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AN ITERATIVE METHOD FOR SOLVING EQUILIBRIUM PROBLEM FIXED POINT PROBLEM AND GENERALIZED VARIATIONAL INEQUALITIES PROBLEM

Lijuan Zhang and Juchun Li

Abstract. In this paper, we introduce a new iterative scheme for finding a common element of the set of an equilibrium problem, the set of fixed points of nonexpansive mapping and the set of solutions of the generalized variational inequality for α-inverse strongly g-monotone mapping in a Hilbert space. Under suitable conditions, strong convergence theorems for approximating a common element of the above three sets are obtained.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H. Recall that a self-mapping $f : C \to C$ is Lipschitz continuous on C if there is a constant $k > 0$ such that $\|f(x) - f(y)\| \leq k\|x - y\|, x, y \in C$; if $k = 1$, the mapping is called nonexpansive. We denoted by $F(f)$ the set of fixed pints of f.

Let $A : C \to H$, $g : C \to C$ two nonlinear operators. We consider a generalized variational inequality (GVI) problem as follows: to find $u \in C$, $g(u) \in C$ such that

$$\langle g(v) - g(u), Au\rangle \geq 0. \tag{1.1}$$

The set of solutions of above variational inequality (which is called Noor [1] variational inequality) problem is denoted by $GVI(C, A, g)$. Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in H$ satisfying $0 = Au$ and so on. An operator A of C into H is said α-inverse-strongly g-monotone if there exists a positive real number α such that

$$\langle g(u) - g(v), Au - Av\rangle \geq \alpha\|Au - Av\|^2$$

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for all $u, v \in C$. It is obvious that A is continuous when g is continuous. The operator $g : C \to C$ is said to be strongly monotone if and only if there exists $\gamma > 0$ such that

$$\langle g(u) - g(v), u - v\rangle \geq \gamma\|u - v\|^2$$

for all $u, v \in C$.

By the strong monotonicity of g with constant $\gamma > 0$, we have

$$\|g(u) - g(v)\|\|u - v\| \geq \langle g(u) - g(v), u - v\rangle \geq \gamma\|u - v\|^2.$$

That is,

$$\|g(u) - g(v)\| \geq \gamma\|u - v\|, \tag{1.2}$$

thus, the above formula implies that g^{-1} is single-valued operator and

$$\|g^{-1}(u) - g^{-1}(v)\| \leq \frac{1}{\gamma}\|u - v\|.$$

When $R(g) = C$, g is bijective on C, and g^{-1} is Lipschitzian continuous. For $g = I$, where I is the identity operator, problem (1.1) is equivalent to finding $u \in C$, such that

$$\langle v - u, Au\rangle \geq 0, \forall v \in C,$$

which is known as the classical variational inequality introduced and studied by Stampacchia [2] in 1964, the set of solutions of above variational inequality problem is denoted by $VI(C, A)$.

Let F be a bifunction of $C \times C$ into R, where R is the set of real numbers. The equilibrium problems for $F : C \times C \to R$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solution of (1.1) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $\hat{x} \in EP(F)$ if and only if $\langle T\hat{x}, y - \hat{x}\rangle \geq 0$ for all $y \in C$, i.e., $\hat{x}$ is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.3). Some methods have been proposed to solve the equilibrium problems.

Very recently, L.Zhang et al [3] introduce an iterative scheme by the general iterative method: arbitrary initial $x_0 \in C$,

$$g(x_{n+1}) = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(g(x_n) - \lambda_n Ax_n).$$

We will prove that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle g(q) - f(q), g(q) - g(p)\rangle \leq 0, p \in g^{-1}(F(S)) \cap GVI(C, A, g).$$

Y. Su et al [4] introduce an iterative scheme given as follows: $x_0 \in H$ and

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(u_n - \lambda_n Au_n). \end{cases} \tag{1.3}$$

for all $n \in N$, where $\{\alpha_n\} \in [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy some appropriate condition. Furthermore, they proved that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap VI(C,A) \cap EP(F)$.

In this paper, motivated and inspired by the above results, we introduce an iterative scheme given as follows: $x_1 \in H$ and

$$\begin{cases} F(g(u_n), y) + \dfrac{1}{r_n}\langle y - g(u_n), g(u_n) - g(x_n)\rangle \geq 0, \quad \forall y \in C, \\ g(x_{n+1}) = \alpha_n f(x_n) + (1-\alpha_n) SP_C(g(u_n) - \lambda_n A u_n). \end{cases} \tag{1.4}$$

for all $n \in N$, where $\{\alpha_n\} \in [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy some appropriate condition. Furthermore, they proved that $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in g^{-1}(F(S)) \cap g^{-1}(EP(F)) \cap GVI(C,A,g)$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the metric projection of H to C. It is well known that P_C satisfies :

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.1}$$

for every $x, y \in H$, and P_C is characterized by the following properties:

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \tag{2.2}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.3}$$

for all $x \in H, y \in C$. In the context of the variational inequality problem, this implies

$$u \in VI(C,A,g) \Leftrightarrow g(u) = P_C(g(u) - \lambda A u), \forall \lambda > 0. \tag{2.4}$$

We have the following Lemma.

Lemma 2.1. ([3]) *Let C be a closed convex subset of a real Hilbert space H, for all $x, y \in C$ and $\lambda > 0$, if $\lambda \leq 2\alpha$. Then the following inequality holds:*

$$\|P_C(g(x) - \lambda A x) - P_C(g(y) - \lambda A y)\| \leq \|g(x) - g(y)\|.$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for every $(y,g) \in G(T)$ implies $f \in Tx$.

Let $g : C \to C$ and $\varphi : C \to R \cup \{+\infty\}$. A vector $w \in H$ is called a g-subgradient of φ at $x \in dom\varphi$. If $\langle g(y) - g(x), w\rangle \leq \varphi(y) - \varphi(x)$, for all $y \in C$. Each φ can be associated with the following g-subdifferential mapping $\partial_g \varphi$ defined by

$$\partial_g\varphi(x) = \begin{cases} w \in H : \langle g(y) - g(x), w\rangle \leq \varphi(y) - \varphi(x), \forall\ y \in C\},\ x \in C, \\ \emptyset,\ x \notin C. \end{cases}$$

Let $N_C(v) = \{w \in H : \langle g(v) - g(u), w\rangle \geq 0, \forall u \in C\}$. When $g = I$, $N_C(v)$ be the normal cone to C at $v \in C$.

Lemma 2.2. ([3]) *Let C be a closed convex subset of a real Hilbert space H, and A an α-inverse-strongly g-monotone operator of C into H. Assume that $g : C \to C$ is bijective mapping with $R(g) = C$. Let T be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C v,\ v \in C, \\ \emptyset,\ v \notin C. \end{cases}$$

Then T is maximal g-monotone and $T^{-1}0 = GVI(C, A, g)$.

For solving the equilibrium problem for a bifunction $F : C \times C \to R$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.3. ([5]) *Let C be a nonempty closed subset of H and F be a bifunction of $C \times C$ into R satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.4. ([6]) *Assume that $F : C \times C \to R$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall\ y \in C\}$$

for all $x \in H$. Then the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(3) *$F(T_r) = EP(f)$;*

(4) *$EP(f)$ is closed and convex.*

Lemma 2.5. ([7]) *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z = Tz$.*

Lemma 2.6. ([8]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers such that:*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \beta_n, n \geq 0,$$

where $\{\lambda_n\}, \{\beta_n\}$ satisfy the conditions:
(i) $\{\lambda_n\} \subset (0, 1)$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\lambda_n} \leq 0$ or $\sum_{n=1}^{\infty} |\beta_n| < \infty$. Then $\lim_{n\to\infty} s_n = 0$.

3. Main result

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A4), Let $f : C \to C$ is Lipschitz continuous with coefficient k $(0 < k)$, and $g : C \to C$ is strongly monotone and weakly continuous with coefficient γ $(k < \gamma)$ and $R(g) = C$, and A an α-inverse-strongly g-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $\Omega = g^{-1}(F(S)) \cap g^{-1}(EP(F)) \cap GVI(C, A, g) \neq \emptyset$. Suppose $\{x_n\}$ be sequences generated by (1.4) for every $n = 0, 1, 2, ...,$ where $\{\lambda_n\} \subset [a, b]$ and $\{\alpha_n\}$ is sequence in $(0, 1)$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$, $\liminf_{n\to\infty} r_n > 0$ $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty, \sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty$. Then $\{x_n\}$ defined in (1.4) converges strongly to $q \in \Omega$, which is the unique solution in the Ω to the following variational inequality*

$$\langle f(q) - g(q), g(p) - g(q)\rangle \leq 0, \quad \forall\ p \in \Omega.$$

Proof. Put $y_n = P_C(g(u_n) - \lambda_n Au_n)$ for every $n = 0, 1, 2, ...$ Let $u \in \Omega$. By (2.4) and Lemma 2.1, we have

$$\begin{aligned} \|y_n - g(u)\| &= \|P_C(g(u_n) - \lambda_n Au_n) - P_C(g(u) - \lambda_n Au)\| \\ &\leq \|(g(u_n) - \lambda_n Au_n) - (g(u) - \lambda_n Au)\| \\ &\leq \|g(u_n) - g(u)\|. \end{aligned}$$

From $g(u_n) = T_{r_n} g(x_n)$, we have

$$\|g(u_n - g(u)\| = \|T_{r_n} g(x_n) - T_{r_n} g(u)\| \leq \|g(x_n) - g(u)\|$$

for every $n \geq 1$. Then we compute that

$$\begin{aligned} \|g(x_{n+1}) - g(u)\| &= \|\alpha_n f(x_n) + (1 - \alpha_n)Sy_n - g(u)\| \\ &\leq \alpha_n \|f(x_n) - g(u)\| + (1 - \alpha_n)\|Sy_n - g(u)\| \end{aligned}$$

$$\begin{aligned}
&\le \alpha_n\|f(x_n)-f(u)\|+\alpha_n\|f(u)-g(u)\|+(1-\alpha_n)\|y_n-g(u)\| \\
&\le \alpha_n k\|x_n-u\|+(1-\alpha_n)\|g(x_n)-g(u)\|+\alpha_n\|f(u)-g(u)\| \\
&\le \frac{\alpha_n k}{\gamma}\|g(x_n)-g(u)\|+(1-\alpha_n)\|g(x_n)-g(u)\|+\alpha_n\|f(u)-g(u)\| \\
&= (1-(1-\frac{k}{\gamma})\alpha_n)\|g(x_n)-g(u)\|+\alpha_n\|f(u)-g(u)\| \\
&\le \max\{\|g(x_n)-g(u)\|,\frac{1}{1-\frac{k}{\gamma}}\|f(u)-g(u)\|\}.
\end{aligned}$$

By induction,

$$\|g(x_n)-g(u)\| \le \max\{\|g(x_0)-g(u)\|,\frac{1}{1-\frac{k}{\gamma}}\|f(u)-g(u)\|\},\quad n\ge 0.$$

Therefore, $\{g(x_n)\}$ is bounded. By (1.2), we have $\{x_n\}$ is bounded. From the property of g, A, S, f, we have $\{y_n\}$,$\{Sy_n\}$, $\{Ax_n\}$, $\{f(x_n)\}$ are also bounded. By (2.4), we have

$$\begin{aligned}
\|y_{n+1}-y_n\| &\le \|g(u_{n+1})-\lambda_{n+1}Au_{n+1}-(g(u_n)-\lambda_n Au_n)\| \\
&\le \|g(u_{n+1})-\lambda_{n+1}Au_{n+1}-(g(u_n)-\lambda_{n+1}Au_n)\| \\
&\quad +|\lambda_n-\lambda_{n+1}|\|Au_n\| \\
&\le \|g(u_{n+1})-g(u_n)\|+|\lambda_n-\lambda_{n+1}|\|Au_n\|
\end{aligned}$$

for every $n=0,1,2,....$ So we obtain

$$\begin{aligned}
&\|g(x_{n+1})-g(x_n)\| \\
&= \|\alpha_n f(x_n)+(1-\alpha_n)Sy_n-\alpha_{n-1}f(x_{n-1})-(1-\alpha_{n-1})Sy_{n-1}\| \\
&= \|(\alpha_n-\alpha_{n-1})(f(x_{n-1})-Sy_{n-1})+(1-\alpha_n)(Sy_n-Sy_{n-1}) \\
&\quad +\alpha_n(f(x_n)-f(x_{n-1}))\| \\
&\le |\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-Sy_{n-1}\|+(1-\alpha_n)\|y_n-y_{n-1}\| \\
&\quad +\alpha_n k\|x_n-x_{n-1}\| \\
&\le (1-\alpha_n)(\|g(u_n)-g(u_{n-1})\|+|\lambda_{n-1}-\lambda_n|\|Au_{n-1}\|) \\
&\quad +|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-Sy_{n-1}\|+\alpha_n\frac{k}{\gamma}\|g(x_n)-g(x_{n-1})\| \\
&\le \frac{k}{\gamma}\alpha_n\|g(x_n)-g(x_{n-1})\|+(1-\alpha_n)\|g(u_n)-g(u_{n-1})\| \\
&\quad +(|\lambda_n-\lambda_{n-1}|+|\alpha_n-\alpha_{n-1}|)M
\end{aligned}$$

for every $n=0,1,2,...$, where

$$M=max\{sup\{\|Au_n\|:n\in N\},sup\{\|f(x_n)\|+\|S(y_n)\|:n\in N\}\}.$$

On the other hand, from $g(u_n)=T_{r_n}g(x_n)$ and $g(u_{n+1})=T_{r_n}g(x_{n+1})$, we have

$$F(g(u_n),y)+\frac{1}{r_n}\langle y-g(u_n),g(u_n)-g(x_n)\rangle\ge 0 \tag{3.1}$$

for all $y \in C$, and

$$F(g(u_{n+1}), y) + \frac{1}{r_{n+1}}\langle y - g(u_{n+1}), g(u_{n+1}) - g(x_{n+1})\rangle \geq 0 \qquad (3.2)$$

for all $y \in C$. Putting $y = g(u_{n+1})$ in (3.1) and $y = g(u_n)$ in (3.2), we have

$$F(g(u_n), g(u_{n+1})) + \frac{1}{r_n}\langle g(u_{n+1}) - g(u_n), g(u_n) - g(x_n)\rangle \geq 0$$

and

$$F(g(u_{n+1}), g(u_n)) + \frac{1}{r_{n+1}}\langle g(u_n) - g(u_{n+1}), g(u_{n+1}) - g(x_{n+1})\rangle \geq 0.$$

So, from (A2) we have

$$\langle g(u_{n+1}) - g(u_n), \frac{g(u_n) - g(x_n)}{r_n} - \frac{g(u_{n+1}) - g(x_{n+1})}{r_{n+1}}\rangle \geq 0$$

and hence

$$\langle g(u_{n+1}) - g(u_n), g(u_n) - g(u_{n+1}) + g(u_{n+1}) - g(x_n) - \frac{r_n}{r_{n+1}}(g(u_{n+1}) - g(x_{n+1}))\rangle \geq 0.$$

Without loss of generality, let us assume there exist a real number m such that $r_n \geq m > 0$ for all $n \in N$. Then, we have

$$\begin{aligned}&\|g(u_{n+1}) - g(u_n)\|^2\\ &\leq \langle g(u_{n+1}) - g(u_n), g(x_{n+1}) - g(x_n) + (1 - \frac{r_n}{r_{n+1}})(g(u_{n+1}) - g(x_{n+1}))\rangle.\end{aligned}$$

Observe that, we have

$$\begin{aligned}\|g(u_{n+1}) - g(u_n)\| &\leq \|g(x_{n+1}) - g(x_n)\| + (1 - \frac{r_n}{r_{n+1}})\|g(u_{n+1}) - g(x_{n+1})\|\\ &\leq \|g(x_{n+1}) - g(x_n)\| + \frac{1}{m}|r_n - r_{n+1}|L\end{aligned}$$

where $L = sup\{\|g(u_n) - g(x_n)\| : n \in N\}$. So we have

$$\begin{aligned}\|g(x_{n+1}) - g(x_n)\| \leq (1 - (1 - \frac{k}{\gamma})\alpha_n)\|g(x_n) - g(x_{n-1})\| + (|\lambda_n - \lambda_{n-1}|\\ + |\alpha_n - \alpha_{n-1}|)M + \frac{L}{m}|r_n - r_{n+1}|.\end{aligned}$$

Since $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ and $\sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty$, in view of Lemma 2.6, we have $\lim_{n\to\infty} \|g(x_{n+1}) - g(x_n)\| = 0$. Then we also obtain $\lim_{n\to\infty} \|g(u_{n+1}) - g(u_n)\| = 0$ and $\|y_{n+1} - y_n\| \to 0$. From the definition of $g(x_n)$, we have

$$\begin{aligned}\|g(x_n) - Sy_n\| &\leq \|g(x_n) - Sy_{n-1}\| + \|Sy_{n-1} - Sy_n\|\\ &\leq \alpha_{n-1}\|f(x_{n-1}) - Sy_{n-1}\| + \|y_{n-1} - y_n\|,\end{aligned}$$

we have $\|g(x_n) - Sy_n\| \to 0$. For $u \in \Omega$, we have

$$\begin{aligned}\|g(u_n) - g(u)\|^2 &= \|Tr_n g(x_n) - Tr_n g(u)\|^2\\ &\le \langle Tr_n g(x_n) - Tr_n g(u), g(x_n) - g(u)\rangle\\ &\le \langle g(u_n) - g(u), g(x_n) - g(u)\rangle\\ &= \frac{1}{2}(\|g(u_n) - g(u)\|^2 + \|g(x_n) - g(u)\|^2 - \|g(x_n) - g(u_n)\|^2)\end{aligned}$$

and hence

$$\|g(u_n) - g(u)\|^2 \le \|g(x_n) - g(u)\|^2 - \|g(x_n) - g(u_n)\|^2.$$

Therefore, from the convexity of $\|.\|^2$

$$\begin{aligned}&\|g(x_{n+1}) - g(u)\|^2\\ &= \|\alpha_n f(x_n) + (1-\alpha_n)Sy_n - g(u)\|^2\\ &\le \alpha_n\|f(x_n) - g(u)\|^2 + (1-\alpha_n)\|y_n - g(u)\|^2\\ &\le \alpha_n\|f(x_n) - g(u)\|^2 + (1-\alpha_n)\|g(u_n) - g(u)\|^2\\ &\le \alpha_n\|f(x_n) - g(u)\|^2 + (1-\alpha_n)(\|g(x_n) - g(u)\|^2 - \|g(x_n) - g(u_n)\|^2)\\ &\le \alpha_n\|f(x_n) - g(u)\|^2 + \|g(x_n) - g(u)\|^2 - (1-\alpha_n)\|g(x_n) - g(u_n)\|^2,\end{aligned}$$

and hence

$$\begin{aligned}&-(1-\alpha_n)\|g(x_n) - g(u_n)\|^2\\ &\le \alpha_n\|f(x_n) - g(u)\|^2 + (\|g(x_n) - g(u)\| + \|g(x_{n+1}) - g(u)\|)\|g(x_n) - g(x_{n+1})\|.\end{aligned}$$

So we have

$$\|g(x_n) - g(u_n)\| \to 0, \|x_n - u_n\| \to 0.$$

Next we show $\|y_n - g(u_n)\| \to 0$, for $u \in \Omega$, we compute that

$$\begin{aligned}&\|g(x_{n+1}) - g(u)\|^2\\ &= \|\alpha_n f(x_n) + (1-\alpha_n)Sy_n - g(u)\|^2\\ &\le \alpha_n\|f(x_n) - g(u)\|^2 + (1-\alpha_n)\|y_n - g(u)\|^2\\ &= \alpha_n\|f(x_n) - g(u)\|^2 + (1-\alpha_n)\|P_C(g(u_n) - \lambda_n Au_n) - P_C(g(u) - \lambda_n Au)\|^2\\ &\le \alpha_n\|f(x_n) - g(u)\|^2 + (1-\alpha_n)\|g(u_n) - \lambda_n Au_n - (g(u) - \lambda_n Au)\|^2\\ &\le \alpha_n\|f(x_n) - g(u)\|^2 + (1-\alpha_n)[\|g(u_n) - g(u)\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Au\|^2]\\ &\le \alpha_n\|f(x_n) - g(u)\|^2 + \|g(u_n) - g(u)\|^2 + (1-\alpha_n)a(b-2\alpha)\|Au_n - Au\|^2\\ &\le \alpha_n\|f(x_n) - g(u)\|^2 + \|g(x_n) - g(u)\|^2 + (1-\alpha_n)a(b-2\alpha)\|Au_n - Au\|^2.\end{aligned}$$

So, we obtain

$$\begin{aligned}&-(1-\alpha_n)a(b-2\alpha)\|Au_n - Au\|^2\\ &\le \alpha_n\|f(x_n) - g(u)\|^2 + (\|g(x_n) - g(u)\| + \|g(x_{n+1}) - g(u)\|)\|g(x_n) - g(x_{n+1})\|.\end{aligned}$$

Since $\alpha_n \to 0$ and $\|g(x_n) - g(x_{n+1})\| \to 0$, then $\|Au_n - Au\| \to 0, n \to \infty$. Further, from (2.1), we obtain

$$\begin{aligned}
\|y_n - g(u)\|^2 &= \|P_C(g(u_n) - \lambda_n Au_n) - P_C(g(u) - \lambda_n Au)\|^2 \\
&\leq \langle g(u_n) - \lambda_n Au_n - (g(u) - \lambda_n Au), y_n - g(u)\rangle \\
&= \frac{1}{2}\{\|(g(u_n) - \lambda_n Au_n) - (g(u) - \lambda_n Au)\|^2 + \|y_n - g(u)\|^2 \\
&\quad - \|g(u_n) - y_n - \lambda_n(Au_n - Au)\|^2\} \\
&\leq \frac{1}{2}\{\|g(u_n) - g(u)\|^2 + \|y_n - g(u)\|^2 - \|g(u_n) - y_n\|^2 \\
&\quad + 2\lambda_n\langle g(u_n) - y_n, Au_n - Au\rangle - \lambda_n^2\|Au_n - Au\|^2\}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|y_n - g(u)\|^2 \leq{}& \|g(u_n) - g(u)\|^2 - \|g(u_n) - y_n\|^2 + 2\lambda_n\langle g(u_n) - y_n, Au_n - Au\rangle \\
&- \lambda_n^2\|Au_n - Au\|^2.
\end{aligned}$$

And hence

$$\begin{aligned}
\|g(x_{n+1}) - g(u)\|^2 &\leq \alpha_n\|f(x_n) - g(u)\|^2 + (1 - \alpha_n)\|Sy_n - g(u)\|^2 \\
&\leq \alpha_n\|f(x_n) - g(u)\|^2 + (1 - \alpha_n)\|y_n - g(u)\|^2 \\
&\leq \alpha_n\|f(x_n) - g(u)\|^2 + \|g(u_n) - g(u)\|^2 - \|g(u_n) - y_n\|^2 \\
&\quad + 2\lambda_n\langle g(u_n) - y_n, Au_n - Au\rangle - \lambda_n^2\|Au_n - Au\|^2 \\
&\leq \alpha_n\|f(x_n) - g(u)\|^2 + \|g(x_n) - g(u)\|^2 - \|g(u_n) - y_n\|^2 \\
&\quad + 2\lambda_n\langle g(u_n) - y_n, Au_n - Au\rangle - \lambda_n^2\|Au_n - Au\|^2.
\end{aligned}$$

Since $\alpha_n \to 0$, $\|g(x_{n+1}) - g(x_n)\| \to 0$ and $\|Au_n - Au\| \to 0$, we obtain $\|g(u_n) - y_n\| \to 0$.

Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(q) - g(q), Sy_n - g(q)\rangle &= \lim_{i\to\infty}\langle f(q) - g(q), Sy_{n_i} - g(q)\rangle \\
&= \lim_{i\to\infty}\langle f(q) - g(q), g(x_{n_i}) - g(q)\rangle.
\end{aligned}$$

As $\{x_{n_i}\}$ is bounded, we have that a subsequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ converges weakly to z. Without loss of generality that $x_{n_i} \rightharpoonup z$, by the weakly continuity of g, we obtain $g(x_{n_i}) \rightharpoonup g(z)$. Then we can obtain $z \in \Omega$. In fact, let us first show that $z \in GVI(C, A, g)$. Let

$$Tv = \begin{cases} Av + N_C v, \ v \in C, \\ \emptyset, \ v \notin C. \end{cases}$$

Then T is maximal g-monotone. Let $(v, w) \in G(T)$. Since $w - Av \in N_C(v)$ and $u_n \in C$, we have

$$\langle g(v) - g(u_n), w - Av\rangle \geq 0.$$

On the other hand, from $y_n = P_C(g(u_n) - \lambda_n Au_n)$, we have $\langle g(v) - y_n, y_n - (g(u_n) - \lambda_n Au_n)\rangle \geq 0$ and hence

$$\langle g(v) - y_n, \frac{y_n - g(u_n)}{\lambda_n} + Au_n\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
&\langle g(v) - g(u_{n_i}), w\rangle \geq \langle g(v) - g(u_{n_i}), Av\rangle \\
&\geq \langle g(v) - g(u_{n_i}), Av\rangle - \langle g(v) - y_{n_i}, \frac{y_{n_i} - g(u_{n_i})}{\lambda_{n_i}} + Au_{n_i}\rangle \\
&= \langle g(v) - g(u_{n_i}), Av\rangle - \langle g(v) - y_{n_i}, \frac{y_{n_i} - g(u_{n_i})}{\lambda_{n_i}}\rangle - \langle g(v) - y_{n_i}, Au_{n_i}\rangle \\
&= \langle g(v) - g(u_{n_i}), Av - Au_{n_i}\rangle - \langle g(v) - y_{n_i}, \frac{y_{n_i} - g(u_{n_i})}{\lambda_{n_i}}\rangle - \langle g(x_{n_i}) - y_{n_i}, Au_{n_i}\rangle \\
&\geq -\langle g(v) - y_{n_i}, \frac{y_{n_i} - g(u_{n_i})}{\lambda_{n_i}}\rangle - \langle g(x_{n_i}) - y_{n_i}, Au_{n_i}\rangle.
\end{aligned}$$

From $\|g(u_n) - y_n\| \to 0$ and the weakly continuity of g. Hence we have $\langle g(v) - g(z), w\rangle \geq 0$ as $i \to \infty$. Since T is maximal g-monotone, by Lemma 2.2, we have $z \in T^{-1}0$ and hence $z \in GVI(C, A, g)$.

$$\begin{aligned}
\|g(x_n) - Sg(x_n)\| &\leq \|g(x_n) - Sy_n\| + \|Sy_n - Sg(x_n)\| \\
&\leq \|g(x_n) - Sy_n\| + \|g(u_n) - y_n\| + \|g(u_n) - g(x_n)\|,
\end{aligned}$$

We have $\|g(x_n) - Sg(x_n)\| \to 0$, in view of Lemma 2.5, we obtain $g(z) \in F(S)$. So we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(q) - g(q), Sy_n - g(q)\rangle &= \lim_{i\to\infty}\langle f(q) - g(q), Sy_{n_i} - g(q)\rangle \\
&= \langle f(q) - g(q), g(z) - g(q)\rangle \leq 0.
\end{aligned}$$

Since $g(u_n) = T_{r_n} g(x_n)$, we derive

$$F(g(u_n), y) + \frac{1}{r_n}\langle y - g(u_n), g(u_n) - g(x_n)\rangle \geq 0, \quad \forall\, y \in C.$$

From (A2), we have

$$\frac{1}{r_n}\langle y - g(u_n), g(u_n) - g(x_n)\rangle \geq F(y, g(u_n))$$

and hence

$$\langle y - g(u_{n_i}), \frac{g(u_{n_i}) - g(x_{n_i})}{r_{n_i}}\rangle \geq F(y, g(u_{n_i}))$$

Since $\|g(u_n) - g(x_n)\| \to 0$ and $\{u_{n_i}\} \rightharpoonup z$, from the weak lower semicontinuity of F and $F(x, y)$ in the second variable y, we have $F(y, g(z)) \leq 0$, for all $y \in C$. For $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $F(y_t, g(z)) \leq 0$. From the convexity of equilibrium bifunction $F(x, y)$ in the second variable y, we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, g(z)) \leq tF(y_t, y),$$

and hence $F(y_t, g(z)) \geq 0$. Then, From (A3), we have $\Theta(g(z), y) \geq 0$ for all $y \in C$ and hence $z \in g^{-1}(EP(F))$ and $z \in \Omega$. We compute that

$$\begin{aligned}
&\|g(x_{n+1}) - g(q)\|^2 \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)Sy_n - g(q)\|^2 \\
&= \alpha_n^2\|f(x_n) - g(q)\|^2 + 2\alpha_n(1-\alpha_n)\langle f(x_n) - g(q), Sy_n - g(q)\rangle \\
&\quad + (1-\alpha_n)^2\|Sy_n - g(q)\|^2 \\
&\leq (1 - 2\alpha_n + \alpha_n^2)\|g(x_n) - g(q)\|^2 + \alpha_n^2\|f(x_n) - g(q)\|^2 + 2\alpha_n(1-\alpha_n) \\
&\quad \langle f(x_n) - f(q), Sy_n - g(q)\rangle + 2\alpha_n(1-\alpha_n)\langle f(q) - g(q), Sy_n - g(q)\rangle \\
&\leq [1 - 2\alpha_n + \alpha_n^2 + 2\frac{k\alpha_n(1-\alpha_n)}{\gamma}]\|g(x_n) - g(q)\|^2 + \alpha_n^2\|f(x_n) - g(q)\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(q) - g(q), Sy_n - g(q)\rangle \\
&= (1 - \bar{\alpha_n})\|g(x_n) - g(q)\|^2 + \bar{\alpha_n}\bar{\beta_n},
\end{aligned}$$

where

$$\begin{aligned}
\bar{\alpha_n} &= \alpha_n[2 - \alpha_n - 2\frac{k(1-\alpha_n)}{\gamma}], \\
\bar{\beta_n} &= \frac{\alpha_n\|f(x_n) - g(q)\|^2 + 2(1-\alpha_n)\langle f(q) - g(q), Sy_n - g(q)\rangle}{2 - \alpha_n - 2\frac{k(1-\alpha_n)}{\gamma}}.
\end{aligned}$$

It is easily seen that $\bar{\alpha_n} \to 0, \sum_{n=1}^{\infty}\bar{\alpha_n} = \infty$, and $\limsup_{n\to\infty}\bar{\beta_n} \leq 0$, by Lemma 2.6, we obtain $g(x_n) \to g(q)$. Therefore $\|x_n - q\| \leq \frac{1}{\gamma}\|g(x_{n+1}) - g(q)\| \to 0$, we obtain $x_n \to q$. This completes the proof.

□

Remark 1. Putting $g = I$, by Theorem 3.1, we can obtain the theorem 3.1 in [4].

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Lijuan Zhang
College of Mathematics and Computer, Hebei University, Baoding, 071002, PR China
E-mail address: zhanglj@hbu.edu.cn

Juchun Li
Baoding Xiangyang Aviation Precision Machinery Co., Ltd, Baoding, 071064, PR China
E-mail address: bd542@sina.com