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HIGHER ORDER GENOCCHI, EULER POLYNOMIALS ASSOCIATED WITH q-BERNSTEIN TYPE POLYNOMIALS

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Abstract. The main aim of this paper is to give some relationships between q-Bernstein, higher order genocchi and Euler polynomials.

1. Introduction, Definitions and Notations

Throughout this paper, $\mathbb{C}$ denotes the complex number field and $\mathbb{N}$ the set of natural numbers. We assume that $q \in \mathbb{C}$ with $|q| < 1$ and that the q-number is defined by $[x]_q = \frac{q^x-1}{q-1}$(see [1],[2],[5]-[18]). The generating functions of the higher order Genocchi , Euler polynomials and q-Bernstein polynomials, respectively, can be defined as follows:

$$D^{(w)}(x,t) = \left(\frac{2t}{e^t+1}\right)^w e^{xt} = \sum_{n=0}^{\infty} G_n^{(w)}(x) \frac{t^n}{n!}, \text{ for } |t| < \pi, \tag{1.1}$$

$$S^{(w)}(x,t) = \left(\frac{2}{e^t+1}\right)^w e^{xt} = \sum_{n=0}^{\infty} E_n^{(w)}(x) \frac{t^n}{n!}, \text{ for } |t| < \pi \tag{1.2}$$

(see [13],[18],[19]) and

(1.3)

$$F_k(x,t;q) = \frac{\left(t[x]_q\right)^k}{k!} e^{t[1-x]_q} = \sum_{n=k}^{\infty} B_{k,n}(x;q) \frac{t^n}{n!}, \; t \in \mathbb{C}, \; k = 0, 1, \cdots, n.$$

where $\lim_{q\to 1} F_k(x,t;q) = F_k(t,x) = \frac{(tx)^k}{k!} e^{t(1-x)}$ (see [3]).

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By (1.1) and (1.2) we easily see that

$$D^{(w)}(x,t) = t^{w} S^{(w)}(x,t) \tag{1.4}$$

From the above, we have

$$\sum_{n=0}^{\infty} G_n^{(w)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n^{(w)}(x) \frac{t^{n+w}}{n!} \tag{1.5}$$

By comparing coefficients t^n in the both sides of the above equation for $n \in \mathbb{N}$,

$$G_{n+w}^{(w)}(x) = \frac{(n+w)!}{(n)!} E_n^{(w)}(x) \tag{1.6}$$

These polynomials have explicit formulas, respectively, as follows:

$$G_n^{(w)}(x) = \sum_{k=0}^{n} \binom{n}{k} G_k^{(w)} x^{n-k} \tag{1.7}$$

$$E_n^{(w)}(x) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(w)} x^{n-k} \tag{1.8}$$

$$B_{k,n}(x;q) = \binom{n}{k} [x]_q^k [1-x]_q^{n-k} \ , \ x \in [0,1] \tag{1.9}$$

where $G_k^{(w)} = G_k^{(w)}(0)$, $E_k^{(w)} = E_k^{(w)}(0)$ are higher order Genocchi and Euler numbers, respectively.

The purpose of this paper is to give some relationships between q-Bernstein, higher order Genocchi and Euler polynomials.We derive the higher order zeta functions from the Mellin transformation of this generating function which interpolates the higher order Genocchi polynomials at negative integers and associated with q-Bernstein polynomials. Furthermore, we define particular higher order zeta functions associated with q-Bernstein polynomials.

2. New Identities On q-Bernstein Type Polynomials

Theorem 1. *Let $n \in \mathbb{N}$ and $0 \leq w \leq n$. We obtain*

$$B_{w,n}(x;q) = \frac{[x]_q^w}{2^{n-w} w!} \sum_{l=0}^{n} \binom{n}{l} G_l^{(w)} \left(2 [1-x]_q\right) E_{n-l}^{(-w)}. \tag{2.1}$$

Proof. By using (1.1),(1.2) and (1.3) we have

$$\sum_{n=k}^{\infty} 2^n B_{w,n}(x;q)\frac{t^n}{n!} = \frac{\left(2[x]_q\right)^w}{w!}\left(\sum_{n=0}^{\infty} G_n^{(w)}\left(2[1-x]_q\right)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} E_n^{(-w)}\frac{t^n}{n!}\right)$$

By using Cauchy product in the above we have

$$\sum_{n=w}^{\infty} 2^n B_{w,n}(x;q)\frac{t^n}{n!} = \frac{\left(2[x]_q\right)^w}{w!}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} G_l^{(w)}\left(2[1-x]_q\right)E_{n-l}^{(-w)}\right)\frac{t^n}{n!}$$

From the above we have

$$\begin{aligned}\sum_{n=w}^{\infty} 2^n B_{w,n}(x;q)\frac{t^n}{n!} &= \frac{\left(2[x]_q\right)^w}{w!}\sum_{n=0}^{w-1}\left(\sum_{l=0}^{n}\binom{n}{l} G_l^{(w)}\left(2[1-x]_q\right)E_{n-l}^{(-w)}\right)\frac{t^n}{n!} \\ &+\frac{\left(2[x]_q\right)^w}{w!}\sum_{n=w}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} G_l^{(w)}\left(2[1-x]_q\right)E_{n-l}^{(-w)}\right)\frac{t^n}{n!}\end{aligned} \tag{2.2}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the both sides of the above equation, we arrive at the desired result. □

From (2.2), we get the following corollary:

Corollary 1. *Let $w \in \mathbb{N}$ with $w \leq n$. Then we have*

$$\frac{\left(2[x]_q\right)^w}{w!}\sum_{n=0}^{w-1}\sum_{l=0}^{n}\binom{n}{l} G_l^{(w)}\left(2[1-x]_q\right)E_{n-l}^{(-w)} = 0. \tag{2.3}$$

From (1.1), we have

$$D^{(w)}(t,x) = \left(\frac{2t}{e^t+1}\right)\cdot\left(\frac{2t}{e^t+1}\right)\cdots\left(\frac{2t}{e^t+1}\right)e^{tx}$$

$$= 2^w t^w e^{xt}\sum_{n_1=0}^{\infty}(-1)^{n_1}e^{n_1 t}\sum_{n_2=0}^{\infty}(-1)^{n_2}e^{n_2 t}\cdots\sum_{n_w=0}^{\infty}(-1)^{n_w}e^{n_w t}$$

$$(2.4)\qquad = 2^w t^w \sum_{n_1,n_2,\ldots,n_{w=0}}^{\infty}(-1)^{n_1+n_2+\ldots+n_w}e^{(x+n_1+n_2+\ldots+n_w)t}$$

$$= \sum_{n=0}^{\infty}G_n^{(w)}(x)\frac{t^n}{n!}.$$

By using (2.4) we easily see that,

$$(2.5)\; G_{n+w}^{(w)}(x)$$
$$= 2^w\frac{(n+w)!}{n!}\sum_{n_1,n_2,\ldots,n_{w=0}}^{\infty}(-1)^{n_1+n_2+\ldots+n_w}(x+n_1+n_2+\ldots+n_w)^n.$$

For $s\in\mathbb{C}$ we have

$$\frac{1}{\Gamma(s)}\int_0^{\infty}D^{(w)}(-t,x)\,t^{s-w-1}dt$$
$$= 2^w\sum_{n_1,\ldots,n_w=0}^{\infty}(-1)^{n_1+n_2+\ldots+n_w}\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-(x+n_1+n_2+\ldots+n_w)t}t^{s-1}dt$$
$$(2.6) = 2^w\sum_{n_1,\ldots,n_w=0}^{\infty}\frac{(-1)^{n_1+n_2+\ldots+n_w}}{(x+n_1+n_2+\ldots+n_w)^s}$$

We define zeta function as follows:

(2.7)
$$\zeta^{(w)}(s,x) = 2^w\sum_{n_1,\ldots,n_w=0}^{\infty}\frac{(-1)^{n_1+n_2+\ldots+n_w}}{(x+n_1+n_2+\ldots+n_w)^s}\text{ for }\Re(s)>0.\text{ (see [7])}$$

Note that $\zeta^{(w)}(s,x)$ can be continued analytically to an entire function of $s\in\mathbb{C}$. By substituting $s=-n$ into (2.5), we obtain (2.5) as follows:

$$(2.8)\qquad \zeta^{(w)}(-n,x) = \frac{n!}{(n+w)!}G_{n+w}^{(w)}(x)\;,\;\text{ (see [7])}$$

$\zeta^{(w)}(s,1)=\zeta^{(w)}(s)$ which is the zeta function, From (1.6) and (2.8) for $x=0$. We can easily derive the following equation:

$$\zeta^{(w)}(-n)=\frac{n!}{(n+w)!}G_{n+w}^{(w)}=E_n^{(w)} \tag{2.9}$$

By using (2.1) and (2.9) we get following theorem:

Theorem 2. *Let $w\in\mathbb{N}$ with $w\leq n$. Then we have*

$$\begin{aligned} &B_{w,n}(x;q) \\ &=\frac{2^{w-n}[x]_q^w}{w!}\sum_{l=0}^{w}\binom{n}{l}\frac{l!}{(n-l)!}\zeta^{(w)}\left(w-l,2[1-x]_q\right)\zeta^{(-w)}(l-n). \end{aligned} \tag{2.10}$$

we define particular zeta function as follows:

$$\begin{aligned} H^{(w,\lambda)}(s\mid a_1,a_2,...,a_w\mid F) &= \sum_{n_i=a_i(\mathrm{mod}F)}\frac{\lambda^{n_1+n_2+...+n_w}}{(n_1+n_2+...+n_w)^s},\ i=1,2,...,w \\ &= \sum_{n_i=0}^{\infty}\frac{\lambda^{\sum_{l=1}^{w}n_lF+a_l}}{\left(\sum_{l=1}^{w}n_lF+a_l\right)^s} \\ &= \frac{\lambda^{\sum_{l=1}^{w}a_l}}{F^s}\sum_{n_i=0}^{\infty}\frac{\lambda^{F\sum_{l=1}^{w}n_l}}{\left(\sum_{l=1}^{w}n_l+\frac{a_l}{F}\right)^s} \end{aligned} \tag{2.11}$$

where $F\equiv 1(\mathrm{mod}2)$, by substituting $\lambda=-1$ into (2.11), we have

$$H^{(w,-1)}(s\mid a_1,a_2,...,a_w\mid F)=\frac{(-1)^{\sum_{l=1}^{w}a_l}}{F^s2^w}\zeta^{(w)}\left(s,\frac{\sum_{l=1}^{w}a_w}{F}\right) \tag{2.12}$$

By using (2.12) we obtain the following theorem:

Theorem 3. *Let $n\in\mathbb{N}$ and $F\equiv 1\,(mod2)$ with $0\leq a_i<F$ for $i=1,2,...,w$. We have*

$$\begin{aligned} &G_{n+w}^{(w)}\left(\frac{a_1+a_2+...+a_w}{F}\right) \\ &=\frac{(-1)^{\sum_{l=1}^{w}a_l}(n+w)!2^w}{n!F^n}H^{(w,-1)}(-n\mid a_1,a_2,...,a_w\mid F) \end{aligned} \tag{2.13}$$

From (2.10), (2.12) and (2.13) we get following theorem:

Theorem 4. *Let $F \equiv 1\,(mod 2)$, $n \in \mathbb{N}$ and $0 \leq w \leq n$. One has*

$$B_{w,n}\left(\frac{\sum_{k=1}^{w} a_k}{F};q\right) = \frac{2^{2w-n}\left(-1\right)^{\sum_{k=1}^{w} a_k}}{w!}\left[\frac{a_1+a_2+...+a_w}{F}\right]_q$$

$$\times\sum_{l=0}^{n}\binom{n}{n-l,n-l}F^{w-l}H^{(w,-1)}$$

$$\times\left(w-l \mid \frac{1}{F},\frac{q}{F},\frac{q^2}{F},...,\frac{q^{-\frac{a_1+a_2+...+a_w}{F}}}{F} \mid F\right)\zeta^{(-w)}\left(l-n\right).$$

where $\binom{n}{n-l,n-l} = \frac{n!}{(n-l)!(n-l)!}$.

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