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ON AN L^∞-VERSION OF A PEXIDERIZED QUADRATIC FUNCTIONAL INEQUALITY

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Abstract. Let $f, g, h, k : \mathbb{R}^n \to \mathbb{C}$ be locally integrable functions. We deal with the L^∞-version of the Hyers-Ulam stability of the quadratic functional inequality and the Pexiderized quadratic functional inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|_{L^\infty(\mathbb{R}^n)} \le \varepsilon,$$
$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\|_{L^\infty(\mathbb{R}^n)} \le \varepsilon,$$

based on the concept of linear functionals on the space of smooth functions with compact support.

1. Introduction

When we consider the Hyers–Ulam stability problems(see Hyers[5], Hyers-Isac-Rassias[6]) whose underlining functions are defined on a measure space it is more natural to deal with the problems in *almost everywhere sense* (or equivalently L^∞*-sense* for measurable functions) than *for all sense*. Recently, some of such stability problems have been studied in the sense of Schwartz distributions [2, 3, 4]. However, the author guesses that the Schwartz theory of distributions would not be interested for the readers. For the reason, in the present article, making use of the same methods as in [2, 3, 4] with possible change of terminologies we consider *an* L^∞*-version* of the stability of generalized quadratic functional equation which would be more interested for the readers(see P. W. Cholewa[1] and F. Skof[7] for classical Hyers-Ulam stability of quadratic functional equations). Throughout this article, we denote by $L^1_{loc}(\mathbb{R}^n)$ the space of all locally integrable functions $f : \mathbb{R}^n \to \mathbb{C}$. Let

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$f \in L^1_{loc}(\mathbb{R}^n)$. Then we consider the inequality

$$\left| \int \Big(f(x+y) + f(x-y) - 2f(x) - 2f(y) \Big) \varphi(x,y) dx\, dy \right| \le \epsilon \|\varphi\|_{L^1}$$

for all φ in the space $\mathbb{C}_c^\infty(\mathbb{R}^{2n})$ of all infinitely differentiable functions with bounded supports, which is equivalent to each of the following inequalities

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|_{L^\infty(\mathbb{R}^{2n})} \le \epsilon, \tag{1.1}$$

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \le \epsilon, \;\; a.\,e.\; (x,y) \in \mathbb{R}^{2n}. \tag{1.2}$$

As a result we first prove that if $f \in L^1_{loc}(\mathbb{R}^n)$ satisfies the inequality (1.1), there exists a unique quadratic function $Q(x) = \sum_{1\le j\le k\le n} a_{jk}\, x_j x_k,\;\; a_{jk} \in \mathbb{C},\; j,k = 1,\ldots,n,$ such that

$$\|f(x) - Q(x)\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{2}\epsilon. \tag{1.3}$$

Generalizing the result we also prove the Hyers-Ulam stability of the Pexider generalization

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\|_{L^\infty(\mathbb{R}^{2n})} \le \epsilon \tag{1.4}$$

of the inequality (1.1), where $f, g, h, k \in L^1_{loc}(\mathbb{R}^n)$.

2. Stability of quadratic functional equation

The classical stability of the quadratic functional equation was proved by P. W. Cholewa[1] and F. Skof[7]:

Theorem 2.1. *Let $f : G \to E$ be a mapping from a group G to a Banach space E satisfying the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \epsilon$$

for all $x,\; y \in G$. Then there exists a unique function $q : G \to E$ satisfying

$$q(x+y) + q(x-y) - 2q(x) - 2q(y) = 0$$

such that

$$\|f(x) - q(x)\| \le \frac{1}{2}\epsilon$$

for all $x \in G$.

In this section, generalizing the above result we consider the stability of quadratic functional inequality (1.1). We denote by $\omega(x)$ the function on $\mathbb{R}^n$,

$$\omega(x) = \begin{cases} Ke^{-\frac{1}{1-|x|^2}}, & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$$

where

$$K^{-1} = \int_{|x|<1} e^{-\frac{1}{1-|x|^2}} dx.$$

It is easy to see that $\omega(x)$ an infinitely differentiable function with support $\{x : |x| \leq 1\}$. We first use the family of functions $\omega_t(x) := t^{-n}\omega(x/t),\ t > 0$. Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then for each $t > 0$, $(f * \omega_t)(x) := \int f(y)\omega_t(x-y)dy$ is a smooth function in $\mathbb{R}^n$ and $(f * \omega_t)(x) \to f(x)$ a. e. $x \in \mathbb{R}^n$ as $t \to 0^+$. It will be very useful to employ the n-dimensional heat kernel

$$E_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t), \quad t > 0.$$

It can be checked that the convolution $(f * E_t)(x)$ is defined for some $f \in L^1_{loc}(\mathbb{R}^n)$ of suitable growth conditions. In particular, it will be proved that the convolutions are well defined for all $f \in L^1_{loc}(\mathbb{R}^n)$ satisfying (1.1). Furthermore the heat kernel $E_t(x)$ enjoys the semigroup property

$$(E_t * E_s)(x) = E_{t+s}(x),$$

which will be very useful later.

Lemma 2.2. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function* (*both the real and imaginary parts of f are measurable functions*) *satisfying the inequality*

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \leq \epsilon$$

for all $x, y \in \mathbb{R}^n$. Then there exists a unique quadratic function

$$Q(x) = \sum_{1\leq j\leq k\leq n} a_{jk}\, x_j x_k,\ a_{jk} \in \mathbb{C},\ j,k = 1,\ldots,n \tag{2.1}$$

such that

$$|f(x) - Q(x)| \leq \frac{\epsilon}{2},$$

for all $x \in \mathbb{R}^n$.

Proof. As we see in the proof of Theorem 2.1, $Q(x)$ is given by

$$Q(x) = \lim_{m\to\infty} 4^{-m} f(2^m x), \tag{2.2}$$

and satisfies the functional equation

$$Q(x+y)+Q(x-y)-2Q(x)-2Q(y)=0. \tag{2.3}$$

In view of (2.2), $Q(x)$ is measurable and the solution $Q(x)$ of (2.3) is given by (2.1). This completes the proof. □

Lemma 2.3. *Let $f:\mathbb{R}^n\times(0,\infty)\to\mathbb{C}$ be a continuous function satisfying the inequality*

$$|f(x+y,t+s)+f(x-y,t+s)-2f(x,t)-2f(y,s)|\le\epsilon \tag{2.4}$$

for all $x,y\in\mathbb{R}^n$, $t,s>0$. Then there exists a unique quadratic function $Q(x)=\sum_{1\le j\le k\le n}a_{jk}\,x_jx_k$ and a unique complex number $c\in\mathbb{C}$ such that

$$|f(x,t)-Q(x)-ct|\le\frac{1}{2}\epsilon$$

for all $x\in\mathbb{R}^n$, $t>0$.

Proof. Putting $x=y=0$ and $s=t$ in (2.4) and dividing the result by 4 we have

$$|2^{-1}f(0,2t)-f(0,t)|\le 4^{-1}\epsilon.$$

By the induction argument we have

$$|2^{-n}f(0,2^nt)-f(0,t)|\le\frac{\epsilon}{2} \tag{2.5}$$

for all $n\in\mathbb{N}$, $t>0$. It follows from the inequality (2.5) that $a(t):=\lim_{m\to\infty}2^{-m}f(0,\,2^mt)$ converges uniformly and is the unique function satisfying

$$a(t+s)=a(t)+a(s), \tag{2.6}$$

$$|f(0,t)-a(t)|\le\frac{\epsilon}{2} \tag{2.7}$$

for all $t,s>0$. On the other hand, putting $y=x$ and $s=t$ in (2.4), dividing the result by 4 and using the induction argument we have

$$|f(x,t)-4^{-n}f(2^nx,\,2^nt)-\sum_{k=1}^{n}4^{-k}f(0,2^kt)|\le\frac{\epsilon}{3}. \tag{2.8}$$

It follows from (2.6) and (2.7) that

$$|\sum_{k=1}^{n}4^{-k}f(0,2^kt)-(1-2^{-n})a(t)|\le\frac{\epsilon}{6}. \tag{2.9}$$

From (2.8) and (2.9), letting $F(x,t)=f(x,t)-a(t)$ we have

$$|F(x,t)-4^{-n}F(2^nx,\,2^nt)|\le\frac{\epsilon}{2}. \tag{2.10}$$

Now it is easy to see that

$$F_0(x,t) := \lim_{m\to\infty} 4^{-m} F(2^m x, 2^m t)$$

satisfies

$$|F(x,t) - F_0(x,t)| \le \frac{\epsilon}{2} \tag{2.11}$$

and the quadratic–additive functional equation

$$F_0(x+y,t+s) + F_0(x-y,t+s) - 2F_0(x,t) - 2F_0(y,s) = 0 \tag{2.12}$$

for all $x, y \in \mathbb{R}^n,\ t,s > 0$.

Let $q(x,t) := F_0(x,t) + a(t)$. Then $q(x,t)$ is a continuous function satisfying the equation(2.12) and has the form

$$q(x,t) = \sum_{1\le j\le k\le n} a_{jk}\, x_j x_k + ct$$

for some $a_{jk}, c \in \mathbb{C}$, $1 \le j \le k \le n$. Thus in view of (2.11) we have

$$|f(x,t) - \sum_{1\le j\le k\le n} a_{jk}\, x_j x_k - ct| \le \frac{\epsilon}{2}.$$

This completes the proof.

□

Theorem 2.4. *Let* $f \in L^1_{loc}(\mathbb{R}^n)$ *satisfy the inequality* (1.1). *Then there exists a unique quadratic function*

$$Q(x) = \sum_{1\le j\le k\le n} a_{jk}\, x_j x_k$$

such that

$$\|f(x) - Q(x)\|_{L^\infty(\mathbb{R}^n)} \le \frac{\epsilon}{2}.$$

Proof. We first prove that f is of polynomial growth. Multiplying $\varphi(x,y) = \omega_t(\xi - x)\omega_s(\eta - y)$ in both sides of (1.1), integrating the result with respect to x and y, and replacing ξ, η by x, y respectively, we have

$$|(f*\omega_t*\omega_s)(x+y) + (f*\omega_t*\omega_s)(x-y) - 2(f*\omega_t)(x) - 2(f*\omega_s)(y)| \le \epsilon \tag{2.13}$$

for all x, $y \in \mathbb{R}^n$, $t, s > 0$. In view of (2.13) it is easy to see that

$$f_0(x) := \limsup_{t\to 0^+} (f*\omega_t)(x)$$

exists. Letting $y = 0$ in (2.13) we have

$$|(f*\omega_t*\omega_s)(x) - (f*\omega_t)(x) - (f*\omega_s)(0)| \le \frac{\epsilon}{2}. \tag{2.14}$$

Fix $x \in \mathbb{R}^n$ and let $t = t_n \to 0^+$ so that $(f * \omega_{t_n})(x) \to f(x)$ in (2.14) to get

$$|(f * \omega_s)(x) - f_0(x) - (f * \omega_s)(0)| \le \frac{\epsilon}{2}. \tag{2.15}$$

From the inequality (2.13), (2.14), (2.15) and the triangle inequality we have

$$|f_0(x+y) + f_0(x-y) - 2f_0(x) - 2f_0(y)| \le 5\epsilon$$

for all $x, y \in \mathbb{R}^n$. By Lemma 2.2, there exists a unique function

$$Q(x) = \sum_{1 \le j \le k \le n} a_{jk}\, x_j x_k,\ a_{jk} \in \mathbb{C},\, j, k = 1, \ldots, n$$

such that

$$|f_0(x) - Q(x)| \le \frac{5}{2}\epsilon \tag{2.16}$$

for all $x \in \mathbb{R}^n$.

From (2.15) and (2.16) we have

$$|(f * \omega_s)(x) - Q(x) - (f * \omega_s)(0)| \le 3\epsilon. \tag{2.17}$$

Letting $s = s_n \to 0^+$ so that $(f * \omega_{s_n})(0) \to f(0)$ in (2.17) we have

$$\|f(x) - Q(x)\|_{L^\infty} \le \frac{7}{2}\epsilon. \tag{2.18}$$

Thus f is of polynomial growth. Now we employ the n-dimensional heat kernel $E_t(x)$, $t > 0$. Convolving $E_t(x)E_s(y)$ in (1.1) and using the semigroup property

$$(E_t * E_s)(x) = E_{t+s}(x)$$

we obtain the *quadratic–additive type* functional inequality

$$|f(x+y, t+s) + f(x-y, t+s) - 2f(x,t) - 2f(y,s)| \le \epsilon \tag{2.19}$$

for x, $y \in \mathbb{R}^n$, $t, s > 0$, where $f(x,t)$ is given by

$$f(x,t) = \int f(y)E_t(x-y)dy.$$

Now using Lemma 2.3, we have

$$|f(x,t) - \sum_{1 \le j \le k \le n} a_{jk}\, x_j x_k - ct| \le \frac{\epsilon}{2}. \tag{2.20}$$

Letting $t \to 0^+$ in (2.20) we get the result.

□

3. Stability of Pexiderized quadratic functional equation

In this section we prove the Hyers-Ulam stability of (1.4), which is a Pexider generalization of (1.1).

Lemma 3.1. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable function satisfying the inequality*

$$|f(x+y) - f(x) - f(y)| \le \epsilon$$

for all $x, y \in \mathbb{R}^n$. Then there exists a unique additive function

$$A(x) = a \cdot x, \;\; a \in \mathbb{C}^n \tag{3.1}$$

such that

$$|f(x) - A(x)| \le \epsilon$$

for all $x \in \mathbb{R}^n$.

Proof. As we see in the proof of the classical Hyers-Ulam stability theorem of additive functional equation[5, 6], the function $A(x)$ is given by

$$A(x) = \lim_{m\to\infty} 2^{-m} f(2^m x), \tag{3.2}$$

and satisfies the functional equation

$$A(x+y) - A(x) - A(y) = 0. \tag{3.3}$$

In view of (3.2), $A(x)$ is measurable and the solution $A(x)$ of (3.3) has the form (3.1). This completes the proof. □

Lemma 3.2. *The inequality* (1.4) *implies*

$$\|\widetilde{h}(x+y) + \widetilde{k}(x-y) - 2f(x) - 2g(y)\|_{L^\infty(\mathbb{R}^{2n})} \le 2\epsilon \tag{3.4}$$

for some $\widetilde{h}, \widetilde{k} \in L^1_{loc}(\mathbb{R}^n)$.

Proof. Composite in (1.4) the linear mapping $L : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by

$$L(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right), \quad x, y \in \mathbb{R}^n.$$

Then we have

$$\left| f(x) + g(y) - 2h\left(\frac{x+y}{2}\right) - 2k\left(\frac{x-y}{2}\right)\right| \le \epsilon, \tag{3.5}$$

for all $(x, y) \in L^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^{2n}$ is the set of all (x, y) for which the inequality (1.4) holds. Since $(L^{-1}(\Omega))^c$ has measure zero the inequality

(3.5) holds for a. e. $x \in \mathbb{R}^n$. Then the inequality (3.5) can be written in the form (3.4). This completes the proof. □

Lemma 3.3. *Let f, g, $h, k \in L^1_{loc}(\mathbb{R}^n)$ satisfy the inequality* (1.4). *Then there exists a polynomial $P(x)$ such that*

$$|f(x)|,\ |g(x)|,\ |h(x)|, |k(x)| \le |P(x)|,\ \ a.e.\ x \in \mathbb{R}^n.$$

Proof. The proof is obtained by the same procedure as that of Lemma 3.4 of [2]. Here we give the sketch of the proof. For a complete proof we refer the reader to [2]. We denote by

$$f(x,t,s) = (f * \omega_t * \omega_s)(x), \quad f(x,t) = (f * \omega_t)(x). \tag{3.6}$$

Multiplying $\varphi(x,y) = \omega_t(\xi - x)\omega_s(\eta - y)$ in both sides of (1.4), integrating the result with respect to x and y and replacing ξ, η by x, y respectively, we have

$$|f(x+y,t,s) + g(x-y,t,s) - 2h(x,t) - 2k(y,s)| \le \epsilon \tag{3.7}$$

for all x, $y \in \mathbb{R}^n$, $t, s > 0$. Let

$$f^e(x,t,s) = \frac{1}{2}(f(x,t,s) + f(-x,t,s)), \quad f^e(x,t) = \frac{1}{2}(f(x,t) + f(-x,t)).$$

Then we have

$$|f^e(x+y,t,s) + g^e(x-y,t,s) - 2h^e(x,t) - 2k^e(y,s)| \le \epsilon. \tag{3.8}$$

From (3.7), using limiting processes we can show that there exist $c_3, c_4 \in \mathbb{C}$ such that

$$|f^e(x,t) + g^e(x,t) - 2h^e(x,t) - 2c_4| \le \epsilon, \tag{3.9}$$

$$|f^e(y,s) + g^e(y,s) - 2k^e(y,s) - 2c_3| \le \epsilon, \tag{3.10}$$

and hence

$$|h^e(x,t) - k^e(x,t) - c_3 + c_4| \le \epsilon. \tag{3.11}$$

Using (3.9), (3.10), (3.11) and the triangle inequality we can show that

$$|h^e(x+y,t,s) + h^e(x-y,t,s) - 2h^e(x,t) - 2h^e(y,s) + 2c_3| \le 4\epsilon. \tag{3.12}$$

Using limiting processes we can prove that there exists a unique quadratic function

$$Q(x) = \sum_{1 \le j \le k \le n} a_{jk}\, x_j x_k,\ \ a_{jk} \in \mathbb{C},\ j, k = 1, \ldots, n$$

and a positive constant M_1 such that

$$|h^e(x,s) - Q(x) - h(0,s)| \le M_1. \tag{3.13}$$

For the odd part of the inequality (3.7) let

$$f^o(x,t,s)=\frac{1}{2}(f(x,t,s)-f(-x,t,s)),\quad f^o(x,t)=\frac{1}{2}(f(x,t)-f(-x,t)).$$

Then we have

$$|f^o(x+y,t,s)+g^o(x-y,t,s)-2h^o(x,t)-2k^o(y,s)|\le\epsilon. \tag{3.14}$$

Putting $y=0,\ s\to 0^+$ and convolving $\omega_s(x)$ in (3.14) we have

$$|f^o(x,t,s)+g^o(x,t,s)-2h^o(x,t,s)|\le\epsilon. \tag{3.15}$$

Using (3.14), (3.15) and the triangle inequality we can prove that

$$|h^o(x+y,t,s)+h^o(x-y,t,s)-2h^o(x,t)|\le 2\epsilon. \tag{3.16}$$

From (3.16), using limiting processes we can show that there exist $a\in\mathbb{C}^n$ and $M_2>0$ such that

$$|h^o(x,s)-a\cdot x|\le M_2. \tag{3.17}$$

It follows from (3.13) and (3.17) that

$$|h(x,s)-Q(x)-a\cdot x-h(0,s)|\le M_1+M_2. \tag{3.18}$$

Letting $s=s_n\to 0^+$ so that $\lim h(0,s_n)=\limsup_{s\to 0}h(0,s):=d$ in (3.18) we have

$$|h(x)-Q(x)-a\cdot x-d\|\le M_1+M_2,\ a.e.\ x\in\mathbb{R}^n, \tag{3.19}$$

which implies $h(x)$ has polynomial growth. Changing the role of h and k we also prove that k has polynomial growth. Finally, it follows from Lemma 3.2 that f,g are of polynomial growth. This completes the proof. □

Theorem 3.4. *Let $f,\ g,\ h,\ k\in L^1_{loc}(\mathbb{R}^n)$ satisfy the inequality* (1.4). *Then f,g,h,k satisfy*

$$\|f(x)-Q(x)+(a+b)\cdot x+c_1\|_{L^\infty(\mathbb{R}^n)}\le 8\epsilon,$$
$$\|g(x)-Q(x)+(a-b)\cdot x+c_2\|_{L^\infty(\mathbb{R}^n)}\le 8\epsilon,$$
$$\|h(x)-Q(x)+a\cdot x+c_3\|_{L^\infty(\mathbb{R}^n)}\le 4\epsilon,$$
$$\|k(x)-Q(x)+b\cdot x+c_4\|_{L^\infty(\mathbb{R}^n)}\le 4\epsilon,$$

where $Q(x)=\sum_{1\le j\le k\le n}a_{jk}\,x_jx_k,\ \ a,b\in\mathbb{C}^n,\ \ c_1,c_2,c_3,c_4\in\mathbb{C}$.

Proof. By Lemma 3.3, f,g,h,k are of polynomial growth. Thus we can use the heat kernel $E_t(x)$ instead of $\omega_t(x)$ in Lemma 3.3. Multiplying $\varphi(x,y)=E_t(\xi-x)E_s(\eta-y)$ in both sides of (1.4), integrating the result with respect to x and y and replacing ξ by y, η by y we have

$$|f(x+y,t+s)+g(x-y,t+s)-2h(x,t)-2k(y,s)|\le\epsilon \tag{3.20}$$

for all $x,\ y \in \mathbb{R}^n,\ t, s > 0$, where

$$f(x,t) = (f * E_t)(x).$$

From the inequality (3.20) we have

$$|h^e(x+y,t+s)+h^e(x-y,t+s)-2h^e(x,t)-2h^e(y,s)+2c_3| \le 4\epsilon. \tag{3.21}$$

By Lemma 2.3 there is a unique quadratic form $Q(x) = \sum_{1\le j\le k\le n} a_{jk}\, x_j x_k$ and a complex number c such that

$$|h^e(x,t) - Q(x) - ct - c_3| \le 2\epsilon. \tag{3.22}$$

for all $x \in \mathbb{R}^n,\ t > 0$. Also it follows from (3.20) that

$$|h^o(x+y,t+s) - h^o(x,t) - h^o(y,s)| \le 2\epsilon \tag{3.23}$$

for all $x,\ y \in \mathbb{R}^n,\ t, s > 0$. In view of Lemma 3.1, there exists $a \in \mathbb{C}^n,\ d_1 \in \mathbb{C}$ such that

$$|h^o(x,t) - a \cdot x - d_1 t| \le 2\epsilon. \tag{3.24}$$

Changing the roles of h and k in (3.20) we also have

$$|k^e(x,t) - Q^*(x) - c^*t - c_4| \le 2\epsilon, \tag{3.25}$$

$$|k^o(x,t) - b \cdot x - d_2 t| \le 2\epsilon \tag{3.26}$$

for some quadratic function Q^* and $c^*, c_4, d_2 \in \mathbb{C},\ b \in \mathbb{C}^n$. As in Lemma 3.4 we have

$$|h^e(x,t) - k^e(x,t) - c_3 + c_4| \le \epsilon. \tag{3.27}$$

From (3.22), (3.25), (3.27), using the triangle inequality we have

$$|Q(x) - Q^*(x) + (c - c^*)t| \le 5\epsilon,$$

and hence $Q = Q^*,\ c = c^*$. Thus it follows from (3.22), (3.24), (3.25) and (3.26) that

$$|h(x,t) - Q(x) - a \cdot x - (c + d_1)t - c_3| \le 4\epsilon, \tag{3.28}$$

$$|k(x,t) - Q(x) - b \cdot x - (c + d_2)t - c_4| \le 4\epsilon. \tag{3.29}$$

Letting $t \to 0^+$ in (3.28) and (3.29) we have

$$|h(x) - Q(x) - a \cdot x - c_3| \le 4\epsilon,\ a.e.\, x \in \mathbb{R}^n \tag{3.30}$$

$$|k(x) - Q(x) - b \cdot x - c_4| \le 4\epsilon,\ a.e.\, x \in \mathbb{R}^n. \tag{3.31}$$

Finally we find the approximations of f and g. In view of Lemma 3.2 we have

$$|f(x,t) - \widetilde{Q}(x) - \widetilde{a} \cdot x - \widetilde{d}_1 t - c_1| \le 8\epsilon, \tag{3.32}$$

$$|g(x,t) - \widetilde{Q}(x) - \widetilde{b} \cdot x - \widetilde{d}_2 t - c_2| \le 8\epsilon. \tag{3.33}$$

for some quadratic function $\widetilde{Q}$ and $c_1, c_2, \widetilde{d_1}, \widetilde{d_2} \in \mathbb{C}$, $\widetilde{a}, \widetilde{b} \in \mathbb{C}^n$. From (3.20), we can show that $\widetilde{Q} = Q$ and $\widetilde{a} = a+b$, $\widetilde{b} = b-a$. This completes the proof.

□

In particular, if $\epsilon = 0$ in Theorem 3.4 we have

Corollary 3.5. *The solutions f, g, h, $k \in L^1_{loc}(\mathbb{R}^n)$ of the equation*

$$f(x+y) + g(x-y) - 2h(x) - 2k(y) = 0, \quad a.e.\,(x,y) \in \mathbb{R}^{2n} \tag{3.34}$$

are of the form

$$f(x) = Q(x) + (a+b)\cdot x + c_1, \tag{3.35}$$

$$g(x) = Q(x) + (a-b)\cdot x + c_2, \tag{3.36}$$

$$h(x) = Q(x) + a\cdot x + c_3, \tag{3.37}$$

$$k(x) = Q(x) + b\cdot x + c_4, \tag{3.38}$$

almost everywhere $x \in \mathbb{R}^n$, where $Q(x) = \sum_{1\le j\le k\le n} a_{jk}\, x_j x_k$, $a, b \in \mathbb{C}^n$, $c_1, c_2, c_3, c_4 \in \mathbb{C}$.

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