the integration constant C in (25) is taken as that in (29).

Substituting the general solutions of (5) into (30) we have three types of travelling wave solutions of the weak equation (23) as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_1(\xi) = \frac{12a_0(\lambda^2 - 4\mu)}{\lambda^2 + 8\mu} \left(\frac{C_1 \sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1 \cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi} \right)^2 - \frac{3a_0\lambda^2}{\lambda^2 + 8\mu} + a_0, \tag{31}$$

where $\xi = x - \frac{\lambda^2+8\mu+\delta a_0}{a_0}t$, C_1 and C_2 are arbitrary constants.

If C_1 and C_2 are taken as special values, the various known results in the literature can be rediscovered, for instance, if $C_1 > 0, C_1^2 > C_2^2$, then $u_1 = u_1(\xi)$ can be written as

$$u_1(\xi) = -\frac{12a_0(\lambda^2 - 4\mu)}{\lambda^2 + 8\mu} \mathrm{sech}^2 \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + \xi_0 \right) + \frac{10(\lambda^2 - 4\mu)a_0}{\lambda^2 + 8\mu}. \tag{32}$$

where $\xi_0 = \tanh^{-1}\frac{C_2}{C_1}, \xi = x - \frac{\lambda^2+8\mu+\delta a_0}{a_0}t.$

When $\lambda^2 - 4\mu < 0$,

$$u_2(\xi) = \frac{3a_0(4\mu - \lambda^2)}{\lambda^2 + 8\mu} \left(\frac{-C_1 \sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2 \cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1 \cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2 \sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi} \right)^2 - \frac{3a_0\lambda^2}{\lambda^2 + 8\mu} + a_0, \tag{33}$$

where $\xi = x - \frac{\lambda^2+8\mu+\delta a_0}{a_0}t$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$,

$$u_3(\xi) = \frac{12a_0}{\lambda^2 + 8\mu} \left(\frac{C_2}{C_1 + C_2\xi} \right)^2 - \frac{3a_0\lambda^2}{\lambda^2 + 8\mu} + a_0, \tag{34}$$

where $\xi = x - \frac{\lambda^2+8\mu+\delta a_0}{a_0}t$, C_1 and C_2 are arbitrary constants.

5. Application to a coupled Higgs equation

Let us consider the Coupled Higgs equation [24]

$$u_{tt} - u_{xx} + |u|^2 u - 2uv = 0, \tag{35}$$

$$v_{tt} + v_{xx} - (|u|^2)_{xx} = 0. \tag{36}$$

Using the wave variables

$$u(x,t) = e^{i\theta} u(\xi),\ v(x,t) = v(\xi),\ \theta = px - rt,\ \xi = x - \omega t, \tag{37}$$

(35) and (36) are carried to the nonlinear ordinary differential equation, respectively:

$$(\omega^2 - 1)u'' + (p^2 - r^2)u - 2uv + u^3 = 0, \tag{38}$$

$$(\omega^2 + 1)v'' - 2(u')^2 - 2uu'' = 0. \tag{39}$$

Considering the homogeneous balance between u^3 and u'' in (38) and that between $(u')^2$ and v'' in (39) ($3m = n + 2, 2m + 2 = n + 2 \Rightarrow m = 1, n = 2$), so we can write the solutions by (4)

$$u(\xi) = a_1 \left(\frac{G'}{G}\right) + a_0, a_1 \neq 0, \tag{40}$$

$$v(\xi) = b_2 \left(\frac{G'}{G}\right)^2 + b_1 \left(\frac{G'}{G}\right) + b_0, b_2 \neq 0, \tag{41}$$

where $G = G(\xi)$ satisfies the second order LODE:

$$G'' + \lambda G' + \mu G = 0, \tag{42}$$

and $a_1, a_0, b_2, b_1, b_0, \lambda$ and μ are constants to be determined later.

By using (40), (41) and (42) it is derived that

$$uv = a_1 b_2 \left(\frac{G'}{G}\right)^3 + (a_1 b_1 + a_0 b_2)\left(\frac{G'}{G}\right)^2 + (a_0 b_1 + a_1 b_0)\left(\frac{G'}{G}\right) + a_0 b_0, \tag{43}$$

$$u^3(\xi) = a_1^3 \left(\frac{G'}{G}\right)^3 + 3a_1^2 a_0 \left(\frac{G'}{G}\right)^2 + 3a_1 a_0 \left(\frac{G'}{G}\right) + a_0^3. \tag{44}$$

$$u' = -a_1 \left(\frac{G'}{G}\right)^2 - a_1 \lambda \left(\frac{G'}{G}\right) - a_1 \mu. \tag{45}$$

$$u'' = 2a_1\left(\frac{G'}{G}\right)^3 + 3a_1\lambda\left(\frac{G'}{G}\right)^2 + (2a_1\mu + a_1\lambda^2)\left(\frac{G'}{G}\right) + a_1\lambda\mu. \tag{46}$$

By substituting (43)-(46) into (38) and (39) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left-hand side of (38) and (39) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of (38) and (39) to zero yields a set of simultaneous algebraic equations for $a_1, a_0, b_2, b_1, b_0, \omega, \lambda$ and μ. Solving the resulting algebraic equations, yields the sets of coefficients the followings:

$$a_1 = \frac{2a_0}{\lambda}, b_2 = -2, b_1 = -2\lambda, b_0 = -\frac{4\mu\lambda^2 - \lambda^2p^2 + \lambda^2r^2 - a_0^2\lambda^2 + 4a_0^2\mu}{2\lambda^2}, \omega = \pm\frac{\sqrt{-2a_0^2 - \lambda^2}}{\lambda}, \tag{47}$$

where λ, μ and a_0 are arbitrary constants.

By using (47), expression (40) and (41) can be written as

$$u(\xi) = \frac{2a_0}{\lambda}\left(\frac{G'}{G}\right) + a_0, \tag{48}$$

$$v(\xi) = -2\left(\frac{G'}{G}\right)^2 - 2\lambda\left(\frac{G'}{G}\right) - \frac{4\mu\lambda^2 - \lambda^2p^2 + \lambda^2r^2 - a_0^2\lambda^2 + 4a_0^2\mu}{2\lambda^2}, \tag{49}$$

where $\xi = x \pm \frac{\sqrt{-2a_0^2-\lambda^2}}{\lambda}t$.

Substituting the general solutions of (42) into (48) and (49) we have three types of travelling wave solutions of the coupled Higgs equation (38) and (39) as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_1(\xi) = \frac{a_0\sqrt{\lambda^2 - 4\mu}}{\lambda}\left(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right) \times\exp[\mathbf{i}(px - rt)], \tag{50}$$

$$v_1(\xi) = -2(\lambda^2 - 4\mu)\left(\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right)^2$$
$$-\frac{-\lambda^4 + 4\mu\lambda^2 - \lambda^2p^2 + \lambda^2r^2 - a_0^2\lambda^2 + 4a_0^2\mu}{2\lambda^2}, \tag{51}$$

where $\xi = x \pm \frac{\sqrt{-2a_0^2-\lambda^2}}{\lambda}t$, a_0, C_1 and C_2 are arbitrary constants.

In particular, if $C_1 \neq 0, C_2 = 0,$, then $u_1 = u_1(\xi)$ can be written as

$$u_1(\xi) = a_0\lambda\tanh\frac{\lambda}{2}\exp[\mathbf{i}(px - rt)], \tag{52}$$

$$v_1(\xi) = 2\lambda\text{sech}^2\frac{\lambda}{2} + 2 + \lambda^2 + p^2 - r^2 - a_0^2, \tag{53}$$

which is the known solitary wave solution of the coupled Higgs equation (35) and (36) (see [24]).

When $\lambda^2 - 4\mu < 0$,

$$u_2(\xi) = \frac{a_0\sqrt{4\mu - \lambda^2}}{\lambda}\left(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right), \tag{54}$$

$$v_2(\xi) = -2(4\mu - \lambda^2)\left(\frac{-C_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi - C_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right)^2$$
$$-\frac{-\lambda^4 + 4\mu\lambda^2 - \lambda^2p^2 + \lambda^2r^2 - a_0^2\lambda^2 + 4a_0^2\mu}{2\lambda^2}, \tag{55}$$

where $\xi = x \pm \frac{\sqrt{-2a_0^2-\lambda^2}}{\lambda}t$, a_0, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$,

$$u_3(\xi) = 0, \tag{56}$$

$$v_3(\xi) = -\frac{-\lambda^4 + 4\mu\lambda^2 - \lambda^2p^2 + \lambda^2r^2 - a_0^2\lambda^2 + 4a_0^2\mu}{2\lambda^2}, \tag{57}$$

where $\xi = x \pm \frac{\sqrt{-2a_0^2-\lambda^2}}{\lambda}t$, a_0, C_1 and C_2 are arbitrary constants.

6. Conclusion

In this paper, we obtained three types of new travelling wave solutions in general form for some nonlinear evolution equations, the Benjamin, Bona and Mahony equation, the weak symmetric equation, the Mindlin equation and the Higgs equations based on the $\left(\frac{G'}{G}\right)$-expansion method, and as a result some of the previously known traveling wave solutions were recovered as special cases. Our results reveal that the $\left(\frac{G'}{G}\right)$-expansion method is concise, direct, easy to apply, yet powerful tool for solving various kinds of nonlinear problems arising in mathematical physics.

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