

EXISTENCE AND ITERATIVE APPROXIMATIONS OF SOLUTIONS FOR STRONGLY NONLINEAR VARIATIONAL-LIKE INEQUALITIES

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ABSTRACT. In this paper, we introduce and study a new class of strongly nonlinear variational-like inequalities. Under suitable conditions, we prove the existence of solutions for the class of strongly nonlinear variational-like inequalities. By making use of the auxiliary principle technique, we suggest an iterative algorithm for the strongly nonlinear variational-like inequality and give the convergence criteria of the sequences generated by the iterative algorithm.

1. Introduction

It is well known that there are lots of iterative type algorithms for finding the approximate solutions of various variational inequalities in Hilbert spaces [3] and [8-13]. Among the most effective numerical technique is the projection method and its variant forms. However, the standard projection technique can no longer be applied to suggest the iterative type algorithm for variational-like inequalities. This fact motivated Gowinski, Lions and Tremoliers [7] to develop the auxiliary principle technique, which does not depend on the projection. Ding [4,5] and Ding and Tan [6] extended the auxiliary principle technique to suggest several iterative algorithms to compute approximate solutions for some classes of general nonlinear mixed variational inequalities and variational-like inequalities.

In this paper, we introduce and study a new class of strongly nonlinear variational-like inequalities. By applying a result due to Chang [1], we prove the existence of solutions for the class of strongly nonlinear variational-like inequalities. Using the auxiliary principle technique, we suggest and analyze a new three-step iterative algorithm for solving the class of strongly nonlinear variational-like inequalities. The convergence criteria of the sequence generated by the iterative algorithm are given.

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2. Preliminaries

Let H be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let K be a nonempty closed convex subset of H , $A, B, C, F : K \rightarrow H$, $N : H \times H \times H \rightarrow H$ and $\eta : K \times K \rightarrow H$ be mappings. Suppose that $a : H \times H \rightarrow (-\infty, \infty)$ is a coercive continuous bilinear form, that is, there exist positive constants c and d such that

- (C1) $a(v, v) \geq c\|v\|^2, \forall v \in H$;
 (C2) $a(u, v) \leq d\|u\|\|v\|, \forall u, v \in H$.

Clearly, $c \leq d$.

Assume that $b : H \times H \rightarrow (-\infty, +\infty)$ is nondifferential and satisfies the following conditions:

- (C3) b is linear in the first argument;
 (C4) b is convex in the second argument;
 (C5) b is bounded, that is, there exists a constant $l > 0$ satisfying

$$|b(u, v)| \leq l\|u\|\|v\|, \quad \forall u, v \in H;$$

- (C6) $b(u, v) - b(u, w) \leq b(u, v - w), \forall u, v, w \in H$.

Now we consider the following strongly nonlinear variational-like inequality problem: Find $u \in K$ such that

$$\begin{aligned} \langle N(Au, Bu, Cu) + Fu, \eta(v, u) \rangle + a(u, v - u) \\ \geq b(u, u) - b(u, v), \quad \forall v \in K. \end{aligned} \quad (2.1)$$

Special Cases

(A) If $N(Au, Bu, Cu) = Au - Bu$, $Fu = 0$, $a(u, v) = 0$ and $b(u, v) = f(v)$ for all $u, v \in K$, then the strongly nonlinear variational-like inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Au - Bu, \eta(v, u) \rangle \geq f(u) - f(v), \quad \forall v \in K, \quad (2.2)$$

which was introduced and studied by Ding [4].

(B) If $N(Au, Bu, Cu) = Au - Bu$, $Fu = 0$, $a(u, v) = 0$, $\eta(u, v) = gu - gv$ and $b(u, v) = f(v)$ for all $u, v \in K$, then the strongly nonlinear variational-like inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Au - Bu, gv - gu \rangle \geq f(u) - f(v), \quad \forall v \in K, \quad (2.3)$$

which was studied by Yao [13].

Remark 2.1. For suitable and appropriate choices of the mappings a, b, A, B, C, F, N and the nonempty closed convex subset K , one can obtain a number of new and previously known nonlinear variational and variational-like inequalities as special cases of the strongly nonlinear variational-like inequality (2.1).

Definition 2.1. Let $A, B, C : K \rightarrow H$, $N : H \times H \times H \rightarrow H$ and $\eta : K \times K \rightarrow H$ be mappings.

(1) A is said to be *Lipschitz continuous* with constant α if there exists a constant $\alpha > 0$ such that

$$\|Au - Av\| \leq \alpha\|u - v\|, \quad \forall u, v \in K;$$

(2) N is said to be *Lipschitz continuous* with constant β in the third argument if there exists a constant $\beta > 0$ such that

$$\|N(w, z, u) - N(w, z, v)\| \leq \beta\|u - v\|, \quad \forall u, v, w, z \in H;$$

(3) A is said to be *strongly monotone* with constant γ if there exists a constant $\gamma > 0$ such that

$$\langle Au - Av, u - v \rangle \geq \gamma\|u - v\|^2, \quad \forall u, v \in K;$$

(4) N is said to be η -*strongly monotone* with constant ξ with respect to A in the first argument if there exists a constant $\xi > 0$ such that

$$\langle N(Au, w, z) - N(Av, w, z), \eta(u, v) \rangle \geq \xi\|u - v\|^2, \quad \forall u, v \in K, w, z \in H;$$

(5) N is said to be η -*relaxed monotone* with constant ξ with respect to A in the second argument if there exists a constant $\zeta > 0$ such that

$$\langle N(Au, w, z) - N(Av, w, z), \eta(u, v) \rangle \geq -\zeta\|u - v\|^2, \quad \forall u, v \in K, w, z \in H;$$

(6) N is said to be η -*monotone* with respect to A in the second argument if

$$\langle N(Au, w, z) - N(Av, w, z), \eta(u, v) \rangle \geq 0, \quad \forall u, v \in K, w, z \in H;$$

(7) η is said to be *Lipschitz continuous* with constant δ if there exists a constant $\delta > 0$ such that

$$\|\eta(u, v)\| \leq \delta\|u - v\|, \quad \forall u, v \in K;$$

(8) η is said to be *strongly monotone* with constant ω if there exists a constant $\omega > 0$ such that

$$\langle u - v, \eta(u, v) \rangle \geq \omega\|u - v\|^2, \quad \forall u, v \in K.$$

(9) the mapping $N(A, B, C) : K \rightarrow H$ is said to be η -*hemicontinuous* on K , if for all $u, v \in K$, the function $t \mapsto \langle N(A(u + t(v - u)), B(u + t(v - u)), C(u + t(v - u))), \eta(v, u) \rangle$ is continuous on $[0, 1]$.

Lemma 2.1. ([1,2]) *Let X be a nonempty closed convex subset of a Hausdorff linear topological space E and $\phi, \psi : X \times X \rightarrow R$ be mappings satisfying the following conditions:*

- (a) $\psi(x, y) \leq \phi(x, y)$, $\forall x, y \in X$, and $\psi(x, x) \geq 0$, $\forall x \in X$;
- (b) for each $x \in X$, $\phi(x, \cdot)$ is upper semicontinuous on X ;
- (c) for each $y \in X$, the set $\{x \in X : \psi(x, y) < 0\}$ is a convex set;
- (d) there exists a nonempty compact set $K \subset X$ and $x_0 \in K$ such that $\psi(x_0, y) < 0$, $\forall y \in X \setminus K$.

Then there exists $\hat{y} \in K$ such that $\phi(x, \hat{y}) \geq 0$, $\forall x \in X$.

3. Existence theorems

In this section, we establish two existence theorems of solutions for the strongly nonlinear variational-like inequality (2.1).

Theorem 3.1. *Suppose that $a : H \times H \rightarrow (-\infty, \infty)$ is a coercive continuous bilinear form with (C1) and (C2), $b : H \times H \rightarrow (-\infty, \infty)$ satisfies (C3)-(C6) and $A, B, C, F : K \rightarrow H$ and $N : H \times H \times H \rightarrow H$ are mappings. Let $\eta : K \times K \rightarrow H$ be Lipschitz continuous with constant δ and strongly monotone with constant ω , for each $v \in K$, $\eta(\cdot, v)$ be continuous and $\eta(v, u) = -\eta(u, v)$ for all $u, v \in K$. Assume that the mapping $N(A, B, C)$ is η -hemicontinuous on K , N is η -strongly monotone with constant α with respect to A in the first argument, η -monotone with respect to B in the second argument and η -relaxed monotone with constant β with respect to C in the third argument. Suppose that F is bounded and completely continuous and for given $x, y, z \in H$ and $w, v \in K$, the mapping $u \mapsto \langle N(x, y, z) + Fw, \eta(v, u) \rangle$ is concave and upper semicontinuous on K . If $c + \alpha > \beta + l$, then the strongly nonlinear variational-like inequality (2.1) has a solution u in K .*

Proof. We first prove that for each fixed $\hat{u} \in K$, there exists a $\hat{w} \in K$ such that

$$\begin{aligned} \langle N(A\hat{w}, B\hat{w}, C\hat{w}) + F\hat{u}, \eta(v, \hat{w}) \rangle + a(\hat{w}, v - \hat{w}) \\ \geq b(\hat{w}, \hat{w}) - b(\hat{w}, v), \quad \forall v \in K. \end{aligned} \quad (3.1)$$

Define the functionals ϕ and $\psi : K \times K \rightarrow R$ by

$$\begin{aligned} \phi(v, w) = \langle N(Av, Bv, Cv) + F\hat{u}, \eta(v, w) \rangle \\ + a(v, v - w) + b(v, v) - b(v, w) \end{aligned}$$

and

$$\begin{aligned} \psi(v, w) = \langle N(Aw, Bw, Cw) + F\hat{u}, \eta(v, w) \rangle \\ + a(w, v - w) + b(w, v) - b(w, w) \end{aligned}$$

for all $v, w \in K$. We check that the functionals ϕ and ψ satisfy all the conditions of Lemma 2.1 in the weak topology. It is easy to see for all $v, w \in K$,

$$\begin{aligned} \phi(v, w) - \psi(v, w) \\ \geq \langle N(Av, Bv, Cv) - N(Aw, Bv, Cv), \eta(v, w) \rangle \\ + \langle N(Aw, Bv, Cv) - N(Aw, Bw, Cv), \eta(v, w) \rangle \\ + \langle N(Aw, Bw, Cv) - N(Aw, Bw, Cw), \eta(v, w) \rangle \\ + a(v - w, v - w) - b(v - w, w - v) \\ \geq (c + \alpha - \beta - l)\|v - w\|^2 \geq 0, \end{aligned}$$

which yields that ϕ and ψ satisfy the condition (a) of Lemma 2.1. Note that b is convex and lower semicontinuous with respect to the second argument and for given $x, y, z \in H$, $w, v \in K$, the mapping $u \mapsto \langle N(x, y, z) + Fw, \eta(v, u) \rangle$ is concave and upper semicontinuous. It follows that $\phi(v, w)$ is weakly upper semicontinuous with respect to w and the set $\{v \in K : \psi(v, w) < 0\}$ is convex

for each $w \in K$. Therefore the conditions (b) and (c) of Lemma 2.1 hold. Let v^* be in K . Put

$$D = (c + \alpha - \beta - l)^{-1}[\delta\|N(Av^*, Bv^*, Cv^*) + F\hat{u}\| + (d + l)\|v^*\|]$$

and

$$T = \{w \in K : \|w - v^*\| \leq D\}.$$

Clearly, T is a weakly compact subset of K and for any $w \in K \setminus T$

$$\begin{aligned} \psi(v^*, w) &= -\langle N(Aw, Bw, Cw) - N(Av^*, Bw, Cw), \eta(w, v^*) \rangle \\ &\quad - \langle N(Av^*, Bw, Cw) - N(Av^*, Bv^*, Cw), \eta(w, v^*) \rangle \\ &\quad - \langle N(Av^*, Bv^*, Cw) - N(Av^*, Bv^*, Cv^*), \eta(w, v^*) \rangle \\ &\quad - \langle N(Av^*, Bv^*, Cv^*) + F\hat{u}, \eta(w, v^*) \rangle \\ &\quad - a(v^* - w, v^* - w) + a(v^*, v^* - w) + b(w, v^*) - b(w, w) \\ &\leq -\|w - v^*\|[(c + \alpha - \beta - l)\|w - v^*\| \\ &\quad - \delta\|N(Av^*, Bv^*, Cv^*) + F\hat{u}\| - (d + l)\|v^*\|] < 0, \end{aligned}$$

which means that the condition (d) of Lemma 2.1 holds. Thus Lemma 2.1 ensures that there exists a $\hat{w} \in K$ such that $\phi(v, \hat{w}) \geq 0$ for all $v \in K$, that is,

$$\begin{aligned} &\langle N(Av, Bv, Cv) + F\hat{u}, \eta(v, \hat{w}) \rangle + a(v, v - \hat{w}) \\ &\geq b(v, v) - b(v, \hat{w}), \quad v \in K. \end{aligned} \tag{3.2}$$

Let t be in $(0, 1]$ and v be in K . Replacing v by $v_t = tv + (1 - t)\hat{w}$ in (3.2), we know that

$$\begin{aligned} &\langle N(Av_t, Bv_t, Cv_t) + F\hat{u}, \eta(v_t, \hat{w}) \rangle + a(v_t, v_t - \hat{w}) \\ &\geq b(v_t, v_t) - b(v_t, \hat{w}). \end{aligned} \tag{3.3}$$

Notice that b is convex with respect to the second argument. From (3.3) we deduce that

$$\begin{aligned} &\langle N(Av_t, Bv_t, Cv_t) + F\hat{u}, \eta(v, \hat{w}) \rangle + a(v_t, v - \hat{w}) \\ &\geq b(v_t, v) - b(v_t, \hat{w}), \quad \forall v \in K. \end{aligned}$$

Letting $t \rightarrow 0^+$ in the above inequality, we conclude that

$$\begin{aligned} &\langle N(A\hat{w}, B\hat{w}, C\hat{w}) + F\hat{u}, \eta(v, \hat{w}) \rangle + a(\hat{w}, v - \hat{w}) \\ &\geq b(\hat{w}, \hat{w}) - b(\hat{w}, v), \quad \forall v \in K. \end{aligned}$$

That is, \hat{w} is a solution of (3.1).

Now we prove that for each fixed $\hat{u} \in K$, there exists a unique $\hat{w} \in K$ such that (3.1). Let $w_1, w_2 \in K$ be two solutions of (3.1) for fixed $\hat{u} \in K$, we know that

$$\begin{aligned} &\langle N(Aw_1, Bw_1, Cw_1) + F\hat{u}, \eta(v, w_1) \rangle + \rho a(w_1, v - w_1) \\ &\geq b(w_1, v) - b(w_1, w_1) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\langle N(Aw_2, Bw_2, Cw_2) + F\hat{u}, \eta(v, w_2) \rangle + a(w_2, v - w_2) \\ &\geq b(w_2, v) - b(w_2, w_2) \end{aligned} \tag{3.5}$$

for all $v \in K$. Taking $v = w_2$ in (3.4) and $v = w_1$ in (3.5), we get that

$$\begin{aligned} & \langle N(Aw_1, Bw_1, Cw_1) + F\hat{u}, \eta(w_2, w_1) \rangle + a(w_1, w_2 - w_1) \\ & \geq b(w_1, w_2) - b(w_1, w_1) \end{aligned}$$

and

$$\begin{aligned} & \langle N(Aw_2, Bw_2, Cw_2) + F\hat{u}, \eta(w_1, w_2) \rangle + a(w_2, w_1 - w_2) \\ & \geq b(w_2, w_1) - b(w_2, w_2). \end{aligned}$$

Adding these inequalities, we deduce that

$$\begin{aligned} & (c + \alpha - \beta - l)\|w_1 - w_2\|^2 \\ & \leq \langle N(Aw_1, Bw_1, Cw_1) - N(Aw_2, Bw_1, Cw_1), \eta(w_1, w_2) \rangle \\ & \quad + \langle N(Aw_2, Bw_1, Cw_1) - N(Aw_2, Bw_2, Cw_1), \eta(w_1, w_2) \rangle \\ & \quad + \langle N(Aw_2, Bw_2, Cw_1) - N(Aw_2, Bw_2, Cw_2), \eta(w_1, w_2) \rangle \\ & \quad + a(w_1 - w_2, w_1 - w_2) - b(w_1 - w_2, w_2 - w_1) \leq 0, \end{aligned}$$

which yields that $w_1 = w_2$. That is, \hat{w} is the unique solution of (3.1). This means that there exists a mapping $G : K \rightarrow K$ satisfying $G(\hat{u}) = \hat{w}$, where \hat{w} is the unique solution of (3.1) for each $\hat{u} \in K$.

Next we show that G is bounded and completely continuous. Let u_1 and u_2 be arbitrary elements in K . Using (3.1), we see that

$$\begin{aligned} & \langle N(A(Gu_1), B(Gu_1), C(Gu_1)) + Fu_1, \eta(v, Gu_1) \rangle + a(Gu_1, v - Gu_1) \\ & \geq b(Gu_1, v) - b(Gu_1, Gu_1) \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} & \langle N(A(Gu_2), B(Gu_2), C(Gu_2)) + Fu_2, \eta(v, Gu_2) \rangle + a(Gu_2, v - Gu_2) \\ & \geq b(Gu_2, v) - b(Gu_2, Gu_2) \end{aligned} \quad (3.7)$$

for all $v \in K$. Letting $v = Gu_2$ in (3.9) and $v = Gu_1$ in (3.10), and adding these inequalities, we arrive at

$$\begin{aligned} 0 & \geq \langle N(A(Gu_1), B(Gu_1), C(Gu_1)) \\ & \quad - N(A(Gu_2), B(Gu_1), C(Gu_1)), \eta(Gu_1, Gu_2) \rangle \\ & \quad + \langle N(A(Gu_2), B(Gu_1), C(Gu_1)) \\ & \quad - N(A(Gu_2), B(Gu_2), C(Gu_1)), \eta(Gu_1, Gu_2) \rangle \\ & \quad + \langle N(A(Gu_2), B(Gu_2), C(Gu_1)) \\ & \quad - N(A(Gu_2), B(Gu_2), C(Gu_2)), \eta(Gu_1, Gu_2) \rangle \\ & \quad + \langle Fu_1 - Fu_2, \eta(Gu_1, Gu_2) \rangle \\ & \quad + a(Gu_1 - Gu_2, Gu_1 - Gu_2) + b(Gu_1 - Gu_2, Gu_2 - Gu_1) \\ & \geq (\alpha + c - \beta - l)\|Gu_1 - Gu_2\|^2 - \delta\|Fu_1 - Fu_2\|\|Gu_1 - Gu_2\|, \end{aligned}$$

that is,

$$\|Gu_1 - Gu_2\| \leq \frac{\delta}{\alpha + c - \beta - l}\|Fu_1 - Fu_2\|. \quad (3.8)$$

Since F is bounded and completely continuous, it follows from (3.8) that $G : K \rightarrow K$ is also bounded and completely continuous. Hence the Schauder fixed point theorem guarantees that G has a fixed point $u \in K$, which means that u is a solution of the strongly nonlinear variational-like inequality (2.1). This completes the proof. \square

Theorem 3.2. *Let a, b, A, B, F and η be as in Theorem 3.1 and $C : K \rightarrow H$ be Lipschitz continuous with constant γ . Assume that $N : H \times H \times H \rightarrow H$ is η -strongly monotone with constant α with respect to A in the first argument, η -monotone with respect to B in the second argument, Lipschitz continuous with constant β in the third argument and $N(A, B, C)$ is η -hemicontinuous on K . Suppose that for given $x, y, z \in H$ and $w, v \in K$, the mapping $u \mapsto \langle N(x, y, z) + Fw, \eta(v, u) \rangle$ is concave and upper semicontinuous on K . If $c + \alpha > \beta\gamma\delta + l$, then the strongly nonlinear variational-like inequality (2.1) has a solution $u \in K$.*

Proof. Put

$$D = (c + \alpha - \beta\gamma\delta - l)^{-1}[\delta\|N(Av^*, Bv^*, Cv^*) + F\hat{u}\| + (d + l)\|v^*\|]$$

and

$$T = \{w \in K : \|w - v^*\| \leq D\}.$$

As in the proof of Theorem 3.1, we conclude that

$$\begin{aligned} \psi(v^*, w) &= -\langle N(Aw, Bw, Cw) - N(Av^*, Bw, Cw), \eta(w, v^*) \rangle \\ &\quad - \langle N(Av^*, Bw, Cw) - N(Av^*, Bv^*, Cw), \eta(w, v^*) \rangle \\ &\quad - \langle N(Av^*, Bv^*, Cw) - N(Av^*, Bv^*, Cv^*), \eta(w, v^*) \rangle \\ &\quad - \langle N(Av^*, Bv^*, Cv^*) + F\hat{u}, \eta(w, v^*) \rangle \\ &\quad - a(v^* - w, v^* - w) + a(v^*, v^* - w) + b(w, v^*) - b(w, w) \\ &\leq -\|w - v^*\|[(c + \alpha - \beta\gamma\delta - l)\|w - v^*\| \\ &\quad - \delta\|N(Av^*, Bv^*, Cv^*) + F\hat{u}\| - (d + l)\|v^*\|] < 0 \end{aligned}$$

for any $w \in K \setminus T$. The rest of the argument is now essentially the same as in the proof of Theorem 3.1 and therefore is omitted. \square

Theorem 3.3. *Let a, b, A, B, C, N and η be as in Theorem 3.1. Suppose that $F : K \rightarrow H$ is Lipschitz continuous with constant ξ . If $0 < \frac{\delta\xi}{\alpha + c - \beta - l} < 1$, then the strongly nonlinear variational-like inequality (2.1) has a unique solution $u \in K$.*

Proof. As in the proof of Theorem 3.1, it follows from (3.8) that

$$\begin{aligned} \|Gu_1 - Gu_2\| &\leq \frac{\delta}{\alpha + c - \beta - l} \|Fu_1 - Fu_2\| \\ &\leq \frac{\delta\xi}{\alpha + c - \beta - l} \|u_1 - u_2\|, \quad \forall u_1, u_2 \in K, \end{aligned}$$

which yields that $G : K \rightarrow K$ is a contraction mapping and hence it has a unique fixed point $u \in K$, which is a unique solution of the strongly nonlinear variational-like inequality (2.1). This completes the proof. \square

It follows from the arguments of Theorems 3.1-3.3 that

Theorem 3.4. *Let a, b, A, B, C, N and η be as in Theorem 3.2. Suppose that $F : K \rightarrow H$ is Lipschitz continuous with constant ξ . If $0 < \frac{\delta\xi}{\alpha+c-\beta\gamma\delta-l} < 1$, then the strongly nonlinear variational-like inequality (2.1) has a unique solution $u \in K$.*

4. Algorithm and convergence theorems

Let's consider the following auxiliary variational-like inequality problem: For any given $u \in K$, find $w \in K$ such that

$$\begin{aligned} \langle w, \eta(v, w) \rangle &\geq \langle u, \eta(v, w) \rangle - \rho \langle N(Aw, Bw, Cw) + Fu, \eta(v, w) \rangle \\ &\quad - \rho a(w, v - w) - \rho b(w, v) + \rho b(w, w), \quad \forall v \in K, \end{aligned} \quad (4.1)$$

where $\rho > 0$ is a constant. Clearly, $w = u$ is a solution of the auxiliary variational-like inequality (4.1). Based on this observation, we suggest the following iterative algorithm.

Algorithm 4.1. *For any given $u_0 \in K$, compute sequences $\{u_n\}_{n \geq 0}$, $\{w_n\}_{n \geq 0}$ and $\{z_n\}_{n \geq 0}$ by the following iterative schemes*

$$\begin{aligned} &\langle w_n, \eta(v, w_n) \rangle \\ &\geq (1 - \alpha_n) \langle u_n, \eta(v, w_n) \rangle \\ &\quad + \alpha_n \langle u_n - \rho N(Aw_n, Bw_n, Cw_n) - \rho Fu_n, \eta(v, w_n) \rangle \\ &\quad - \alpha_n \rho a(w_n, v - w_n) - \alpha_n \rho b(w_n, v) + \alpha_n \rho b(w_n, w_n) + \langle q_n, \eta(v, w_n) \rangle, \end{aligned} \quad (4.2)$$

$$\begin{aligned} &\langle z_n, \eta(v, z_n) \rangle \\ &\geq (1 - \beta_n) \langle u_n, \eta(v, z_n) \rangle \\ &\quad + \beta_n \langle w_n - \rho N(Az_n, Bz_n, Cz_n) - \rho Fw_n, \eta(v, z_n) \rangle \\ &\quad - \alpha_n \rho a(z_n, v - z_n) - \beta_n \rho b(z_n, v) + \beta_n \rho b(z_n, z_n) + \langle r_n, \eta(v, z_n) \rangle \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} &\langle u_{n+1}, \eta(v, u_{n+1}) \rangle \\ &\geq (1 - \gamma_n) \langle u_n, \eta(v, u_{n+1}) \rangle \\ &\quad + \gamma_n \langle z_n - \rho N(Au_{n+1}, Bu_{n+1}, Cu_{n+1}) - \rho Fz_n, \eta(v, z_n) \rangle \\ &\quad - \alpha_n \rho a(u_{n+1}, v - u_{n+1}) - \gamma_n \rho b(u_{n+1}, v) + \gamma_n \rho b(u_{n+1}, u_{n+1}) \\ &\quad + \langle s_n, \eta(v, u_{n+1}) \rangle \end{aligned} \quad (4.4)$$

for all $v \in K$ and $n \geq 0$, where $\{\alpha_n\}_{n \geq 0}$, $\{\beta_n\}_{n \geq 0}$ and $\{\gamma_n\}_{n \geq 0} \subset [0, 1]$ satisfy $\sum_{n=0}^{\infty} \gamma_n = \infty$, $\{q_n\}_{n \geq 0}$, $\{r_n\}_{n \geq 0}$ and $\{s_n\}_{n \geq 0} \subset H$ and $\rho > 0$ is a constant.

Theorem 4.1. *Let a, b, A, B, C, N, F and η be as in Theorem 3.3. Let F be strongly monotone with constant ζ and $0 < \frac{\delta\xi}{\alpha+c-\beta-l} < 1$. Assume that*

$$\lim_{n \rightarrow \infty} \|q_n\| = \lim_{n \rightarrow \infty} \|r_n\| = \lim_{n \rightarrow \infty} \|s_n\| = 0 \tag{4.5}$$

and

$$\inf\{\alpha_n, \beta_n, \gamma_n : n \geq 0\} > 0. \tag{4.6}$$

If there exists a constant $\rho > 0$ satisfying

$$\frac{\delta - \omega}{(\alpha + c - \beta - l) \inf\{\alpha_n, \beta_n, \gamma_n : n \geq 0\}} \leq \rho < \frac{2\zeta}{\xi^2}, \tag{4.7}$$

then the strongly nonlinear variational-like inequality (2.1) possesses a unique solution $u \in K$ and the iterative sequence $\{u_n\}_{n \geq 0}$ generated by Algorithm 4.1 converges strongly to u .

Proof. It follows from Theorem 3.3 that the strongly nonlinear variational-like inequality (2.1) has a unique solution $u \in K$ such that

$$\begin{aligned} &\langle u, \eta(v, u) \rangle \\ &\geq (1 - \alpha_n)\langle u, \eta(v, u) \rangle + \alpha_n\langle u - \rho N(Au, Bu, Cu) - \rho Fu, \eta(v, u) \rangle \\ &\quad - \alpha_n \rho a(u, v - u) - \alpha_n \rho b(u, v) + \alpha_n \rho b(u, u), \end{aligned} \tag{4.8}$$

$$\begin{aligned} &\langle u, \eta(v, u) \rangle \\ &\geq (1 - \beta_n)\langle u, \eta(v, u) \rangle + \beta_n\langle u - \rho N(Au, Bu, Cu) - \rho Fu, \eta(v, u) \rangle \\ &\quad - \beta_n \rho a(u, v - u) - \beta_n \rho b(u, v) + \beta_n \rho b(u, u) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\langle u, \eta(v, u) \rangle \\ &\geq (1 - \gamma_n)\langle u, \eta(v, u) \rangle + \gamma_n\langle u - \rho N(Au, Bu, Cu) - \rho Fu, \eta(v, u) \rangle \\ &\quad - \gamma_n \rho a(u, v - u) - \gamma_n \rho b(u, v) + \gamma_n \rho b(u, u) \end{aligned} \tag{4.10}$$

for all $v \in K$ and $n \geq 0$. Taking $v = u$ in (4.2), $v = w_n$ in (4.8) and adding these inequalities, we get that

$$\begin{aligned} &\omega \|w_n - u\|^2 \\ &\leq (1 - \alpha_n)\langle u_n - u, \eta(w_n, u) \rangle \\ &\quad - \alpha_n \rho \langle N(Aw_n, Bw_n, Cw_n) - N(Au, Bw_n, Cw_n), \eta(w_n, u) \rangle \\ &\quad - \alpha_n \rho \langle N(Au, Bw_n, Cw_n) - N(Au, Bu, Cw_n), \eta(w_n, u) \rangle \\ &\quad - \alpha_n \rho \langle N(Au, Bu, Cw_n) - N(Au, Bu, Cu), \eta(w_n, u) \rangle \\ &\quad + \alpha_n \langle u_n - u - \rho(Fu_n - Fu), \eta(w_n, u) \rangle \\ &\quad - \alpha_n \rho a(w_n - u, w_n - u) + \alpha_n \rho b(w_n - u, u - w_n) + \langle q_n, \eta(u, w_n) \rangle \\ &\leq \delta \{1 - \alpha_n [1 - \sqrt{1 - 2\rho\zeta + (\rho\xi)^2}]\} \|u_n - u\| \|w_n - u\| \\ &\quad - \rho \alpha_n (\alpha + c - \beta - l) \|w_n - u\|^2 + \delta \|q_n\| \|w_n - u\|, \quad \forall n \geq 0, \end{aligned}$$

that is,

$$\begin{aligned} \|w_n - u\| &\leq \theta_1 [1 - \alpha_n(1 - \theta_2)] \|u_n - u\| + \theta_1 \|q_n\| \\ &\leq \|u_n - u\| + \|q_n\|, \quad \forall n \geq 0, \end{aligned} \quad (4.11)$$

$\theta_1 = \frac{\delta}{\omega + \rho(\alpha + c - \beta - l) \inf\{\alpha_n, \beta_n, \gamma_n : n \geq 0\}} \leq 1$ and $\theta_2 = \sqrt{1 - 2\rho\zeta + (\rho\xi)^2} < 1$ by (4.7). It follows from (4.3), (4.9) and (4.11) that

$$\begin{aligned} \omega \|z_n - u\|^2 &\leq \delta(1 - \beta_n) \|u_n - u\| \|z_n - u\| + \delta\beta_n\theta_2 \|w_n - u\| \|z_n - u\| \\ &\quad - \rho\alpha \|z_n - u\|^2 + \delta \|r_n\| \|z_n - u\|, \quad \forall n \geq 0, \end{aligned}$$

that is,

$$\|z_n - u\| \leq [1 - \beta_n(1 - \theta_2)] \|u_n - u\| + \|q_n\| + \|r_n\|, \quad \forall n \geq 0.$$

Similarly, we have that

$$\begin{aligned} \|u_{n+1} - u\| &\leq [1 - \gamma_n(1 - \theta_2)] \|u_n - u\| + \|q_n\| + \|r_n\| + \|s_n\| \\ &\leq \prod_{i=0}^n [1 - \gamma_i(1 - \theta_2)] \|u_0 - u\| \\ &\quad + \sum_{m=0}^n \prod_{i=m+1}^n [1 - \gamma_i(1 - \theta_2)] (\|q_m\| + \|r_m\| + \|s_m\|), \quad \forall n \geq 0. \end{aligned}$$

It follows from (4.5) and (4.6) that $\lim_{n \rightarrow \infty} \|u_{n+1} - u\| = 0$. This completes the proof. \square

Similarly we have the following result.

Theorem 4.2. *Let a, b, A, B, C, N, F and η be as in Theorem 3.4. Let F be strongly monotone with constant ζ and $0 < \frac{\delta\xi}{\alpha + c - \beta\gamma\delta - l} < 1$ and (4.5) and (4.6) hold. If there exists a constant $\rho > 0$ satisfying*

$$\frac{\delta - \omega}{(\alpha + c - \beta\gamma\delta - l) \inf\{\alpha_n, \beta_n, \gamma_n : n \geq 0\}} \leq \rho < \frac{2\zeta}{\xi^2}, \quad (4.12)$$

then the strongly nonlinear variational-like inequality (2.1) possesses a unique solution $u \in K$ and the iterative sequence $\{u_n\}_{n \geq 0}$ generated by Algorithm 4.1 converges strongly to u .

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