Honam Mathematical J. **33** (2011), No. 2, pp. 231–246

CATEGORY WHICH IS SUITABLE FOR STUDYING KHALIMSKY TOPOLOGICAL SPACES WITH DIGITAL CONNECTIVITY

Sang-Eon Han

Abstract. Let $X_{n,k}$ be a Khalimsky topological n dimensional subspace with digital k-connectivity. In relation to the classification of spaces $X_{n,k}$, by comparing several kinds of continuities and homeomorphisms, the paper proposes a category which is suitable for studying the spaces $X_{n,k}$.

1. Introduction

In relation to the study of subsets of the Euclidean nD space with integer coordinates, we have often used the Khalilmsky topological structure [5, 9, 10, 12, 15]. Let $\mathbf{Z}$, $\mathbf{N}$ and $\mathbf{Z}^n$ represent the sets of integers, natural numbers and points in the Euclidean n-dimensional space with integer coordinates, respectively. Motivated by the Alexandroff space [1], the Khalimsky nD space (briefly, $(\mathbf{Z}^n, T^n)$) was established and the study of its properties includes the papers [9, 12, 15]. In this paper we consider a subset $X \subset \mathbf{Z}^n$ as a subspace of $(\mathbf{Z}^n, T^n)$, denoted by (X, T^n_X), $n \geq 1$ [5, 9, 10].

Even though Khalimsky topology of $\mathbf{Z}^n$ has strong merits of studying objects in $\mathbf{Z}^n$, a Khalimsky continuous map need not preserve digital connectivity [9]. However, if a map $f : (X, T^{n_0}_X) \to (Y, T^{n_1}_Y)$ performs Khalimsky continuity and digital connectivity, then it can be very useful in digital topology. Thus, we have often studied a Khalimsky topological space with digital k-connectivity of $\mathbf{Z}^n, n \in \mathbf{N}$ [9, 10] (see Definition 1) and further, used a map preserving digital connectivity [5].

Received April 13, 2011. Accepted May 23, 2011.

2000 Mathematics Subject Classification. 57M10; Secondary 57S30, 55Q20, 68T10, 68U05, 68U10.

Key words and phrases. Khalimsky topology, Khalismky continuity,digital connectivity, homeomorphism.

In the classification of Khalimsky topological space with digital k-connectivity, we study a special kind of category, denoted by CTC, consisting of the following two classes:
(1) a class $Ob(C)$ of $X_{n,k}$;
(2) a class $Mor(X, Y)$ of (k_0, k_1)-continuous maps in Definition 5 for each pair X_{n_0,k_0} and Y_{n_1,k_1} in $Ob(C)$ as morphisms.
Finally, by using a special kind of homeomorphism in CTC (see Definition 5), we can classify spaces $X_{n,k}$.

This paper is organized as follows. Section 2 provides some basic notions. Section 3 investigates some properties of several kinds of categories for studying the spaces $X_{n,k}$. Section 4 compares several kinds of homeomorphisms and shows some merits of CTC. Section 5 concludes the paper with a concluding remark and further work.

2. Preliminaries

In order to classify Khalimsky topological spaces with digital k- connectivity, we can consider several kinds of homeomorphisms from the viewpoint of Khalimsky topology. To do this work, we need to recall some basic notions as follows. *Khalimsky line topology* on $\mathbf{Z}$ is induced from the subbasis $\{[2n-1, 2n+1]_{\mathbf{Z}} : n \in \mathbf{Z}\}$ [1] (see also [12, 15]), where for $\{a, b\} \subset \mathbf{Z}$ with $a \lneq b$, we use the notation $[a, b]_{\mathbf{Z}} = \{a \leq n \leq b : n \in \mathbf{Z}\}$. Furthermore, the typical product topology on $\mathbf{Z}^n$ induced from $(\mathbf{Z}, T)$ is called the *Khalimsky product topology* on $\mathbf{Z}^n$ (or the *Khalimsky nD space*), denoted by $(\mathbf{Z}^n, T^n)$. In this paper each space $X \subset \mathbf{Z}^n$ will be considered to be a subspace (X, T^n_X) induced from $(\mathbf{Z}^n, T^n)$.
In $(\mathbf{Z}^n, T^n)$ a point $x = (x_1, x_2, \cdots, x_n) \in \mathbf{Z}^n$ is *pure open* if all coordinates are odd, *pure closed* if each of the coordinates is even [12] and the other points in $\mathbf{Z}^n$ is called *mixed* [12]. In each of the spaces in Figures 1 and 2, a black big dot stands for a pure open point and the symbols ■ and • mean a pure closed point and a mixed point, respectively.

It is meaningful to study a multidimensional Khalimsky topological space (X, T_X) with digital k-connectivity denoted by $X_{n,k}$ (see Definition 1). Thus let us recall some basic notions, as follows. Owing to the *digital k-connectivity paradox* [14], a set $X \subset \mathbf{Z}^n$ with one of the k-adjacency relations of $\mathbf{Z}^n$ is usually considered in a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$, where $n \in \mathbf{N}$, $X \subset \mathbf{Z}^n$ is the set of points we regard as belonging to the set depicted, k represents an adjacency relation for X and $\bar{k}$ represents an adjacency relation for $\mathbf{Z}^n - X$ [16]. But the paper is not concerned with the $\bar{k}$-adjacency of X. We say that the pair (X, k) is *a digital space with k-adjacency* (briefly, *a digital space*) in $\mathbf{Z}^n$.

As a generalization of the commonly used 4- and 8-adjacency relations of $\mathbf{Z}^2$ and further, 6-, 18- and 26-adjacency relations of $\mathbf{Z}^3$ [14, 16], the k-adjacency relations of $\mathbf{Z}^n$ are represented, as follows.
For a natural number m with $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n,$$

are $k(m, n)$-(*briefly*, k-)adjacent if
• there are at most m indices i such that $|p_i - q_i| = 1$ and
• for all other indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.

In this operator $k := k(m, n)$ is the number of points q which are k-adjacent to a given point p according to the numbers m and n in $\mathbf{N}$, where " :=" means *equal* by definition. Indeed, this $k(m, n)$-adjacency is a generalization of the k-adjacency of [16]. Consequently, this operator leads to the k-adjacency relations of $\mathbf{Z}^n$ [8] (for more details, see [9]):

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n,$$

where $C_i^n = \frac{n!}{(n-i)!\ i!}$.
For example, 8-, 32-, 64- and 80-adjacency relations of $\mathbf{Z}^4$ are considered and further, 10-, 50-, 130-, 210- and 242-adjacency relations of $\mathbf{Z}^5$ are used. For $\{a, b\} \subset \mathbf{Z}$ with $a \lneq b$, $[a, b]_{\mathbf{Z}} = \{a \leq n \leq b : n \in \mathbf{Z}\}$ with 2-adjacency can be recognized to be *digital interval* as a set or a subspace of $(\mathbf{Z}, T)$.

We say that a set of points in $(\mathbf{Z}^n, k)$ is *k-connected* if it is not a union of two disjoint non-empty sets not k-adjacent to each other [14]. For a set (X, k) in $\mathbf{Z}^n$, two distinct points $x, y \in X$ are called *k-connected* if there is a k-path $f : [0, m]_{\mathbf{Z}} \to X$ whose image is a sequence $(x_0, x_1, \cdots, x_m)$ from the set of points $\{f(0) = x_0 = x, f(1) = x_1, \cdots, f(m) = x_m = y\}$ such that x_i and x_{i+1} are k-adjacent, $i \in [0, m-1]_{\mathbf{Z}}, m \geq 1$. For a digital space (X, k) and a point $x \in X$, we say that the maximal k-connected subset of (X, k) containing the point $x \in X$ is the *k-(connected) component* of a point $x \in X$ [14]. The number m is called the *length* of this k-path [14]. For an adjacency relation k, a *simple k-path* in X is the sequence $(x_i)_{i \in [0,m]_{\mathbf{Z}}}$ such that x_i and x_j are k-adjacent if and only if either $j = i + 1$ or $i = j + 1$ [14].

3. Several Kinds of Khalimsky Topological Categories

Since the paper [5] established several kinds of homeomorphisms which can be used in studying Khalimsky topological spaces with digital connectivity, let us now recall them and discuss several kinds of Khalimsky topological categories.

Definition 1. [9, 10] *For a set $X \subset \mathbf{Z}^n$, consider the subspace (X, T^n_X) induced from $(\mathbf{Z}^n, T^n)$. Assuming (X, T^n_X) with one of the k-adjacency relations of $\mathbf{Z}^n$, we denote it (X, k, T^n_X) (briefly, $X_{n,k}$ called a space if there is no confusion).*

Let us now recall several kinds of continuities from the viewpoint of Khalimsky topology as follows. For two Khalimsky topological spaces $(X, T^{n_0}_X) := X$ and $(Y, T^{n_1}_Y) := Y$, we can consider a Khalimsky continuous map at a point $x \in X$.

Using Khalimsky continuous maps, we obtain the *Khalimsky topological category*, briefly KTC, consisting of two classes:
(1) A class of objects (X, T^n_X);
(2) For every ordered pair of objects $(X, T^{n_0}_X)$ and $(Y, T^{n_1}_Y)$, a class of all Khalimsky continuous maps $f : (X, T^{n_0}_X) \to (Y, T^{n_1}_Y)$ as morphisms.

The paper [5] (see also [9, 10]) shows that a Khalimsky continuous map $f : X_{n_0,k} \to Y_{n_1,k_1}$ need not preserve the k_0-connectivity of $X_{n_0,k}$ into the k_1-connectivity of Y_{n_1,k_1}. But, if a map preserves digital connectivity, then it can be very useful in digital topology. Thus we need a map which preserves both digital connectivity and Khalimsky continuity. For a space $X_{n,k}$, let us now recall the notion of special kind of (Khalimsky topological) k-neighborhood.

Definition 2. [4] *(see also* [9, 10]*) For space $X_{n,k} := X$, consider two distinct points $x, y \in X$ and $\varepsilon \in \mathbf{N}$.*

(1) By $l_k(x, y)$ we denote the length of a shortest k-path from x to y in X. Furthermore, we assume that $l_k(x, y) = \infty$ if there is no k-path from x to y.

(2) A subset V of X is called a neighborhood of x if there exists an open set $O_x \in T^n_X$ such that $x \in O_x \subseteq V$.

(3) By $N_k(x, \varepsilon)$ we denote the set

$$\{y \in X : l_k(x, y) \leq \varepsilon\} \cup \{x\}$$

Furthermore, if $\{x\}$ is a singleton set up to a k-adjacency, then we assume that $N_k(x, \varepsilon) = \{x\}$ for any $\varepsilon \in \mathbf{N}$.

(4) If the set $N_k(x, \varepsilon)$ is a (Khalimsky product topological) neighborhood of x in X, then this set is called a Khalimsky topological k-neighborhood of x with radius ε and is denoted by $N_k^*(x_0, \varepsilon)$ instead of $N_k(x, \varepsilon)$.

The paper denotes by $SC_k^{n,l,*}$ a simple closed k-curve with l elements in $\mathbf{Z}^n$ of which some point $x \in SC_k^{n,l,*}$ does not have $N_k^*(x, 1) \subset SC_k^{n,l,*}$. Similarly, we denote by $C_k^{n,l,*}$ a closed k-curve with l elements in $\mathbf{Z}^n$ of which some point $x \in C_k^{n,l,*}$ does not have $N_k^*(x, 1) \subset C_k^{n,l,*}$.
For instance, consider the space $SC_4^{2,8,*} := (e_i)_{i \in [0,7]_\mathbf{Z}}$ in Figure 1(b). Then we observe that $N_4^*(e_6, 2) = SC_4^{2,8,*} - \{e_i | i \in [1, 3]_\mathbf{Z}\}$ and it does not have $N_4^*(e_6, 1)$ because the smallest open set of the point e_6 is exactly $N_4^*(e_6, 2)$.
Meanwhile, in $SC_4^{2,8} := (v_i)_{i \in [0,7]_\mathbf{Z}}$ of Figure 1(b), for every point $v_i \in SC_4^{2,8}$ we observe that $N_4^*(v_i, 1) = \{v_{i-1(mod\,8)}, v_i, v_{i+1(mod\,8)}\}$. In addition, the Khalimsky line space $(\mathbf{Z}, T)$ is assumed with 2-adjacency.
Indeed, we observe that $N_k(x_0, \varepsilon)$ of Definition 2(4) is different from $N_k^*(x_0, \varepsilon)$ of Definition 2(3) if there is no $N_k^*(x_0, \varepsilon)$ in $X_{n,k}$ owing to the topological structure of a given Khalimsky topological space.

Definition 3. [4]*(see also* [6]*) For two digital spaces (X, k_0) and (Y, k_1), we say that f is digitally (k_0, k_1)-continuous at a point $x_0 \in X$ if $f(N_{k_0}(x_0, 1)) \subset N_{k_1}(f(x_0), 1)$.*
If f is (k_0, k_1)-continuous at every point $x \in X$, then f is called a (k_0, k_1)-continuous map. If $k_0 = k_1$, then we call it a k_0-continuous map.

We say that a map $f : (X, x_0) \to (Y, y_0)$ is a *pointed map* if $f(x_0) = y_0$ in this paper. The current presentation of the digital continuity has been often used in digital k-curve and digital k-surface theory. Unlike the pasting property of classical continuity in topology, the digital (k_0, k_1)-continuity of Definition 3 has some intrinsic features [13]: Digital (k_0, k_1)-continuity has *the almost pasting property* instead of *the pasting property* of classical topology.

Using the digital (k_0, k_1)-continuity, we obtain the *digital topological category*, briefly *DTC*, consisting of two classes [4] (see also [9]):
(1) A class of objects (X, k) in $\mathbf{Z}^n$;
(2) For every ordered pair of objects (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a class of all (k_0, k_1)-continuous maps $f : (X, k_0) \to (Y, k_1)$ as morphisms.

Owing to the limitation of Khalimsky continuity related to the preservation of digital connectivity, the following three kinds of continuities

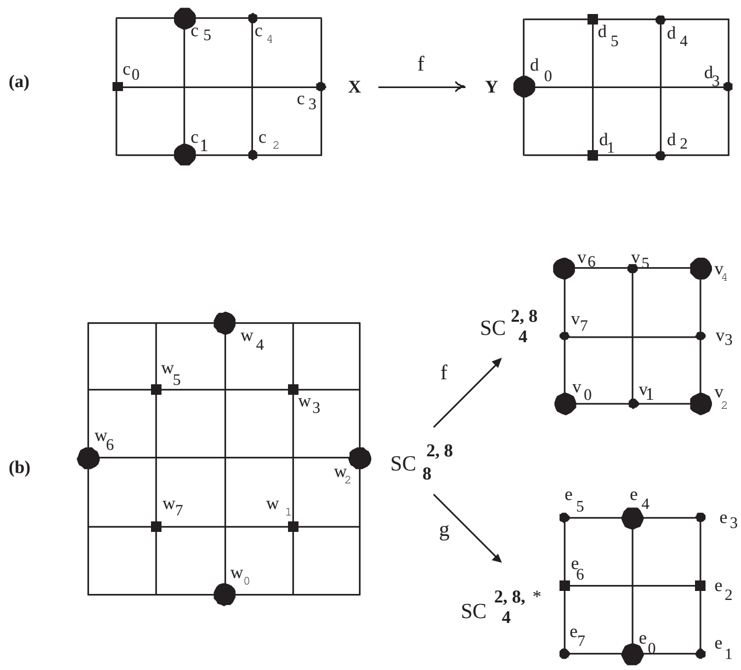

FIGURE 1. Several kinds of simple closed k-curves [5]

were developed. By using Definitions 1 and 3, we have established the following:

Definition 4 (Khalimsky digital(briefly, KD-)(k_0, k_1)-continuity)**.** [5]*(see also* [9]*)*
For two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, a function $f : X \to Y$ is said to be KD-(k_0, k_1)-continuous at a point $x \in X$ if
(1) f is (Khalimsky) continuous at the point x and
(2) f is digitally (k_0, k_1)-continuous at the point $x \in X$.
Furthermore, we say that a map $f : X \to Y$ is KD-(k_0, k_1)-continuous if the map f is KD-(k_0, k_1)-continuous at every point $x \in X$.

None of the conditions (1) and (2) of Definition 4 implies the other [5]. Indeed, the property (2) of Definition 4 is not related to the Khalimsky topology. However, in order to preserve the k_0-connectedness of X_{n_0,k_0} into the k_1-connectedness of Y_{n_0,k_1}, the current KD-(k_0, k_1)-continuity can be used for studying $X_{n,k}$.

By the same method as the establishment of DTC, the paper [5] established a Khalimsky topological category (briefly, $KDTC$) consisting of two classes:
(1) A class $Ob(C)$ of $X_{n,k}$;
(2) A class $Mor(X, Y)$ of KD-(k_0, k_1)-continuous maps as morphisms.

Now, by using the Khalimsky topological k-neighborhood in Definition 2(4), we obtain the following continuity [5]:

Definition 5 ((k_0, k_1)-Continuity). [5] *For two spaces* $X_{n_0,k_0} := X$ *and* $Y_{n_1,k_1} := Y$, *a function* $f : X \to Y$ *is said to be* (k_0, k_1)*-continuous at a point* $x \in X$ *if* $f(N^*_{k_0}(x, r)) \subset N^*_{k_1}(f(x), s)$, *where the number* r *is the least element of* $\mathbf{N}$ *such that* $N^*_{k_0}(x, r)$ *contains an open set including the point* x *(so* $N^*_{k_0}(x, r) = N_{k_0}(x, r)$*) and* s *is the least element of* $\mathbf{N}$ *such that* $N^*_{k_1}(f(x), s)$ *contains an open set including the point* $f(x)$ *(so* $N^*_{k_1}(f(x), s) = N_{k_1}(f(x), s)$*).*
Furthermore, we say that a map $f : X \to Y$ *is* (k_0, k_1)*-continuous if the map* f *is* (k_0, k_1)*-continuous at every point* $x \in X$.

Owing to the difference between $N_k(x, \varepsilon)$ and $N^*_k(x, \varepsilon)$, we clearly observe that KD-(k_0, k_1)-continuity of Definition 4 is different from (k_0, k_1)-continuity of Definition 5 [5].

By using the (k_0, k_1)-continuity of Definition 5, we can establish a Khalimsky topological category (briefly, CTC) consisting of two classes:
(1) A class $Ob(C)$ of $X_{n,k}$;
(2) A class $Mor(X, Y)$ of (k_0, k_1)-continuous maps for each pair X_{n_0,k_0} and Y_{n_1,k_1} in $Ob(C)$ as morphisms.

Using Definitions 1 and 5, we obtain the following:

Definition 6 (Khalimsky (k_0, k_1)-(briefly, K-(k_0, k_1)-)continuity). [5] *For two spaces* $X_{n_0,k_0} := X$ *and* $Y_{n_1,k_1} := Y$, *a function* $f : X \to Y$ *is said to be* K-(k_0, k_1)*-continuous at a point* $x \in X$ *if*
(1) f *is (Khalimsky) continuous at the point* x *and*
(2) f *is* (k_0, k_1)*-continuous at the point* x.
Furthermore, we say that a map $f : X \to Y$ *is* K-(k_0, k_1)*-continuous if* f *is* K-(k_0, k_1)*-continuous at every point* $x \in X$.

We observe that none of the properties (1) and (2) of Definition 6 implies the other [5]. In Definition 6, if such a kind of neighborhood $N^*_{k_1}(f(x), \varepsilon)$ or $N^*_{k_0}(x, \delta)$ does not exist, then we clearly say that f cannot be K-(k_0, k_1)-continuous at the point x. Owing to the difference between $N_k(x, \varepsilon)$ and $N^*_k(x, \varepsilon)$, we observe that KD-(k_0, k_1)-continuity is different from K-(k_0, k_1)-continuity [5].

By using the notion of Definition 6, the paper [5] established a Khalimsky topological category (briefly, *KCTC*) consisting of the following two classes:
(1) A class $Ob(C)$ of $X_{n,k}$;
(2) A class $Mor(X, Y)$ of K-(k_0, k_1)-continuous maps for each pair X_{n_0,k_0} and Y_{n_1,k_1} in $Ob(C)$ as morphisms.

As discussed above, for a map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ the notions of Khalimsky continuity, (k_0, k_1)-continuity, KD-(k_0, k_1)-continuity and K-(k_0, k_1)-continuity are different from each other and further, their usages depend on the spaces X_{n_0,k_0} and Y_{n_1,k_1}.

4. Merits of the Category *CTC* for Studying Khalimsky Topological Spaces

The paper [5] established several kinds of homeomorphisms in *CTC, KDTC* and *KCTC* to classify spaces $X_{n,k}$ up to a KD-k-homeomorphism, a k-homeomorphism or a K-k-homeomorphism. In this section, comparing them, we observe some merits of a homeomorphism in *CTC*.

As usual, a digital space (X, k) can be considered as a digital k-graph $G_k := (V_k, E_k)$ [16]. Thus, digital graph versions of (k_0, k_1)-continuity and (k_0, k_1)-homeomorphism were presented [5]. Thus we can represent a digital (k_0, k_1)-homeomorphism in [2] as a (k_0, k_1)-isomorphism in [11](see also [4]), as follows.

In *DTC*, in order to classify digital spaces (X, k) up to a digital k-isomorphism, we have used the following:

Definition 7 (Digital (k_0, k_1)-isomorphism)**.** [2] *(see also* [4]*) In DTC for two digital spaces (X, k_0) and (Y, k_1), a function $f : X \to Y$ is said to be a digital (k_0, k_1)-isomorphism if*
(1) the map f is bijective and
(2) the map f is a digitally (k_0, k_1)-continuous map and further, f^{-1} is a digitally (k_1, k_0)-continuous map.
Then we say that the space X is digitally (k_0, k_1)-isomorphic to Y.

Let us now recall several kinds of homeomorphisms in Khalimsky topology, as follows.

Definition 8 (Khalimsky homeomorphism)**.** [5] *For two Khalimsky spaces $(X, T_X^{n_0}) := X$ and $(Y, T_Y^{n_1}) := Y$, a map $h : X \to Y$ is called a Khalimsky homeomorphism if h is a Khalimsky continuous bijection and further, $h^{-1} : Y \to X$ is Khalimsky continuous.*

Definition 9 (KD-(k_0, k_1)-homeomorphism)**.** [5] *In KDTC, for two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, a function $f : X \to Y$ is said to be a KD-(k_0, k_1)-homeomorphism if*
(1) the map f is bijective and
(2) the map f is a KD-(k_0, k_1)-continuous map and further, f^{-1} is a KD-(k_1, k_0)-continuous map.
Then we say that the space X is KD-(k_0, k_1)-homeomorphic to Y.

Definition 10 ((k_0, k_1)-homeomorphism)**.** [5] *In CTC, for two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, a map $h : X \to Y$ is called a (k_0, k_1)-homeomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous.*
Then we say that the space X is (k_0, k_1)-homeomorphic to Y.

Definition 11 (K-(k_0, k_1)-homeomorphism)**.** [9] *In KCTC, for two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, a map $h : X \to Y$ is called a K-(k_0, k_1)-homeomorphism if*
(1) h is a K-(k_0, k_1)-continuous bijection and
(2) $h^{-1} : Y \to X$ is K-(k_1, k_0)-continuous.
Then we say that the space X is K-(k_0, k_1)-homeomorphic to Y.

In Definitions 8, 9, 10 and 11, by $X \approx_{Kh} Y$, $X \approx_{KD\cdot(k_0,k_1)} Y$, $X \approx_{(k_0,k_1)} Y$, and $X \approx_{K\cdot(k_0,k_1)} Y$, we denote a Khalimsky homeomorphism, a KD-(k_0, k_1)-, a (k_0, k_1)- and a K-(k_0, k_1)-homeomorphism from X_{n_0,k_0} to Y_{n_1,k_1}, respectively [9]. If $n_0 = n_1$ and $k_0 = k_1$, then we use the notation '$\approx_{k_0}$' instead of '$\approx_{(k_0,k_0)}$'.

Let us now investigate some intrinsic features of the homeomorphisms in Definitions 9, 10 and 11, as follows.

Example 4.1. *For the three spaces $SC_8^{2,8} := \{w_i | i \in [0, 7]_{\mathbf{Z}}\}$, $SC_4^{2,8} := \{v_i | i \in [0, 7]_{\mathbf{Z}}\}$ and $SC_4^{2,8,*} := \{e_i | i \in [0, 7]_{\mathbf{Z}}\}$ in Figure 1(b), we obtain the following:*

(1) $SC_8^{2,8}$ is $(8, 4)$-homeomorphic to $SC_4^{2,8}$.

(2) $SC_8^{2,8}$ is neither KD-$(8, 4)$-, K-$(8, 4)$- nor $(8, 4)$-homeomorphic to $SC_4^{2,8,*}$.

(3) $SC_4^{2,8}$ is neither 4-, KD-4- nor K-4-homeomorphic to $SC_4^{2,8,*}$.

(4) $SC_4^{2,8}$ is neither K-4- nor KD-4-homeomorphic to $SC_4^{2,8,*}$.

Proof: (1) Consider the $(8, 4)$-continuous map $f : SC_8^{2,8} := \{w_i | i \in [0, 7]_{\mathbf{Z}}\} \to SC_4^{2,8} := \{v_i | i \in [0, 7]_{\mathbf{Z}}\}$ given by $f(w_i) = v_i$, $i \in [0, 7]_{\mathbf{Z}}$ (see Figure 1(b)). Then the proof is completed.

(2) The smallest open sets containing the points e_2 and e_6 in $SC_4^{2,8,*}$ in Figure 1(b) are $\{e_0, e_1, e_2, e_3, e_4\} = N_4(e_2, 2)$ and $\{e_4, e_5, e_6, e_7, e_0\} = N_4(e_6, 2)$, respectively. This implies that any KD-$(8,4)$-, K-$(8,4)$- and $(8,4)$-continuous bijections $g : SC_8^{2,8} \to SC_4^{2,8,*}$ cannot have their associating KD-$(4,8)$-, K-$(4,8)$- and $(4,8)$-continuities of g^{-1}, respectively. More precisely, consider the map $g : SC_8^{2,8} \to SC_4^{2,8,*}$ given by $g(w_i) = e_i, i \in [0,7]_{\mathbf{Z}}$. Then, while the bijection g is $(8,4)$-, KD-$(8,4)$- and K-$(8,4)$-continuous, the inverse map cannot be $(4,8)$-, KD-$(4,8)$- and K-$(4,8)$-continuous at the two points e_2 and e_6 in $SC_4^{2,8,*}$.

(3) Owing to the two points e_2 and $e_6 \in SC_4^{2,8,*}$, by the same method as the proof of (2), the proof is completed. For instance, even though there is a 4-continuous bijection $h : SC_4^{2,8} \to SC_4^{2,8,*}$ given by $h(v_i) = e_i, i \in [0,7]_{\mathbf{Z}}$, the inverse map h^{-1} cannot be 4-continuous at the points e_2 and e_6 in $SC_4^{2,8,*}$.

(4) Owing to the points e_2 and e_6 in $SC_4^{2,8,*}$, by the same method as the above, the proof is completed. □

As discussed in Example 4.1, in $(\mathbf{Z}^2, T^2)$ it turns out that each of several homeomorphisms for studying spaces in *KDTC, KCTC* and *CTC* has its own intrinsic merits. In $(\mathbf{Z}^2, T^2)$ let us now investigate some properties of KD-(k_0, k_1)-, K-(k_0, k_1)- and (k_0, k_1)-homeomorphism, as follows.

Theorem 4.2. *Consider the four types of simple closed* 18*-curves in Figure 2, i.e. three types of* $SC_{18}^{3,6}$ *in (a), (b) and (c) of Figure 2, and the other* $SC_{18}^{3,6,*}$*. Then, we obtain the following:*

(1) $SC_{18}^{3,6,*}$ *is neither KD-*18*-, K-*18*- nor* 18*-homeomorphic to* $SC_{18}^{3,6}$*.*

*(2) None of the spaces in (a), (b) and (c) of Figure 2 is KD-*18*-homeomorphic to the other.*

*(3) None of the spaces in (a), (b) and (c) of Figure 2 is K-*18*-homeomorphic to the other.*

(4) Each of the spaces in (a), (b) and (c) of Figure 2 is 18*-homeomorphic to the others.*

Proof: (1) While the point $a_0 \in SC_{18}^{3,6,*}$ has the smallest open set $N_{18}^*(a_0, 2)$, each point $x \in SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2 has the smallest open sets in $N_{18}^*(x, 1) \subset SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2, respectively. This implies that $SC_{18}^{3,6,*}$ cannot be KD-$(18,18)$-, K-18- and 18-homeomorphic to each of the spaces $SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2 because there cannot be KD-18-, K-18- and 18-continuous maps from $SC_{18}^{3,6,*}$ onto each of the spaces $SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2.

In order to prove the cases (2), (3) and (4), we need to investigate each of Khalimsky topological structures of $SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2, as follows. First, we can observe that the set

$$\{\{b_i|i \in \{1,3,5\}\}, \{b_1, b_2\}, \{b_4, b_5\}\} \tag{4.1}$$

is a subbase of the Khalimsky topological structure of $SC_{18}^{3,6}$ in (a) of Figure 2.

Second, the space $SC_{18}^{3,6}$ in (b) of Figure 2 has the following subbase:

$$\{\{c_i|i \in \{2,3,4\}\}, \{c_0, c_1, c_2\}, \{c_0, c_4, c_5\}\}. \tag{4.2}$$

Third, the space $SC_{18}^{3,6}$ in (c) of Figure 2 has the following subbase:

$$\{\{d_i|i \in \{0,1,3,5\}, \{d_1, d_2\}, \{d_4, d_5\}\}. \tag{4.3}$$

In terms of (4.1), (4.2) and (4.3), it turns out that each of the spaces $SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2 is different from each other from the viewpoints of *KTC, KDTC, KCTC and CTC.*

Let us now prove the assertions (2), (3) and (4).

(2) In view of (4.1), (4.2) and (4.3), we can observe that none of the spaces in (a), (b) and (c) of Figure 2 is Khalimsky homeomorphic to the other. Thus, the proof is completed.

(3) By the same reason as (2) above, the proof is completed.

(4) Since each point x of the spaces $SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2 has $N_{18}^*(x,1) \subset SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2, respectively, each of the spaces in (a), (b) and (c) of Figure 2 is 18-homeomorphic to the others in *CTC*. □

In view of Theorem 4.2, we obtain the following:

Theorem 4.3. *For two spaces* $SC_{k_0}^{n_0,l_0,*}$ *and* $SC_{k_1}^{n_1,l_1,*}$, *even though* $l_0 = l_1$, $SC_{k_0}^{n_0,l_0,*}$ *need not be KD-*(k_0,k_1)*-,* K-(k_0,k_1)*- and* (k_0,k_1)*-homeomorphic to* $SC_{k_1}^{n_1,l_1,*}$.

Proof: Depending on both the smallest numbers $\varepsilon_i \geq 2$ of $N_{k_i}^*(x_i, \varepsilon_i) \subset SC_{k_i}^{n_i,l_i,*}$, $i \in \{0,1\}$ and an arrangement of the point x_i of the $N_{k_i}^*(x_i, \varepsilon_i) \subset SC_{k_i}^{n_i,l_i,*}$, the Khalimsky topological structures of $SC_{k_0}^{n_0,l_0,*}$ and $SC_{k_1}^{n_1,l_1,*}$ are determined. □

Let us now investigate some properties of KD-(k_0,k_1)-, K-(k_0,k_1)- and (k_0,k_1)-continuous surjections $f : ([0,m_f]_{\mathbf{Z}}, T_{[0,m_f]_{\mathbf{Z}}}) \to SC_{k_0}^{n_0,l,*}$ or $SC_{k_0}^{n_0,l}$.

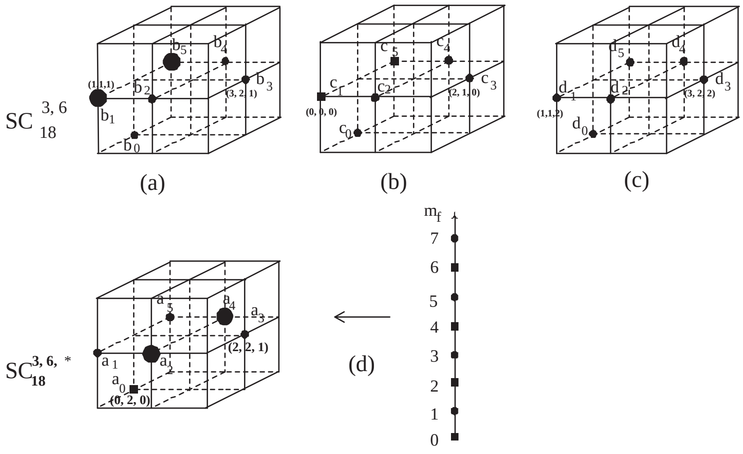

FIGURE 2. Comparison between $SC_{18}^{3,6,*}$ and $SC_{18}^{3,6}$ up to several kinds of homeomorphisms.

Theorem 4.4. *Consider the three types of* $SC_{18}^{3,6}$ *in (a), (b) and (c) of Figure 2, and* $SC_{18}^{3,6,*}$ *in Figure 2. Then, we obtain some properties of several kinds of continuities established in KDTC, KCTC and CTC.*

(1) There are KD-$(2,18)$*-, K-*$(2,18)$*- and* $(2,18)$*-continuous surjections* $f : ([0,m_f]_{\mathbf{Z}}, T_{[0,m_f]_{\mathbf{Z}}}) \to SC_{18}^{3,6,*}$ *in Figure 2(d) and the smallest cardinality of* $Dom(f)$ *is greater than six, i.e.* $m_f - 1 \gneq 6$.

(2) No K-$(2,18)$*-continuous surjection from* $([0,m_f]_{\mathbf{Z}}, T_{[0,m_f]_{\mathbf{Z}}})$ *into each of the three spaces* $SC_{18}^{3,6}$ *in (a), (b) and (c) of Figure 2 exists.*

(3) No KD-$(2,18)$*-continuous surjection from* $([0,m_f]_{\mathbf{Z}}, T_{[0,m_f]_{\mathbf{Z}}})$ *into each of the three spaces* $SC_{18}^{3,6}$ *in (a), (b) and (c) of Figure 2 exists.*

(4) There is a $(2,18)$*-continuous bijection* $f : ([0,m_f]_{\mathbf{Z}}, T_{[0,m_f]_{\mathbf{Z}}}) \to SC_{18}^{3,6}$ *in (a), (b) and (c) of Figure 2 such that the smallest cardinality of* $Dom(f)$ *is equal to six.*

Proof:(1) We observe that the space $SC_{18}^{3,6,*}$ in Figure 2 has the following subbase:

$$\{\{a_2\},\{a_4\},\{a_0,a_1,a_2,a_4,a_5\},\{a_2,a_3,a_4\},\{a_1,a_2\},\{a_4,a_5\}\}. \quad (4.4)$$

Owing to the topological structure of (4.4), we can establish KD-$(2,18)$-,
K-$(2,18)$- and $(2,18)$-continuous surjection $f : ([0,m_f]_{\mathbf{Z}}, T_{[0,m_f]_{\mathbf{Z}}}) \to SC_{18}^{3,6,*}$ and further, each of these maps has the smallest cardinality of

$Dom(f)$ which is greater than six.
For instance, consider a map $f : [0, m_f]_{\mathbf{Z}} \to SC_{18}^{3,6,*} := (a_i)_{i \in [0,5]_{\mathbf{Z}}}$ such that $Dom(f)$ is minimal in such a way:

$$f(0) = a_0, f(1) = a_1 = f(2), f(3) = a_2, f(4) = a_3, f(5) = a_4, f(6) = f(7) = a_5.$$

Then we observe the following:
First, the map f is a Khalimsky continuous surjection with $Dom(f)$ minimal and $\sharp Dom(f) = 8 \gneqq 6$, where $\sharp$ means the cardinality of a given set.
Second, the map f is $(2, 18)$-continuous in CTC.
Third, the map f is digitally $(2, 18)$-continuous in DTC.
Thus, the map f is both a KD-$(2, 18)$- and a K-$(2, 18)$-continuous surjections with $Dom(f)$ minimal such that the smallest cardinality of $Dom(f)$ is greater than six, *i.e.* $m_f - 1 \gneqq 6$.

(2) Let us examine if there is a Khalimsky continuous surjection $g : [0, m_f]_{\mathbf{Z}} \to SC_{18}^{3,6}$ in Figure 2(a), as follows.
First, as shown in (4.1), recall that the space $SC_{18}^{3,6}$ in Figure 2(a) has the following subbase

$$\{\{b_i | i \in \{0, 1, 3, 5\}\}, \{b_1, b_2\}, \{b_4, b_5\}\}.$$

Then, owing to the point $b_3 \in SC_{18}^{3,6}$ in Figure 2(a), we cannot establish a K-$(2, 18)$-continuous surjection from $g : ([0, m_g]_{\mathbf{Z}}, T_{[0,m_g]_{\mathbf{Z}}}) \to SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2.
For instance,

$$g(0) = b_0, g(1) = b_1, g(2) = b_2 = g(3), \cdots . \tag{4.5}$$

Then, while the map g is Khalimsky continuous at $[0, 3]_{\mathbf{Z}} \subset [0, m_g]_{\mathbf{Z}}$, we now have some difficulty in finding a Khalimsky continuous surjection $g : [0, m_g]_{\mathbf{Z}} \to SC_{18}^{3,6}$ in Figure 2(a)(in particular, see the point $4 \in [0, m_g]_{\mathbf{Z}}$), as follows.

(Case 1): Assume $g(4) = b_3$. Then, owing to (4.1), the mapping cannot be Khalimsky continuous at the point $4 \in [0, m_g]_{\mathbf{Z}}$.

(Case 2): Assume $g(4) = b_2$. Then, owing to (4.1), we should take $g(5) = b_2$. Then, even though the map g is Khalimsky continuous at the points 4 and 5 in $[0, m_g]_{\mathbf{Z}}$, we shall meet an obstacle of the mapping $g(6)$ for being a Khalimsky continuous surjection $g : [0, m_g]_{\mathbf{Z}} \to SC_{18}^{3,6}$ in Figure 2(a), which is similar to that of Case 1 above.

Thus, in view of Cases 1 and 2 above, the establishment of a Khalimsky continuous surjection $g : [0, m_g]_{\mathbf{Z}} \to SC_{18}^{3,6}$ in Figure 2(a) cannot hold.

Even though we assume another map $g : [0, m_g]_{\mathbf{Z}} \to SC_{18}^{3,6}$ in Figure 2(a) instead of (4.5), we cannot have a Khalimsky continuous surjection $g : [0, m_g]_{\mathbf{Z}} \to SC_{18}^{3,6}$ in Figure 2(a), either. Thus, the proof is completed.

By the same method as the non-existence of a Khalimsky continuous surjection $g : [0, m_g]_{\mathbf{Z}} \to SC_{18}^{3,6}$ in Figure 2(a), the other cases with codomain $SC_{18}^{3,6}$ in Figure 2 (b) and (c) are also proved.

(3) By the same method as the proof of (2), since there is no Khalimsky continuous surjection from $([0, m_f]_{\mathbf{Z}}, T_{[0,m_f]_{\mathbf{Z}}})$ into each of the three spaces $SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2, the proof is completed.

(4) Consider the map $g : [0, 5]_{\mathbf{Z}} \to SC_{18}^{3,6} := (x_i)_{i \in [0,5]_{\mathbf{Z}}}$ in (a), (b) and (c) of Figure 2 such that $g(i) = x_i, i \in [0, 5]_{\mathbf{Z}}$. Then, for any point $x_i \in SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2, since there are both $N_{18}^*(x_i, 1) \subset SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2 and $N_2^*(t, 1) \subset (\mathbf{Z}, T)$ such that $g(N_2^*(t, 1)) \subset N_{18}^*(x_i, 1)$, the proof is completed. □

In view of Theorems 4.2 and 4.4, we can state some similarity between a (k_0, k_1)-homeomorphism in *CTC* and a digital (k_0, k_1)-isomorphism in *DTC*: The paper [9] establishes forgetful functors from each of *KCTC, KDTC* and *CTC* into *DTC*, denoted by

$$F^* : KCTC \to DTC, \quad F^* : KDTC \to DTC, \quad F^* : CTC \to DTC. \tag{4.6}$$

In terms of the forgetful functors F^* of (4.6), a space $X_{n,k}$ in *KCTC, KDTC*, or *CTC* can be transformed into a digital space (X, k). Furthermore, each of the K-(k_0, k_1)-, (k_0, k_1)- and KD-(k_0, k_1)-continuities are also changed into the digital (k_0, k_1)-continuity in *DTC*.

By comparing each of several kinds of homeomorphisms in Definitions 9, 10 and 11 with a digital (k_0, k_1)-isomorphism, we can observe some merits of a (k_0, k_1)-homeomorphism, as follows.

Remark 4.5. **[Merits of the Category** *CTC*] *As discussed in Example 4.1 and Theorem 4.2, while the three spaces $SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2 are neither KD-$(18, 18)$- nor K-18-homeomorphic to each other, they are all 18-homeomorphic to each other in CTC. Furthermore, in terms of the forgetful functor F^* we observe that the three spaces $SC_{18}^{3,6}$ in (a), (b) and (c) of Figure 2 are digitally 18-isomorphic to each other in DTC. Consequently, in relation with the classification of spaces $X_{n,k}$, we observe some similarity between a (k_0, k_1)-homeomorphism in*

CTC and a digital (k_0, k_1)*-isomorphism in DTC. Finally, we can observe that both KD-*(k_0, k_1)*- and K-*(k_0, k_1)*-homeomorphisms are so specific and a* (k_0, k_1)*-homeomorphism has some merits of classifying spaces* $X_{n,k}$.

5. Concluding Remark and Further Work

As discussed in Example 4.1, Theorems 4.2, 4.3 and 4.4, in relation to the preservation of digital k-connectivity it turns out that both Khalimsky continuity and Khalimsky homeomorphism have some drawback. Thus, we have studied several kinds of continuities and homeomorphisms of Definitions 4, 5, 6, 9, 10 and 11. In particular, by comparing the several kinds of homeomorphisms, we observed some merits of (k_0, k_1)-homeomorphism in *CTC*. By using this homeomorphism, we can substantially compare Khalimsky topological spaces with digital k-connectivity.

References

[1] P. Alexandorff, Diskrete Räume, *Mat. Sb.* **2** (1937) 501-518.

[2] L. Boxer, A classical construction for the digital fundamental group, *Jour. of Mathematical Imaging and Vision* **10** (1999) 51-62.

[3] S. E. Han, Non-product property of the digital fundamental group, *Information Sciences* **171**(1-3) (2005) 73-91.

[4] S.E. Han, On the simplicial complex stemmed from a digital graph, *Honam Mathematical Journal* **27**(1) (2005) 115-129.

[5] S.E. Han, Continuities and homeomorphisms in computer topology and their applications, *Journal of the Korean Mathematical Society* **45**(4)(2008) 923-952.

[6] S.E. Han, Equivalent (k_0, k_1)-covering and generalized digital lifting, *Information Sciences* **178**(2)(2008) 550-561.

[7] S.E. Han, Map preserving local properties of a digital image, *Acta Applicandae Mathematicae*, **104**(2) (2008) 177-190.

[8] S.E. Han, The k-homotopic thinning and a torus-like digital image in $\mathbf{Z}^n$, *Journal of Mathematical Imaging and Vision* **31**(1) (2008) 1-16.

[9] S.E. Han, KD-(k_0, k_1)-homotopy equivalence and its applications, *Journal of Korean Mathematical Society* **47**(5)(2010) 1031-1054.

[10] S.E. Han, N.D. Georgiou, On computer topological function space, *Journal of the Korean Mathematical Society* **46**(4) (2009) 841-857.

[11] S.E. Han and B.G. Park, Digital graph (k_0, k_1)-isomorphism and its applications, *http://atlas-conferences.com/c/a/k/b/36.htm*(2003).

[12] E. Khalimsky, R. Kopperman, P. R. Meyer, Computer graphics and connected topologies on finite ordered sets, *Topology Applications*, **36(1)**(1990) 1-17.

[13] In-Soo Kim, S.E. Han, C.J. Yoo, The pasting property of digital continuity, *Acta Applicandae Mathematicae* **110** (2010) 399-408.

[14] T. Y. Kong, A. Rosenfeld, Topological Algorithms for the Digital Image Processing, Elsevier Science, Amsterdam, 1996.

[15] E. Melin, Extension of continuous functions in digital spaces with the Khalimsky topology, *Topology and Its Applications* **153**(2005) 52-65.

[16] A. Rosenfeld, Continuous functions on digital pictures, *Pattern Recognition Letters*, **4** (1986) 177-184.

Sang-Eon Han
Faculty of Liberal Education,Institute of Pure and Applied Mathematics, Chonbuk National University,
Jeonju-si, Jeollabuk-do 561-756 Korea.
E-mail: sehan@jbnu.ac.kr