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LINEARLY INDEPENDENT SOLUTIONS FOR THE HYPERGEOMETRIC EXTON FUNCTIONS X_1 AND X_2

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Abstract. In investigation of boundary-value problems for certain partial differential equations arising in applied mathematics, we often need to study the solution of system of partial differential equations satisfied by hypergeometric functions and find explicit linearly independent solutions for the system. Here we choose the Exton functions X_1 and X_2 among his twenty functions to show how to find the linearly independent solutions of partial differential equations satisfied by these functions X_1 and X_2.

1. Introduction and Preliminaries

Solutions of many applied problems involving thermal conductivity and dynamics, electromagnetic oscillation and aerodynamics, quantum mechanics and potential theory are obtainable with the help of hypergeometric (higher and special or transcendent) functions [2, 3, 4, 5]. Functions of such kind are often referred to as special functions of mathematical physics. They mainly appear in the solution of partial differential equations which are dealt with harmonic analysis method (see [6]). In view of various applications, it is interesting in itself and seems to be very important to conduct a continuous research of multiple hypergeometric functions. For instance, in [18], a comprehensive list of hypergeometric functions of three variables as many as 205 is recorded, together with their regions of convergence. It is noted that Riemann's functions and the fundamental solutions of the degenerate second-order partial differential equations are expressible by means of hypergeometric functions

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of several variables (see [7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 19, 20]). Therefore, in investigation of boundary-value problems for these partial differential equations, we need to study the solution of the system of hypergeometric functions and find explicit linearly independent solutions (see [12, 13, 14, 15, 16]).

Here, we choose the Exton functions X_1 and X_2 defined, respectively, by the following triple series (see [10]):

$$X_1\left(a_1, a_2; c_1, c_2; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+2n+p}\,(a_2)_p}{(c_1)_m\,(c_2)_{n+p}\,m!n!p!} x^m y^n z^p \tag{1.1}$$

and

$$X_2\left(a_1, a_2; c_1, c_2, c_3; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+2n+p}\,(a_2)_p}{(c_1)_m\,(c_2)_n\,(c_3)_p\,m!n!p!} x^m y^n z^p \tag{1.2}$$

to find the linearly independent solutions of partial differential equations satisfied by these functions. The regions of convergence of these functions X_1 and X_2 are given in [18].

2. The system of partial differential equations for X_1

According to the theory of multiple hypergeometric functions (see [1]), the system of partial differential equations for the Exton hypergeometric function X_1 is readily seen to be given as follows:

$$\begin{cases} \left(c_1 + x\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} + 1\right)x^{-1}u \\ \qquad - \left(a_1 + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 1\right)\left(a_1 + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)u = 0 \\ \left(c_2 + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial y} + 1\right)y^{-1}u \\ \qquad - \left(a_1 + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 1\right)\left(a_1 + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)u = 0 \\ \left(c_2 + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(z\frac{\partial}{\partial z} + 1\right)z^{-1}u - \left(a_1 + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(a_2 + z\frac{\partial}{\partial z}\right)u = 0, \end{cases} \tag{2.1}$$

where $u = X_1\left(a, b; c_1, c_2, c_3; x, y, z\right)$.
Now by making use of some elementary calculations, we find the following system of second order partial differential equations:

$$
(2.2)\quad \begin{cases}
x\left(1-4x\right)u_{xx} - 8xyu_{xy} - 4xzu_{xz} - 4yzu_{yz} - 4y^2u_{yy} - z^2u_{zz} + \left[c_1 - 2\left(2a_1+3\right)x\right]u_x \\
\qquad - 2\left(2a_1+3\right)yu_y - 2\left(a_1+1\right)zu_z - a_1\left(a_1+1\right)u = 0 \\
y\left(1-4y\right)u_{yy} - 4x^2u_{xx} - 8xyu_{xy} - 4xzu_{xz} + z\left(1-4y\right)u_{yz} - z^2u_{zz} - 2\left(2a_1+3\right)xu_x \\
\qquad + \left[c_2 - 2\left(2a_1+3\right)y\right]u_y - 2\left(a_1+1\right)zu_z - a_1\left(a_1+1\right)u = 0 \\
z\left(1-z\right)u_{zz} - 2xzu_{xz} + y\left(1-2z\right)u_{yz} - a_2 2xu_x - 2a_2yu_y \\
\qquad + \left[c_2 - \left(a_1+a_2+1\right)z\right]u_z - a_1a_2u = 0.
\end{cases}
$$

It is noted that three equations of the system (2.2) are simultaneous, because the hypergeometric function X_1 satisfies the system. Now in order to find the linearly independent solutions of the system (2.2) we consider u as in the form $u = x^{\tau}y^{\nu}z^{\lambda}w$, where w is an unknown function, and τ, ν, and λ are constants which are to be determined. So, substituting $u = x^{\tau}y^{\nu}z^{\lambda}w$ into the system (2.2), we obtain

$$
\begin{cases}
x\left(1-4x\right)w_{xx} - 8xyw_{xy} - 4xzw_{xz} - 4yzw_{yz} - 4y^2w_{yy} - z^2w_{zz} \\
+ \left\{\left(2\tau+c_1\right) - 2\left[2\left(2\tau+2\nu+\lambda+a_1\right)+3\right]x\right\}w_x - 2\left[2\left(2\tau+2\nu+\lambda+a_1\right)+3\right]yw_y \\
\qquad -2\left[\left(2\tau+2\nu+\lambda+a_1\right)+1\right]zw_z \\
\qquad\qquad - \left\{-\dfrac{\tau\left(\tau-1+c_1\right)}{x} + \left(2\tau+2\nu+\lambda+a_1\right)\left[\left(2\tau+2\nu+\lambda+a_1\right)+1\right]\right\}w = 0 \\
y\left(1-4y\right)w_{yy} - 4x^2w_{xx} - 8xyw_{xy} - 4xzw_{xz} + z\left(1-4y\right)w_{yz} - z^2w_{zz} \\
-2\left[2\left(2\tau+2\nu+\lambda+a_1\right)+3\right]xw_x + \left\{\left(2\nu+\lambda+c_2\right) - 2\left[2\left(2\nu+2\tau+\lambda+a_1\right)+3\right]y\right\}w_y \\
- \left\{-\dfrac{\nu}{y} + 2\left[\left(2\tau+2\nu+\lambda+a_1\right)+1\right]\right\}zw_z \\
\qquad\qquad - \left\{-\dfrac{\nu\left(\nu+\lambda-1+c_2\right)}{y} + \left(2\tau+2\nu+\lambda+a_1\right)\left[\left(2\tau+2\nu+\lambda+a_1\right)+1\right]\right\}w = 0 \\
z\left(1-z\right)w_{zz} - 2xzw_{xz} + y\left(1-2z\right)w_{yz} - 2\left(\lambda+a_2\right)xw_x + \left[\dfrac{\lambda}{z} - 2\left(\lambda+a_2\right)\right]yu_y \\
\qquad + \left\{\left(\nu+2\lambda+c_2\right) - \left[\left(2\tau+2\nu+\lambda+a_1\right)+\left(\lambda+a_2\right)+1\right]z\right\}w_z \\
\qquad\qquad - \left\{\dfrac{\lambda\left(\nu+\lambda-1+c_2\right)}{z} + \left(2\tau+2\nu+\lambda+a_1\right)\left(\lambda+a_2\right)\right\}w = 0.
\end{cases}
\tag{2.3}
$$

It is noted that the system (2.3) is analogical to the system (2.2). Therefore, it is required that the following conditions should be satisfied:

$$
(2.4)\qquad \begin{cases}
\tau\left(\tau-1+c_1\right) = 0 \\
\nu = 0 \\
\nu\left(\nu+\lambda-1+c_2\right) = 0 \\
\lambda = 0 \\
\lambda\left(\nu+\lambda-1+c_2\right) = 0.
\end{cases}
$$

It is not difficult to see that the system (2.4) satisfies the following solutions:

$$\begin{array}{lcc} & 1 & 2 \\ \tau: & 0 & 1-c_1 \\ \nu: & 0 & 0 \\ \lambda: & 0 & 0 \end{array} \tag{2.5}$$

Finally, substituting 2 solutions of the system (2.4) into (2.3), we find the following linearly independent solutions of the system (2.2):

$$u_1 = X_1\left(a_1, a_2; c_1, c_2; x, y, z\right), \tag{2.6}$$

$$u_2 = x^{1-c_1} X_1\left(2-2c_1+a_1, a_2; 2-c_1, c_2; x, y, z\right). \tag{2.7}$$

Therefore, it is seen that the global solution of the system (2.2) is combined to be in the form

$$u = k_1 u_1 + k_2 u_2 \tag{2.8}$$

where k_i $(i = 1,\ 2)$ are constants.

3. The system of partial differential equations for X_2

By using the same method as in Section 2, we obtain the following system of partial differential equations satisfied by the Exton function X_2:

$$\left\{\begin{array}{l} x\left(1-4x\right)u_{xx} - 4y^2 u_{yy} - z^2 u_{zz} - 8xyu_{xy} - 4xzu_{xz} - 4yzu_{yz} + \\ \qquad \left[c_1 - 2\left(2a_1+3\right)x\right]u_x - 2\left(2a_1+3\right)yu_y - 2\left(a_1+1\right)zu_z - a_1\left(a_1+1\right)u = 0 \\ y\left(1-4y\right)u_{yy} - 4x^2 u_{xx} - z^2 u_{zz} - 8xyu_{xy} - 4xzu_{xz} - 4yzu_{yz} + \\ \qquad \left[c_2 - 2\left(2a_1+3\right)y\right]u_y - 2\left(2a_1+3\right)xu_x - 2\left(a_1+1\right)zu_z - a_1\left(a_1+1\right)u = 0 \\ z\left(1-z\right)u_{zz} - 2xzu_{xz} - 2yzu_{yz} + \left[c_3 - \left(a_1+a_2+1\right)z\right]u_z \\ \qquad - 2a_2xu_x - 2a_2yu_y - a_1a_2u = 0, \end{array}\right. \tag{3.1}$$

where

$$u = X_2\left(a_1, a_2; c_1, c_2, c_3; x, y, z\right).$$

Now in order to find the linearly independent solutions of the system (3.1) we consider u as in the form $u = x^{\tau} y^{\nu} z^{\lambda} w$, where w is an unknown function, and τ, ν, and λ are constants which are to be determined. So, substituting $u = x^{\tau} y^{\nu} z^{\lambda} w$ into the system (3.1), we obtain

$$\left\{\begin{array}{l} x(1-4x)w_{xx} - 4y^2 w_{yy} - z^2 w_{zz} - 8xyw_{xy} - 4xzw_{xz} - 4yzw_{yz} \\ \quad + \{2\tau + c_1 - 2[2(2\tau+2\nu+\lambda+a_1)+3]x\} w_x - 2[2(2\tau+2\nu+\lambda+a_1)+3] yw_y \\ -2[(2\tau+2\nu+\lambda+a_1)+1] zw_z \\ \quad - \left\{-\tau(\tau-1+c_1)x^{-1} + (2\tau+2\nu+\lambda+a_1)[(2\tau+2\nu+\lambda+a_1)+1]\right\} w = 0 \\ y(1-4y)w_{yy} - 4x^2 w_{xx} - z^2 w_{zz} - 8xyw_{xy} - 4xzw_{xz} - 4yzw_{yz} \\ \quad + \{2\nu + c_2 - 2[2(2\nu+2\tau+\lambda+a_1)+3]y\} w_y - 2[2(2\tau+2\nu+\lambda+a_1)+3] xw_x \\ -2[(2\tau+2\nu+\lambda+a_1)+1] zw_z \\ \quad - \left\{-\nu(\nu-1+c_2)y^{-1} + (2\tau+2\nu+\lambda+a_1)[(2\tau+2\nu+\lambda+a_1)+1]\right\} w = 0 \\ z(1-z)w_{zz} - 2xzw_{xz} - 2yzw_{yz} + \{2\lambda + c_3 - [(2\tau+2\nu+\lambda+a_1)+(\lambda+a_2)+1]z\} w_z \\ -2(\lambda+a_2)xw_x - 2(\lambda+a_2)yw_y - \left[-\lambda(\lambda-1+c_3)z^{-1} + (2\tau+2\nu+\lambda+a_1)(\lambda+a_2)\right] w = 0. \end{array}\right. \tag{3.2}$$

Naturally, it is required that the following conditions should be satisfied:

$$\left\{\begin{array}{l} \tau(\tau-1+c_1) = 0 \\ \nu(\nu-1+c_2) = 0 \\ \lambda(\lambda-1+c_3) = 0. \end{array}\right. \tag{3.3}$$

It is readily seen that the system (3.3) has the following eight solutions:

	1	2	3	4	5	6	7	8
τ :	0	$1-c_1$	0	0	$1-c_1$	$1-c_1$	0	$1-c_1$
ν :	0	0	$1-c_2$	0	$1-c_2$	0	$1-c_2$	$1-c_2$
λ :	0	0	0	$1-c_3$	0	$1-c_3$	$1-c_3$	$1-c_3$

(3.4)

Furthermore, substituting 8 solutions of the system (3.4) into the system (3.2), we get the following linearly independent solutions:

$$u_1 = X_2(a_1, a_2; c_1, c_2, c_3; x, y, z), \tag{3.5}$$

$$u_2 = x^{1-c_1} X_2(2-2c_1+a_1, a_2; 2-c_1, c_2, c_3; x, y, z), \tag{3.6}$$

$$u_3 = y^{1-c_2} X_2(2-2c_2+a_1, a_2; c_1, 2-c_2, c_3; x, y, z), \tag{3.7}$$

$$u_4 = z^{1-c_3} X_2(1-c_3+a_1, 1-c_3+a_2; c_1, c_2, 2-c_3; x, y, z), \tag{3.8}$$

$$u_5 = x^{1-c_1} y^{1-c_2} X_2(4-2c_1-2c_2+a_1, a_2; 2-c_1, 2-c_2, c_3; x, y, z), \tag{3.9}$$

$$u_6 = x^{1-c_1} z^{1-c_3} X_2(3-2c_1-c_3+a_1, 1-c_3+a_2; 2-c_1, c_2, 2-c_3; x, y, z), \tag{3.10}$$

$$u_7 = y^{1-c_2} z^{1-c_3} X_2(3-2c_2-c_3+a_1, 1-c_3+a_2; c_1, 2-c_2, 2-c_3; x, y, z), \tag{3.11}$$

$$u_8 = x^{1-c_1} y^{1-c_2} z^{1-c_3} X_2(5-2c_1-2c_2-c_3+a_1, 1-c_3+a_2; 2-c_1, 2-c_2, 2-c_3; x, y, z). \tag{3.12}$$

Thus, the global solution of the system for the Exton function X_2 is presented in the following form:

$$u = \sum_{j=1}^{8} k_j u_j, \tag{3.13}$$

where k_i $(i = 1, 2, \ldots, 8)$ are constants.

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