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EXTENDING THE APPLICATION OF THE SHADOWING LEMMA FOR OPERATORS WITH CHAOTIC BEHAVIOUR

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Abstract. We use a weaker version of the celebrated Newton–Kantorovich theorem [3] reported by us in [1] to find solutions of discrete dynamical systems involving operators with chaotic behavior. Our results are obtained by extending the application of the shadowing lemma [4], and are given under the same computational cost as before [4]–[6].

1. Introduction

It is well known that complicated behaviour of dynamical systems can easily be detected via numerical experiments. However, it is very difficult to prove mathematically in general that a given system behaves chaotically.

Several authors have worked on various aspects of this problem, see, e.g., [4]–[6], and the references therein. In particular the shadowing lemma [4, p. 1684] proved via the celebrated Newton–Kantorovich theorem [3] was used in [4] to present a computer-assisted method that allows us to prove that a discrete dynamical system admits the shift operator as a subsystem. Motivated by this work and using a weaker version of the Newton–Kantorovich theorem reported by us in [1], [2] (see Theorem 2.1 that follows) we show that it is possible to weaken the shadowing Lemma on on which the work in [4] is based. In particular we show that under weaker hypotheses and the same computational cost a larger upper bound on the crucial norm of operator M^{-1} (see (7)) is found and the information on location of the shadowing orbit is more precise. Other advantages have already been reported in [1]. Clearly this apporach widens the applicability of the shadowing lemma.

2. The shadowing lemma

We need the definitions: Let $D \subseteq \mathbf{R}^k$ be an open subset of $\mathbf{R}^k$ (k a natural number), and let $f : D \to D$ be an injective operator. Then the pair (D, f) is a discrete dynamical system. Denote by $S = l^\infty\left(\mathbf{Z}, \mathbf{R}^k\right)$ the space of $\mathbf{R}^k$

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valued bounded sequences $x = \{x_n\}$ with norm $\|x\| = \sup_{n\in\mathbf{Z}} |x_n|_2$. Here we use the Euclidean norm in $\mathbf{R}^k$ and denote it by $|\cdot|$, omitting the index 2. A δ_0–pseudo–orbit is a sequence $y = \{y_n\} \in D^{\mathbf{Z}}$ with $|y_{n+1} - f(y_n)| \leq \delta_0$ $(n \in \mathbf{Z})$. A r-shadowing orbit $x = \{x_n\}$ of a δ_0–pseudo–orbit y is an orbit of (D, f) with $|y_n - x_n| \leq 2$ $(n \in \mathbf{Z})$.

We need the following semilocal convergence theorem for Newton method [1, page 132, Case 3 for $\delta = \delta_0$].

Theorem 2.1. *Let $F : D \subseteq X \to Y$ be a Fréchet differentiable operator. Assume there exist $x_0 \in D$, positive constant η, β, L_0 and L such that:*

$$F'(x_0)^{-1} \in L(Y, X),$$

$$\left\|F'(x_0)^{-1}\right\| \leq \beta, \tag{1}$$

$$\left\|F'(x_0)^{-1} F(x_0)\right\| \leq \eta, \tag{2}$$

$$\|F'(x) - F'(y)\| \leq L\|x - y\|, \text{ for all } x, y \in D, \tag{3}$$

$$\|F'(x) - F'(x_0)\| \leq L_0\|x - x_0\|, \text{ for all } x \in D, \tag{4}$$

$$h_A = \beta L_1 \eta \leq 1, \tag{5}$$

and

$$\bar{U}(x_0, s^*) = \{x \in X : \|x - x_0\| \leq s^*\} \subseteq D,$$

where

$$s^* = \lim_{n\to\infty} s_n,$$

$$s_0 = 0, s_1 = \eta, s_{n+2} = s_{n+1} + \frac{L(s_{n+1} - s_n)}{2(1 - L_0 s_{n+1})} \quad (n \geq 0),$$

$$L_1 = \frac{1}{4}(L + 4L_0 + \sqrt{L^2 + 8L_0 L}).$$

Then, sequence $\{y_n\}$ $(n \geq 0)$ generated by Newton's method

$$y_{n+1} = y_n - F'(y_n)^{-1} F(y_n) \quad (n \geq 0)$$

is well defined, remains in $\bar{U}(x_0, s^*)$ for all $n \geq 0$ and converges to a unique solution $y^* \in \bar{U}(x_0, s^*)$, so that estimates

$$\|y_{n+1} - y_n\| \leq s_{n+1} - s_n$$

and

$$\|y_n - y^*\| \leq s^* - s_n \leq 2\eta - s_n$$

hold for all $n \geq 0$.

Moreover y^* is the unique solution of equation $F(y) = 0$ in $U(x_0, R)$ provided that

$$L_0(s^* + R) \leq 2$$

and

$$U(x_0, R) \subseteq D.$$

The advantages of Theorem 2.1 over the Newton-Kantorovich theorem [3] have been explained in detail in [1], [2].

From now on we set $X = Y = \mathbf{R}^k$.

Sufficient conditions for a δ_0-pseudo-orbit y to admit a unique r-shadowing orbit are given in the following main result.

Theorem 2.2. *(Weak version of the shadowing lemma) Let $D \subseteq \mathbf{R}^k$ be open, $f \in C^{1,Lip}(D, D)$ be injective, $y = \{y_n\} \in D^{\mathbf{Z}}$ be a given sequence, $\{A_n\}$ be a bounded sequence of $k \times k$ matrices and let $\delta_0, \delta, \ell_0, \ell$ be positive constants. Assume that for the operator*

$$M : S \to S \text{ with } \{M z\}_n = z_{n+1} - Az_n \tag{6}$$

is invertible and

$$\left\|M^{-1}\right\| \leq a = \frac{1}{\delta + \sqrt{\ell_1 \delta_0}}., \tag{7}$$

where

$$\ell_1 = \frac{1}{4} (\ell + 4 \ell_0 + \sqrt{\ell^2 + 8 \ell_0 \ell}).$$

Then, the numbers t^*, R given by

$$t^* = \lim_{n \to \infty} t_n \tag{8}$$

and

$$R = \frac{2}{\ell_0} - t^* \tag{9}$$

satisfy $0 < t^* \leq R$, where sequence $\{t_n\}$ is given by

$$t_0 = 0, t_1 = \eta, t_{n+2} = t_{n+1} + \frac{\ell (t_{n+1} - t_n)^2}{2(1 - \ell_0 t_{n+1})} \quad (n \geq 0) \tag{10}$$

and

$$\eta = \frac{\delta_0}{\frac{1}{\|M^{-1}\|} - \delta}. \tag{11}$$

Let $r \in [t^*, R]$. Moreover, assume that

$$\overline{\bigcup_{n \in \mathbf{Z}} U(y_n, r)} \subseteq D \tag{12}$$

and for every $n \in \mathbf{Z}$

$$|y_{n+1} - f(y_n)| \leq \delta_0, \tag{13}$$

$$|A_n - Df(y_n)| \leq \delta, \tag{14}$$

$$|F'(u) - F'(0)| \leq \ell_0 |u| \tag{15}$$

and

$$|F'(u) - F'(v)| \leq \ell |u - v|, \tag{16}$$

for all $u, v \in U(y_n, r)$.

Then there is a unique t^*-shadowing orbit $x^* = \{x_n\}$ of y. Moreover, there is no orbit $\bar{x}$ other than x^* such that

$$\|\bar{x} - y\| \le r. \tag{17}$$

Proof. We shall solve the difference equation

$$x_{n+1} = f(x_n) \quad (n \ge 0) \tag{18}$$

provided that x_n is close to y_n. Setting

$$x_n = y_n + z_n \tag{19}$$

and

$$g_n(z_n) = f(z_n + y_n) - A_n z_n - y_{n+1} \tag{20}$$

we can have

$$z_{n+1} = A_n z_n + g_n(z_n). \tag{21}$$

Define $D_0 = \{z = \{z_n\} : \|z\| \le 2\}$ and nonlinear operator $G : D_0 \to S$, by

$$(G(z))_n = g_n(z_n). \tag{22}$$

Operator G can naturally be extended to a neighborhood of D_0. Equation (21) can be rewritten as

$$F(x) = M\ x - G(x) = 0, \tag{23}$$

where F is an operator from D_0 into S.

We will show the existence and uniqueness of a solution $x^* = \{x_n\}$ $(n \ge 0)$ of equation (23) with $\|x^*\| \le r$ using Theorem 2.1. Clearly we need to express η, L_0, L and β in terms of $\left\|M^{-1}\right\|, \delta_0, \delta, \ell_0$ and ℓ.

(i) $\left\|F'(0)^{-1} F(0)\right\| \le \eta$.

Using (13), (14) and (20) we get $\|F(0)\| \le \delta_0$ and $\|G'(0)\| \le \delta$, since $[G'(0)(w)]_n = (F'(y_n) - A_n) w_n$.

By (7) and the Banach lemma on invertible operators [3] we get $F'(0)^{-1}$ exists and

$$\left\|F'(0)^{-1}\right\| \le \left(\frac{1}{\|M^{-1}\|} - \delta\right)^{-1}. \tag{24}$$

That is, η can be given by (11).

(ii) $\left\|F'(0)^{-1}\right\| \le \beta$.

By (24) we can set

$$\beta = \left(\frac{1}{\|M^{-1}\|} - \delta\right)^{-1}. \tag{25}$$

(iii) $\|F'(u) - F'(v)\| \le L\|u - v\|$.

We can have using (16)

$$\begin{aligned} \left|(F'(u) - F'(v))(w)_n\right| &= \left|(F'(y_n + u_n) - F'(y_n + v_n)) w_n\right| \\ &\le \ell\,|u_n - v_n|\,|w_n|. \end{aligned} \tag{26}$$

Hence we can set $L = \ell$.

(iv) $\|F'(u) - F'(0)\| \leq L_0 \|u\|$.

By (17) we get

$$\begin{aligned} |(F'(u) - F'(0))(w)_n| &= |(F'(y_n + u_n) - F'(y_n + 0)) w_n| \\ &\leq \ell_0 |u_n| |w_n|. \end{aligned} \tag{27}$$

That is, we can take $L_0 = \ell_0$.

Crucial condition (5) is satisfied by (7) and with the above choices of η, β, L and L_0.

Therefore the claims of Theorem 2.2 follow immediately from the conclusions of Theorem 2.1.

That completes the proof of the theorem. □

Remark 1. In general

$$\ell_0 \leq \ell \tag{28}$$

holds and $\frac{\ell}{\ell_0}$ can be arbitrarily large [1]. If $\ell_0 = \ell$, Theorem 2.2 reduces to Theorem 1 in [4, p. 1684]. Otherwise our Theorem 2.2 improves Theorem 1 in [4]. Indeed, the upper bound in [4, p. 1684] is given by

$$\|M^{-1}\| \leq b = \frac{1}{\delta + \sqrt{2\ell\delta_0}}. \tag{29}$$

By comparing (7) with (29) we deduce

$$b < a$$

(if $\ell_0 < \ell$).

That is, we have justified the claims made in the introduction.

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