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COMMON FIXED POINT POINT THEOREM FOR A PAIR OF WEAKLY COMPATIBLE MAPPINGS ON BANACH SPACES

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Abstract. In this paper, we prove a common fixed point theorem for noncompatible, discontinuous mappings on a Banach space. Our main theorem extends, improves, generalizes some known results on Banach spaces.

1. Introduction and Preliminaries

Husain and Sehgal [1] proved common fixed point theorems for a family of mappings. Khan and Imdad [5] extended result of Husain and Sehgal [1]. Imdad, Khan and Sessa [2] extended above results and proved common fixed points for three mappings defined on a closed subset of a uniformly convex Banach space.

Rashwan [9] extended result of Imdad, Khan and Sessa [2] by employing four compatible mappings of type (A) instead of weakly commuting mappings and by using one continuous mapping as opposed to two.

Sharma and Bamboria [12] improved results of Rashwan [9] by removing the condition of continuity and replacing the compatibility of mappings of type (A) by weak compatibility. Sharma and Tilwankar [13] proved a common fixed point theorem for four mappings under the condition of weak compatible mappings by using the new property (E.A).

For the study of discontinuous and noncompatible mappings in fixed point theory we refer to Sharma and Deshpande [14], [15] and Sharma, Deshpande and Tiwari [16]. Several observations motivated us to prove common fixed point theorem for noncompatible, discontinuous mappings on a Banach space. Our main theorem extends, improves, generalizes some known results on Banach spaces.

Now, we begin with some known definitions:

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Definition 1. ([10]) Let S and T be self-mappings on X. Then the pair $\{S, T\}$ is called a weakly commuting pair on X if

$$\| STx - TSx \| \leq \| Sx - Tx \| \quad \text{for all } \; x \in X.$$

Definition 2. ([3]) Let S, $T : X \longrightarrow X$ be mappings. S and T are said to be compatible if

$$\lim_{n \longrightarrow \infty} \| STx_n - TSx_n \| = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \longrightarrow \infty} Sx_n = \lim_{n \longrightarrow \infty} Tx_n = t \; \text{ for some } \; t \in X.$$

Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible. On the other hand, examples are given by Jungck [3] and Sessa [10] to show neither of the above implications are reversible.

Definition 3. ([4]) Two self mappings S and T are said to be weakly compatible if they commute at their coincidence points; i.e., if $Tu = Su$ for some $u \in X$, then $TSu = STu$.

Definition 4. ([6]) Let A, $S : X \to X$ be mappings. Then the pair $\{A, S\}$ is said to be compatible of type (I) if $d(t, St) \leq \limsup d(t, ASx_n)$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some $t \in X$.

Definition 5. ([6]) Let A, $S : X \to X$ be mappings. Then the pair $\{A, S\}$ is said to be compatible of type (II) if and only if $\{S, A\}$ is compatible of type (I).

It is well known that a compatible pair of maps is weakly compatible pair but converse need not be true see [4]. However it is interesting to note that the concepts of weakly compatible maps and compatible maps of type (I) are independent from each other. To show this, we illustrate the following examples:

Example 1. Let $X = [0, \infty)$ and A, $S : X \to X$ defined by

$$Ax = \begin{cases} \cos x & \text{when} \quad x \neq 1 \\ 0 & \text{when} \quad x = 1, \end{cases}$$

and

$$Sx = \begin{cases} e^x & \text{when} \quad x \neq 1 \\ 0 & \text{when} \quad x = 1. \end{cases}$$

Then it is clear that $Ax = Sx$ iff $x = 0$ and $x = 1$. Also at these points $ASx = SAx$.

It means the mappings (A, S) are weakly compatible.

Now we suppose that $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to t \in X$. Here $t = 1$ by definition of A and S. Now $d(t, St) = 1$ and $\limsup d(t, ASx_n) = (1 - cos 1) < 1$. Therefore, the pair (A, S) is not compatible of type (I).

Example 2. Let $X = [0, \infty)$ and A, $S : X \to X$ defined by $Ax = 2x + 1$ and $Sx = x^2 + 1$. Then at $x = 0$, $Ax = Sx$. Also at $x = 0$, $ASx = 3$ and $SAx = 2$, which shows that the pair (A, S) is not weakly compatible. Now suppose that $\{x_n\}$ be a sequence in X such that $\lim Ax_n = \lim Sx_n = t \in X$. By definition of

A and S, $t = 1$. For this value, we have $d(t, St) = 1$ and $\limsup d(t, ASx_n) = 2$, which shows that the pair (A, S) is compatible mappings of type (I).

It may be remarked that we have suitable examples to show that the concepts of weakly compatible maps and compatible maps of type (II) are also independent from each other.

Moreover, Pathak, Mishra and Kalinde in [6], have shown that the concepts of compatibility of type (I) and type (II) are independent. Thus, the concept of weak compatibility of maps is independent from compatibility of maps of type (I) as well as type (II).

In a paper Imdad, Khan and Seesa [2] proved the following theorem:

Theorem A. *Let X be uniformly convex and K a non-empty closed subset of X. Let A, S and T be three self-mappings of K satisfying the following conditions:*

(1) *S and T are continuous, $AK \subset SK \cap TK$,*

(2) *$\{A, S\}$ and $\{A, T\}$ are weakly commuting pairs on K,*

(3) *there exists a function $f \in F$ such that for every x, $y \in K$:*

$$\| Ax - Ay \| \leq f(\| Sx - Ty \|, \| Sx - Ax \|, \| Sx - Ay \|, \| Ty - Ax \|, \| Ty - Ay \|)$$

where f has the additional requirements:

(a) *for $t > 0$, $f(t, t, 0, \alpha t, t) \leq \beta t$ and $f(t, t, \alpha t, 0, t) \leq \beta t$ being $\beta < 1$ for $\alpha < 2$ and $\beta = 1$ for $\alpha = 2$, α, $\beta \in R^+$,*

(b) *$f(t, 0, t, t, 0) < t$ for $t > 0$.*

Then there exists a point u in K such that

(c) *u is the unique common fixed point of A, S and T.*

(d) *For any $x_0 \in K$, the sequence $\{Ax_n\}$ defined by*

$$Tx_{2n} = Ax_{2n-1}, \quad Sx_{2n+1} = Ax_{2n}, \quad \text{for } n = 0, 1, 2 \cdots$$

converges strongly to u.

Rashwan [9] extended Theorem A for compatible mappings of type (A) and proved the following:

Theorem B. *Let X and K be as in Theorem A. Let A, B, S and T be mappings on K satisfying the following conditions:*

(1) *one of A, B, S and T is continuous and $AK \subset TK$, $BK \subset SK$,*

(2) *$\{A, S\}$ and $\{B, T\}$ are compatible of type (A),*

(3) *there exists a function $f \in F$ such that for every x, $y \in K$:*

$$\| Ax - By \| \leq f(\| Sx - Ty \|, \| Sx - Ax \|, \| Sx - By \|, \| Ty - Ax \|, \| Ty - By \|),$$

where f satisfies the conditions (a) and (b) as in Theorem A. Then there exists a point u in K such that

(a) *u is the unique common fixed point of A, B, S and T,*

(b) *for any $x_0 \in K$, the sequence $\{y_n\}$ defined by*

$$y_{2n} = Sx_{2n} = Bx_{2n-1},$$
$$y_{2n+1} = Tx_{2n+1} = Ax_{2n}, \qquad n = 1, 2, 3, \cdots$$

converges strongly to u.

Sharma and Bamboria [12] proved the following.

Theorem C. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S and T be mappings on K satisfying the following conditions:*

(1) $AK \subset TK$ *and* $BK \subset SK$,

(2) *there exists a function $f \in F$ such that for every x, $y \in K$:*

$$\| Ax - By \| \leq f(\| Sx - Ty \|, \| Sx - Ax \|, \| Sx - By \|, \| Ty - Ax \|, \| Ty - By \|)$$

where f satisfies the conditions (a) and (b) as in Theorem A,

(3) *one of AK, BK, SK or TK is complete subspace of X, then*

(a) *A and S have a coincidence point,*

(b) *B and T have a coincidence point,*

Further if

(4) *the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then A, B, S and T have a common fixed point z in K.*

Further z is the unique common fixed point of A and S and of B and T.

Sharma and Tilwankar [13] proved the following using (E.A) property.

Theorem D. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S and T be mappings on K satisfying the following conditions:*

(1) $AK \subset TK$ *and* $BK \subset SK$,

(2) $\{A, S\}$ *or* $\{B, T\}$ *satisfies the property (E.A),*

(3) *for every x, $y \in K$:*

$$||Ax - By|| \leq \max(||Sx - Ty||, ||Sx - By||, ||Ty - By||),$$

(4) *one of AK, BK, SK or TK is closed subset of X, then*

(a) *A and S have a coincidence point ,*

(b) *B and T have a coincidence point,*

Further if

(5) *the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then*

(c) *A, B, S and T have a common fixed point z in K.*

Further z is the unique common fixed point of A and S and of B and T.

Throughout the chapter, suppose F denotes the collection of mappings $f : [0, 1) \to [0, 1)$ which are upper semi-continuous, non decreasing in each co-ordinate variables and $f(t) < t$ for all $t > 0$.

Pathak, Khan and Tiwari [8] proved the following:

Theorem E. *Let A, B, S and T be self mappings of a complete metric space (X, d). Suppose any one of the maps A or B is continuous, satisfying*

(1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

(2)

$$d^{2p}(Ax,By) \le a\phi_0(d^{2p}(Sx,Ty)) + (1-a)\max\{\phi_1(d^{2p}(Sx,Ty)),$$
$$\phi_2(d^q(Sx,Ax).d^{q'}(Ty,By)), \phi_3(d^r(Sx,By).d^{r'}(Ty,Ax)),$$
$$\phi_4(d^s(Sx,Ax).d^{s'}(Ty,Ax)), \phi_5(d^l(Sx,By).d^{l'}(Ty,By)\},$$

for all $x,\ y \in X$, *where* $\phi_i \in \Phi$, $i = 1,2,3,4,5$, $0 \le a \le 1$, $0 < p,q,q',l,l',r,r',s,s' \le 1$, *such that*

$$2p = q + q' = r + r' = s + s' = l + l',$$

(3) *If the pairs* $\{A,S\}$ *and* $\{B,T\}$ *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point* y' *in* X. *Furhter* y' *is the common fixed point of* A *and* S *and of* B *and* T.

To prove our main theorem we need the following lemmas:

Lemma 1. ([7]) *If* $\phi_i \in \Phi$ *and* $i \in I$ *where* I *is a finite indexing set, then there exists some* $\phi_i \in \Phi$ *such that* $\max\{\phi_i(t) : i \in I\} \le \phi(t)$ *for all* $t > 0$. *Let* A, B, S *and* T *be mappings on* K *satisfying the following conditions:*

$$AK \subset TK \ \text{ and } \ BK \subset SK, \tag{1.1}$$

$$||Ax - By||^{2p} \le a\phi_0(||Sx - Ty||^{2p}) + (1-a)\max\{\phi_1||(||Sx - Ty||^{2p}),$$
$$\phi_2(||Sx - Ax||^q||Ty - By||^{q'}), \phi_3(||Sx - By||^r||Ty - Ax||^{r'}),$$
$$\phi_4(1/2[||Sx - Ax||^s||Ty - Ax||^{s'})]),$$
$$\phi_5(1/2[||Sx - By||^l||Ty - By||^{l'})]\}, \tag{1.2}$$

for all $x,\ y \in K$, *where* $\phi_i \in \Phi$, $i = 0,1,2,3,4,5$, $0 \le a \le 1$, $0 < p$, q, q', r, r', s, s', l, $l' \le 1$ *such that* $2p = q + q' = r + r' = s + s' = l + l'$. *Then for arbitrary point* x_0 *in* X, *by (1.1), we choose a point* x_1 *such that* $Tx_1 = Ax_0$ *and for this point* x_1, *there exists a point* x_2 *in* X *such that* $Sx_2 = Bx_1$ *and so on.*

Continuing in this manner, we can define a sequence $\{y_n\}$, *for* $n = 1,2,3,\cdots$, *in* X *such that*

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \ \text{ and } \ y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}. \tag{$*$}$$

Lemma 2. ([8]) *Let* $\phi_i \in \Phi$ *and* $\{y_n\}$ *be a sequence of non-negative real numbers. If* $y_{n+1} \le \phi(y_n)$ *for* $n \in N$, *then the sequence converges to 0. Now we will prove the following lemmas which help us to establish our results*:

Lemma 3. ([12]) *If we denote* $d_n = ||y_n - y_{n+1}||$ *then* $\lim_{n\to\infty} d_n = 0$.

Proof. The inequality (1.2) implies

$$||y_{2n} - y_{2n+1}||^{2p}$$
$$\le a\phi_0(||y_{2n-1} - y_{2n}||^{2p}) + (1-a)\max\{\phi_1(||y_{2n-1} - y_{2n}||^{2p}),$$
$$\phi_2(||y_{2n-1} - y_{2n}||^q||y_{2n} - y_{2n+1}||^{q'}), \phi_3(||y_{2n-1} - y_{2n+1}||^r||y_{2n} - y_{2n}||^{r'}),$$

$$\begin{aligned}
&\phi_4(1/2[||y_{2n-1}-y_{2n}||^s||y_{2n}-y_{2n}||^{s\prime}]),\\
&\phi_5(1/2[||y_{2n-1}-y_{2n+1}||^l||y_{2n}-y_{2n+1}||^{l\prime}]\},(d_{2n})^{2p}\\
&\le a\phi_0(d_{2n-1})^{2p}+(1-a)\max\{\phi_1(d_{2n-1})^{2p},\phi_2(d_{2n-1})^q(d_{2n})^{q\prime},\phi_3(0),\\
&\phi_4(0),\phi_5(1/2([d_{2n-1}]^l+[d_{2n}]^l)[d_{2n}]^{l\prime})\}\\
&\le a\phi_0(d_{2n-1})^{2p}+(1-a)\max\{\phi_1(d_{2n-1})^{2p},\phi_2(d_{2n-1})^q(d_{2n})^{q\prime},\phi_3(0),\\
&\phi_4(0),\phi_5(1/2[(d_{2n-1})^l(d_{2n})^{l\prime}+(d_{2n})^l)(d_{2n})^{l\prime}]\}.
\end{aligned}$$

If $d_{2n}>d_{2n-1}$, then we have

$$\begin{aligned}
(d_{2n})^{2p}&\le a\phi_0(d_{2n})^{2p}+(1-a)\max\{\phi_1(d_{2n})^{2p},\phi_2(d_{2n})^{q+q\prime},\phi_3(0),\\
&\quad\phi_4(0),\phi_5(1/2[(d_{2n})^{l+l\prime}+(d_{2n})^{l+l\prime}]\}(d_{2n})^{2p}\\
&\le a\phi_0(d_{2n})^{2p}+(1-a)\max\{\phi_1(d_{2n})^{2p},\phi_2(d_{2n})^{2p},\phi_3(0),\\
&\quad\phi_4(0),\phi_5(d_{2n})^{2p}\}\\
&\le(d_{2n})^{2p}
\end{aligned}$$

a contradiction. Thus, we must have $d_{2n}\le d_{2n-1}$. Then using this inequality the condition (1.2) yields

$$d_{2n}\le\phi(d_{2n-1}). \tag{1.3}$$

Similarly taking $x=x_{2n+2}$ and $y=x_{2n+1}$ in (1.2), we get

$$\begin{aligned}
&||y_{2n+1}-y_{2n+2}||^{2p}\\
&\le a\phi_0(||y_{2n}-y_{2n+1}||^{2p})+(1-a)\max\{\phi_1(||y_{2n}-y_{2n+1}||^{2p}),\\
&\phi_2(||y_{2n+1}-y_{2n+2}||^q||y_{2n}-y_{2n+1}||^{q\prime}),\phi_3(||y_{2n+1}-y_{2n+1}||^r||y_{2n}-y_{2n+1}||^{r\prime}),\\
&\phi_4(1/2[||y_{2n+1}-y_{2n+2}||^s||y_{2n}-y_{2n+2}||^{s\prime}]),\\
&\phi_5(1/2[||y_{2n+1}-y_{2n+1}||^l||y_{2n}-y_{2n+1}||^{l\prime}]\},(d_{2n+1})^{2p}\\
&\le a\phi_0(d_{2n})^{2p}+(1-a)\max\{\phi_1(d_{2n})^{2p},\phi_2(d_{2n+1})^q(d_{2n})^{q\prime},\\
&\phi_3(0),\phi_4(1/2[(d_{2n+1})^s((d_{2n})^{s\prime}+(d_{2n+1})^{s\prime})]),\phi_5(0)\}.
\end{aligned}$$

or

$$\begin{aligned}
(d_{2n+1})^{2p}&\le a\phi_0(d_{2n})^{2p}+(1-a)\max\{\phi_1(d_{2n})^{2p},\phi_2(d_{2n+1})^q(d_{2n})^{q\prime},\\
&\quad\phi_3(0),\phi_4(1/2[(d_{2n+1})^s(d_{2n})^{s\prime}+(d_{2n+1})^{s\prime}(d_{2n+1})^s]),\phi_5(0)\}.
\end{aligned}$$

$d_{2n+1}>d_{2n}$, then we have

$$\begin{aligned}
(d_{2n+1})^{2p}&\le a\phi_0(d_{2n+1})^{2p}+(1-a)\max\{\phi_1(d_{2n+1})^{2p},\phi_2(d_{2n+1})^{q+q\prime},\\
&\quad\phi_3(0),\phi_4(0),\phi_5(1/2[(d_{2n+1})^{l+l\prime}+(d_{2n+1})^{l+l\prime}])\},
\end{aligned}$$

which implies

$$(d_{2n+1})^{2p}\le\phi(d_{2n+1})^{2p}<(d_{2n+1})^{2p},$$

a contradiction. Thus, we must have $d_{2n+1}\le d_{2n}$.

Again from (1.2), we obtain

$$d_{2n+1}\le\phi(d_{2n}). \tag{1.4}$$

From (1.3) and (1.4), we obtain

$$d_{n+1} \leq \phi(d_n), \quad \text{for } n = 0, 1, 2, \cdots ,$$

And so, by Lemma 3, we get $\lim_{n\to\infty} d_n = 0$. □

Lemma 4. ([11]) *The sequence* $\{y_n\}$ *defined by*

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \tag{1.5}$$

is a Cauchy sequence.

We prove Theorem E without using condition of continuity on Banach spaces. We point out that for the existence of fixed point the continuity of any mapping is not required.

2. Main Result

Theorem 1. *Let X be a Banach space space and K a non-empty closed subset of X. Let A, B, S and T be mappings on K satisfying the following conditions:*

(i) $AK \subset TK$ *and* $BK \subset SK$,

(ii)

$$\begin{aligned}||Ax - By||^{2p} \leq a\phi_0(||Sx - Ty||^{2p}) + (1-a)\max\{&\phi_1(||Sx - Ty||^{2p}),\\ &\phi_2(||Sx - Ax||^q||Ty - By||^{q'}), \phi_3(||Sx - By||^r||Ty - Ax||^{r'}),\\ &\phi_4(1/2[||Sx - Ax||^s||Ty - Ax||^{s'})]),\\ &\phi_5(1/2[||Sx - By||^l||Ty - By||^{l'})]\},\end{aligned}$$

for all x, $y \in X$, where $\phi_i \in \Phi$, $i = 0, 1, 2, 3, 4, 5$, $0 \leq a \leq 1$, $0 < p$, q, q', r, r', s, s', l, $l' \leq 1$, such that

$$2p = q + q' = r + r' = s + s' = l + l'.$$

Then A, B, S and T have a common fixed point z in K.

Proof. Since K is closed subset of a Banach space X, therefore, by Lemma 4, the sequence $\{y_n\}$ converges to a point z in K. On the other hand, the sub sequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converges to the point z.

Now suppose that the subsequence $\{y_{2n+1}\}$ is contained in SK and has a limit in SK call it z.

Let $w \in S^{-1}(z)$. Then $Sw = z$. By (ii), we have

$$\begin{aligned}||Aw - Bx_{2n+1}||^{2p} \leq a\phi_0(&||Sw - Tx_{2n+1}||^{2p}\\ &+ (1-a)\max\{\phi_1(||Sw - Tx_{2n+1}||^{2p}),\\ &\phi_2(||Sw - Aw||^q.||Tx_{2n+1} - Bx_{2n+1}||^{q'},\\ &\phi_3(||Sw - Bx_{2n+1}||^r.||Tx_{2n+1} - Aw||^{r'},\end{aligned}$$

$$\phi_4(1/2[||Sw - Aw||^s.||Tx_{2n+1} - Aw||^{s'}]),$$
$$\phi_5(1/2[||Sw - Bx_{2n+1}||^l.||Tx_{2n+1} - Bx_{2n+1}||^{l'}])\},$$

letting $n \to \infty$, we have

$$\begin{aligned} ||Aw - z||^{2p} \leq a\phi_0(||z - z||^{2p} + (1-a)\max\{&\phi_1(||z - z||^{2p}) \\ &\phi_2(||z - Aw||^q.||z - z||^{q'}, \phi_3(||z - z||^r.||z - Aw||^{r'}, \\ &\phi_4(1/2[||z - Aw||^s.||z - Aw||^{s'}]), \\ &\phi_5(1/2[||z - z||^l.||z - z||^{l'}])\}, \end{aligned}$$

or

$$\begin{aligned} ||Aw - z||^{2p} \leq a\phi_0(0) + (1-a)\max\{&\phi_1(0), \phi_2(0), \phi_3(0), \\ &\phi_4(1/2||z - Aw||^{2p}), \phi_5(0)\}, \end{aligned}$$

$$||Aw - z||^{2p} < ||Aw - z||^{2p},$$

which is a contradiction. Hence $Aw = z$. Therefore $Aw = Sw = z$.

Let $v \in T^{-1}(z)$. Then $Tv = z$. By (ii), we have

$$\begin{aligned} &||Ax_{2n+2} - Bv||^{2p} \\ &\leq a\phi_0(||Sx_{2n+2} - Tv||^{2p}) + (1-a)\max\{\phi_1(||Sx_{2n+2} - Tv||^{2p}) \\ &\quad \phi_2(||Sx_{2n+2} - Ax_{2n+2}||^q.||Tv - Bv||^{q'}, \\ &\quad \phi_3(||Sx_{2n+2} - Bv||^r.||Tv - Ax_{2n+2}||^{r'}, \\ &\quad \phi_4(1/2[||Sx_{2n+2} - Ax_{2n+2}||^s.||Tv - Ax_{2n+2}||^{s'}]), \\ &\quad \phi_5(1/2[||Sx_{2n+2} - Bv||^l.||Tv - Bv||^{l'}])\}, \end{aligned}$$

letting $n \to \infty$, we have

$$\begin{aligned} &||z - Bv||^{2p} \\ &\leq a\phi_0(||z - z||^{2p}) + (1-a)\max\{\phi_1(||z - z||^{2p}), \phi_2(||z - z||^q.||z - Bv||^{q'}, \\ &\quad \phi_3(||z - Bv||^r.||z - z||^{r'}, \phi_4(1/2[||z - z||^s.||z - z||^{s'}]), \\ &\quad \phi_5(1/2[||z - Bv||^l.||z - Bv||^{l'}])\}, ||z - Bv||^{2p} \\ &\leq a\phi_0(0) + (1-a)\max\{\phi_1(0), \phi_2(0), \phi_3(0), \phi_4(0), \phi_5(1/2||z - Bv||^{2p})\}, \end{aligned}$$

or

$$\begin{aligned} d^{2p}(z, Bv) &\leq (1-a)\phi(1/2||z - Bv||^{2p}) \\ &\qquad ||z - Bv||^{2p} < ||z - Bv||, \end{aligned}$$

which is a contradiction. Thus $z = Bv$. Therefore $Bv = Tv = z$ and so

$$Bv = Tv = z = Aw = Sw.$$

Since the pair $\{A, S\}$ is weakly compatible therefore, A and S commute at their coincidence point, i.e., if $ASw = SAw$ or $Az = Sz$.

Similarly, since the pair $\{B,T\}$ is weakly compatible therefore, B and T commute at their coincidence point, i.e., if $BTw = TBw$ or $Bz = Tz$. Now, we prove $Az = z$. By (ii), we have

$$\begin{aligned}
&||Az - Bx_{2n+1}||^{2p}\\
&\leq a\phi_0(||Sz - Tx_{2n+1}||^{2p} + (1-a)\max\{\phi_1(||Sz - Tx_{2n+1}||^{2p})\\
&\quad \phi_2(||Sz - Az||^q.||Tx_{2n+1} - Bx_{2n+1}||^{q'},\\
&\quad \phi_3(||Sz - Bx_{2n+1}||^r.||Tx_{2n+1} - Az||^{r'},\\
&\quad \phi_4(1/2[||Sz - Az||^s.||Tx_{2n+1} - Az||^{s'}]),\\
&\quad \phi_5(1/2[||Sz - Bx_{2n+1}||^l.||Tx_{2n+1} - Bx_{2n+1}||])\},
\end{aligned}$$

letting $n \to \infty$, we have

$$\begin{aligned}
&||Az - z||^{2p}\\
&\leq a\phi_0(||Az - z||^{2p} + (1-a)\max\{\phi_1(||Az - z||^{2p})\\
&\quad \phi_2(||Az - Az||^q.||z - z||^{q'}, \phi_3(||Az - z||^r.||z - Az||^{r'},\\
&\quad \phi_4(1/2[||Az - Az||^s.||z - Az||^{s'}]), \phi_5(1/2[||Az - z||^l.||z - z||^{l'}])\},
\end{aligned}$$

or

$$\begin{aligned}
||Az - z||^{2p} &\leq a\phi_0(0) + (1-a)\max\{\phi_1(0), \phi_2(0), \phi_3(0),\\
&\qquad \phi_4(1/2||z - Az||^{2p}), \phi_5(0)\},\\
&\qquad ||Az - z||^{2p} < ||Az - z||^{2p},
\end{aligned}$$

which is a contradiction. Hence $Az = z$. Therefore $Az = Sz = z$.

Now, by (ii), we have

$$\begin{aligned}
&||Ax_{2n+2} - Bz||^{2p}\\
&\leq a\phi_0(||Sx_{2n+2} - Tz||^{2p}) + (1-a)\max\{\phi_1(||Sx_{2n+2} - Tz||^{2p})\\
&\quad \phi_2(||Sx_{2n+2} - Ax_{2n+2}||^q.||Tz - Bz||^{q'}, \phi_3(||Sx_{2n+2} - Bz||^r.||Tz - Ax_{2n+2}||^{r'},\\
&\quad \phi_4(1/2[||Sx_{2n+2} - Ax_{2n+2}||^s.||Tz - Ax_{2n+2}||^{s'}]),\\
&\quad \phi_5(1/2[||Sx_{2n+2} - Bz||^l.||Tz - Bz||^{l'}])\},
\end{aligned}$$

letting $n \to \infty$, we have

$$\begin{aligned}
&||z - Bz||^{2p}\\
&\leq a\phi_0(||z - Bz||^{2p}) + (1-a)\max\{\phi_1(||z - Bz||^{2p})\\
&\quad \phi_2(||z - z||^q.||z - Bz||^{q'}, \phi_3(||z - Bz||^r.||z - Bz||^{r'},\\
&\quad \phi_4(1/2[||z - z||^s.||z - z||^{s'}]), \phi_5(1/2[||z - Bz||^l.||z - Bz||^{l'}])\}, ||z - Bz||^{2p}\\
&\leq a\phi_0(||z - Bz||^{2p}) + (1-a)\max\{\phi_1(||z - Bz||^{2p}),\\
&\quad \phi_3(||z - Bz||^{r+r'}), \phi_4(0), \phi_5(1/2[||z - Bz||^{l+l'}]\},
\end{aligned}$$

or

$$||z - Bz||^{2p} < ||z - Bz||^{2p},$$

which is a contradiction. Thus $z = Bz$. Therefore $Bz = Tz = z$ and so $Bz = Tz = z = Az = Sz$. i.e., z is a common fixed point of A, B, S and T.

For uniqueness let z' $(z \neq z')$ be another common fixed point of A, B, S and T. By (ii), we have

$$\begin{aligned}
&||Az - Bz'||^{2p} \\
&\leq a\phi_0(||Sz - Tz'||^{2p}) + (1-a)\max\{\phi_1(||Sz - Tz'||^{2p}), \\
&\quad \phi_2(||Sz - Az||^q||Tz' - Bz'||^{q'}), \phi_3(||Sz - Bz'||^r||Tz' - Az||^{r'}), \\
&\quad \phi_4(1/2[||Sz - Az||^s||Tz' - Az||^{s'}]), \phi_5(1/2[||Sz - Bz'||^l||Tz' - Bz'||^{l'})]\},
\end{aligned}$$

or

$$\begin{aligned}
&||z - z'||^{2p} \\
&\leq a\phi_0(||z - z'||^{2p}) + (1-a)\max\{\phi_1(||z - z'||^{2p}), \\
&\quad \phi_2(||z - z||^q||z' - z'||^{q'}), \phi_3(||z - z'||^r||z' - z||^{r'}), \\
&\quad \phi_4(1/2[||z - z||^s||z' - z||^{s'})]), \phi_5(1/2[||z - z'||^l||z' - z'||^{l'})]\}, \\
&||z - z'||^{2p} \leq a\phi_0(||z - z'||^{2p}) + (1-a)\max\{\phi_1(||z - z'||^{2p}), \\
&\qquad \phi_2(0), \phi_3(||z - z'||^{r+r'}), \\
&\qquad \phi_4(0), \phi_5(0)\}.
\end{aligned}$$

It follows from the above that

$$||z - z'||^{2p} \leq \phi||z - z'||^{2p},$$

$$||z - z'||^{2p} < ||z - z'||^{2p},$$

which clearly implies that $z = z'$. This completes the proof of the Theorem. □

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