ON DUALITY FOR NONCONVEX QUADRATIC OPTIMIZATION PROBLEMS

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Abstract. In this paper, we consider an optimization problem which consists a nonconvex quadratic objective function and two nonconvex quadratic constraint functions. We formulate its dual problem with semi-definite constraints, and we establish weak and strong duality theorems which hold between these two problems. And we give an example to illustrate our duality results. It is worth while noticing that our weak and strong duality theorems hold without convexity assumptions.

1. Introduction and Preliminaries

We begin the notations and definitions that will be used throughout this paper. The real line is denoted by $\mathbb{R}$ and the n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The space of all $(n \times n)$ symmetric matrices is denoted by S^n. The notation $A \succeq B$ means that the matrix $A - B$ is positive semidefinite. Moreover, the notation $A \succ B$ means that the matrix $A - B$ is positive definite.

Consider the following quadratic optimization problems:

$$\text{(P)} \qquad \text{Minimize} \qquad f(x)$$
$$\text{subject to} \qquad g(x) \leq 0, \ h(x) \leq 0,$$

where $f,\ g,\ h : \mathbb{R}^n \to \mathbb{R} \ (n \geq 3)$ are defined by $f(x) = \frac{1}{2}x^T A_f x + b_f^T x + c_f$, $g(x) = \frac{1}{2}x^T A_g x + b_g^T x + c_g$ and $h(x) = \frac{1}{2}x^T A_h x + b_h^T x + c_h$, $A_f, A_g, A_h \in S^n$, $b_f, b_g, b_h \in \mathbb{R}^n$ and $c_f, c_g, c_h \in \mathbb{R}$. Denote $F_P = \{x \in \mathbb{R}^n \ : g(x) \leq 0, \ h(x) \leq 0\}$

The problem (P) is said to be regular whenever there exist $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\gamma_1 H_g + \gamma_2 H_h \succ 0,$$

where

$$H_g = \begin{pmatrix} A_g & b_g \\ b_g^T & 2c_g \end{pmatrix} \quad \text{and} \quad H_h = \begin{pmatrix} A_h & b_h \\ b_h^T & 2c_h \end{pmatrix}.$$

Recently many authors [1, 2, 4, 5, 6, 7, 8] have studied optimality conditions for quadratic optimization problems. In particular, Jeyakumar et al. [4] formulate modified Wolfe dual problems for scalar quadratic optimization problems using the semidefinite constraint and then proved strong duality theorems which have no duality gaps. In this paper, we consider an optimization problem which consists a quadratic objective function and two quadratic constraint functions. Following approaches in [4], we formulate its dual problem with semidefinite constraints. We establish weak and strong duality theorems which hold between these two problems. And we give an example to illustrate our duality results.

The following result is Jeyakumar at al. [3]: Theorem 1.1 give necessary optimality conditions for a regular problem (P).

Theorem 1.1. ([3]) *Suppose that problem (P) is regular and that $\bar{x}$ is a global minimizer. Then the following Fritz John type necessary condition holds; that is, there exists $(\mu, \lambda_1, \lambda_2) \in \mathbb{R}^3_+ \setminus \{(0,0,0)\}$ such that $\nabla(\mu f + \lambda_1 g + \lambda_2 h)(\bar{x}) = 0$, $\lambda_1 g(\bar{x}) = \lambda_2 h(\bar{x}) = 0$, and*

$$\mu A_f + \lambda_1 A_g + \lambda_2 A_h \succeq 0.$$

Moreover, if the Slater condition holds, i.e., there exists $x_0 \in \mathbb{R}^n$ such that $g(x_0) < 0$ and $h(x_0) < 0$, then a feasible point $\bar{x}$ is a global minimizer if and only if there exist $\lambda_1, \lambda_2 \geq 0$ such that

$$\nabla(f + \lambda_1 g + \lambda_2 h)(\bar{x}) = 0, \tag{1}$$
$$\lambda_1 g(\bar{x}) = \lambda_2 h(\bar{x}) = 0, \tag{2}$$
$$A_f + \lambda_1 A_g + \lambda_2 A_h \succeq 0. \tag{3}$$

The above Theorem 1.1 may not hold if (P) has more than three nonconvex quadratic functions as constraint function.

Now, we define a Karush-Kuhn-Tucker (shortly, KKT) point for (P) as follows:

Definition 1. A point $\bar{x} \in F_P$ is called a KKT point for (P) if there exists $\lambda_1, \lambda_2 \geq 0$ such that (1)–(3) holds.

2. Duality theorems

We define a Wolfe type dual problem for (P) as follows:

(D)   Maximize $f(u) + \lambda_1 g(u) + \lambda_2 h(u)$

subject to $A_f u + b_f + \lambda_1(A_g u + b_g) + \lambda_2(A_h u + b_h) = 0,$

$A_f + \lambda_1 A_g + \lambda_2 A_h \succeq 0,$

$\lambda_1 \geq 0, \ \lambda_2 \geq 0.$

The following theorem states that weak duality theorem holds between (P) and (D).

Theorem 2.1. *(Weak Duality) Let x be feasible for (P) and let $(u, \lambda_1, \lambda_2)$ be feasible for (D). Then the following inequality holds:*

$$f(x) \geq f(u) + \lambda_1 g(u) + \lambda_2 h(u).$$

Proof. Let x be feasible for (P) and let $(u, \lambda_1, \lambda_2)$ be feasible for (D). Note that $f(\cdot) + \lambda_1 g(\cdot) + \lambda_2 h(\cdot)$ is convex (since $A_f + \lambda_1 A_g + \lambda_2 A_h \succeq 0$). It follows that

$$f(x) - \Big(f(u) + \lambda_1 g(u) + \lambda_2 h(u) \Big)$$
$$\geq f(x) + \lambda_1 g(x) + \lambda_2 h(x) - \Big(f(u) + \lambda_1 g(u) + \lambda_2 h(u) \Big)$$
$$\geq \nabla \Big(f(u) + \lambda_1 g(u) + \lambda_2 h(u) \Big)(x - u)$$
$$= \{A_f u + b_f + \lambda_1 (A_g u + b_g) + \lambda_2 (A_h u + b_h)\}(x - u)$$
$$= 0.$$

$\square$

Now we give a strong duality theorem which holds between (P) and (D).

Theorem 2.2. *(Strong Duality) Suppose that problem (P) is regular. Let $\bar{x}$ be an optimal solution of (P) and the Slater condition holds. Then there exists $\bar{\lambda}_1 \geq 0$, $\bar{\lambda}_2 \geq 0$ such that $(\bar{x}, \bar{\lambda}_1, \bar{\lambda}_2)$ is a feasible for (D). Moreover, weak duality holds, $(\bar{x}, \bar{\lambda}_1, \bar{\lambda}_2)$ is an optimal solution of (D).*

Proof. Let $\bar{x}$ be an optimal solution of (P). Then, it follows from Theorem 1.1 that there exists $\bar{\lambda}_1 \geq 0$, $\bar{\lambda}_2 \geq 0$ such that

$$A_f \bar{x} + b_f + \bar{\lambda}_1 (A_g \bar{x} + b_g) + \bar{\lambda}_2 (A_h \bar{x} + b_h) = 0,$$
$$A_f + \bar{\lambda}_1 A_g + \bar{\lambda}_2 A_h \succeq 0,$$
$$\bar{\lambda}_1 \left(\frac{1}{2} \bar{x}^T A_g \bar{x} + b_g^T \bar{x} + c_g \right) = 0,$$
$$\bar{\lambda}_2 \left(\frac{1}{2} \bar{x}^T A_h \bar{x} + b_h^T \bar{x} + c_h \right) = 0.$$

Thus, $(\bar{x}, \bar{\lambda}_1, \bar{\lambda}_2)$ is a feasible for (D). By weak duality, for any feasible $(u, \lambda_1, \lambda_2)$ for (D),

$$\frac{1}{2} \bar{x}^T A_f \bar{x} + b_f^T \bar{x} + c_f \geq \frac{1}{2} u^T A_f u + b_f^T u + c_f + \lambda_1 \left(\frac{1}{2} u^T A_g u + b_g^T u + c_g \right)$$
$$+ \lambda_2 \left(\frac{1}{2} u^T A_h u + b_h^T u + c_h \right).$$

Since $\frac{1}{2} \bar{x}^T A_f \bar{x} + b_f^T \bar{x} + c_f = \frac{1}{2} \bar{x}^T A_f \bar{x} + b_f^T \bar{x} + c_f + \bar{\lambda}_1 (\frac{1}{2} \bar{x}^T A_g \bar{x} + b_g^T \bar{x} + c_g) + \bar{\lambda}_2 (\frac{1}{2} \bar{x}^T A_h \bar{x} + b_h^T \bar{x} + c_h)$, $(\bar{x}, \bar{\lambda}_1, \bar{\lambda}_2)$ is an optimal solution for (D). $\square$

It is worth while noticing that our weak and strong duality theorems hold without convexity assumptions.

Following remark of Jeyakumar et al. [1] for the trust-region problem, we make the following example which illustrates Theorems 2.1 and 2.2.

Example 2.1. Consider the following quadratic optimization problem:

$$(P) \qquad \text{Minimize} \quad x_1^2 - x_2^2$$
$$\text{subject to} \quad -x_1^2 - x_2^2 - x_3^2 + 1 \leq 0,$$
$$x_1^2 + x_2^2 + x_3^2 - 3 \leq 0,$$
$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0.$$

Let $f(x_1, x_2, x_3) = x_1^2 - x_2^2$, $g(x_1, x_2, x_3) = -x_1^2 - x_2^2 + x_3^2 + 1$ and $h(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 3$. The feasible set $F_p = \{(x_1, x_2, x_3) : 1 \leq x_1^2 + x_2^2 + x_3^2 \leq 3, \ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0\}$. Then $f(x) = \frac{1}{2} x^T A_f x + b_f^T x + c_f$, $g(x) = \frac{1}{2} x^T A_g x + b_g^T x + c_g$ and $h(x) = \frac{1}{2} x^T A_h x + b_h^T x + c_h$ where $x = (x_1, x_2) \in \mathbb{R}^2$, $A_f = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$A_g = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, $A_h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $b_f = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $b_g = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $b_h = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $c_f = 0$, $c_g = 1$, $c_h = -3$. Let $\gamma_1 \in \mathbb{R}$ and $\gamma_2 \in \mathbb{R}$ be such that $\gamma_1 < 0$, $\gamma_1 < \gamma_2 < \frac{\gamma_1}{3}$. Then

$$\gamma_1 H_g + \gamma_2 H_h = \begin{pmatrix} -2\gamma_1 + 2\gamma_2 & 0 & 0 & 0 \\ 0 & -2\gamma_1 + 2\gamma_2 & 0 & 0 \\ 0 & 0 & -2\gamma_1 + 2\gamma_2 & 0 \\ 0 & 0 & 0 & 2\gamma_1 - 6\gamma_2 \end{pmatrix} \succ 0.$$

This gives us that the problem (P) is regular. It is clear that the Slater condition holds for (P).

Let $\bar{x} = (0, \sqrt{3}, 0)$. Then $\bar{x}$ is an optimal solution of (P). Thus, from Theorem 1.1, there exist $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ such that $\nabla(f + \lambda_1 g + \lambda_2 h)(\bar{x}) = 0$, $\lambda_1 g(\bar{x}) = \lambda_2 h(\bar{x}) = 0$, and

$$A_f + \lambda_1 A_g + \lambda_2 A_h \succeq 0.$$

Let $C := \{(\bar{x}, \lambda_1, \lambda_2) \mid \bar{x}$ is a KKT point for (P) with $(\lambda_1, \lambda_2)\}$. Then $C = \{(\bar{x}, 0, 1)\}$. The dual problem can be formulated as follows:

$$(D) \qquad \text{Maximize} \quad u_1^2 - u_2^2 + \lambda_1(-u_1^2 - u_2^2 - u_3^2 + 1) + \lambda_2(u_1^2 + u_2^2 + u_3^2 - 3)$$
$$\text{subject to} \quad 2(1 - \lambda_1 + \lambda_2)u_1 = 0,$$
$$-2(1 + \lambda_1 - \lambda_2)u_2 = 0,$$
$$-2(\lambda_1 - \lambda_2)u_3 = 0$$
$$\lambda_1 \geq 0, \ \lambda_2 \geq 1 + \lambda_1.$$

Let F_D be the feasible set of (D). Then

$$
\begin{aligned}
F_D \quad = \quad & \{(0,0,0,\lambda_1,\lambda_2) \in \mathbb{R}^5 \mid \lambda_1 \geq 0,\ \lambda_2 > 1 + \lambda_1\} \\
& \bigcup \{(0,u_2,0,\lambda_1,\lambda_2) \in \mathbb{R}^5 \mid u_2 \in \mathbb{R}, \lambda_1 \geq 0,\ \lambda_2 = 1 + \lambda_1\}.
\end{aligned}
$$

It is clear that $C \subset F_D$ and we can check that the weak duality holds between (P) and (D). By the weak duality $(\bar{x},0,1)$ is an optimal solution of (D). So, the strong duality holds for (P) and (D).

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