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FILTERS OF MTL-ALGEBRAS BASED ON VAGUE SET THEORY

Young Sik Park and Chul Hwan Park

Abstract. In this paper, we introduce the concept of a vague filter of MTL-algebra, and then some related properties are investigated.

1. Introduction

Zadeh[13] introduced the concept of fuzzy set as a new mathematical tool for dealing with uncertainties, several researches were conducted on the generalization of the notion of fuzzy sets. The idea of "vague set" was first published by Gau and Buehrer [3], as a generalization of the notion of fuzzy set. Esteva and Godo[2] introduced a new algebra, called an *MTL-algebra*, and studied several basic properties. MTL-algebras are algebraic structures for monoidal t-norm based logic (MTL), a many-valued propositional calculus that formalizes the structure of the real interval $[0, 1]$, induced by a left-continuous t-norm. They also introduced the notion of filters in MTL-algebras. Zhang [12] studied further properties of filters in MTL-algebras. Using the vague set, Biswas [1] studied vague groups. Jun and Park [6, 8] studied vague ideals and vague deductive systems in subtraction algebras. In this paper, we introduce the notion of vague filters in MTL-algebras, and then some related properties are investigated.

2. Preliminaries

In this section, we collect some definition and results that have been used in the sequel.

Definition 1. ([4]) An algebra $(L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ with four binary operation and two constant is a *residuated lattice* if it satisfies:

(R1) $(L, \leq, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the largest element 1,

(R2) $\odot$ is a commutative semigroup with the unit element 1,

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(R3) The Galois correspondence holds, that is,

$$(\forall x, y, z \in L)\,(x \odot y \leq z \iff x \leq y \to z).$$

Proposition 2.1. ([10]) *Let* $(L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ *be a residuated lattice. Then the following properties hold:*

(a1) $x \leq y \iff x \to y = 1$,
(a2) $0 \to x = 1$, $1 \to x = x$, $x \to (y \to x) = 1$,
(a3) $y \leq (y \to x) \to x$,
(a4) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$,
(a5) $x \to y \leq (z \to x) \to (z \to y)$, $\quad x \to y \leq (y \to z) \to (x \to z)$,
(a6) $y \leq x \Rightarrow x \to z \leq y \to z$, $z \to y \leq z \to x$,
(a7) $(\bigvee_{i \in \Gamma} y_i) \to x = \bigwedge_{i \in \Gamma} (y_i \to x)$.

We define $x^* = \bigvee\{y \in L \mid x \odot y = 0\}$, *equivalently,* $x^* = x \to 0$. *Then*

(a8) $0^* = 1$, $\quad 1^* = 0$, $\quad x \leq x^{**}$, *and* $x^* = x^{***}$.

Definition 2. ([2]) An *MTL-algebra* is a residuated lattice $L = (L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ satisfying the pre-linearity equation:

$$(x \to y) \vee (y \to x) = 1.$$

Proposition 2.2. ([12]) *In an MTL-algebra, the following are true :*

(a9) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,
(a10) $x \odot y \leq x \wedge y$.

Definition 3. ([2]) Let L be an MTL-algebra. A nonempty subset F of L is called a *filter* of L if it satisfies

(f1) $(\forall x, y \in F)$ $(x \odot y \in F)$.
(f2) $(\forall x \in F)$ $(\forall y \in L)$ $(x \leq y \Rightarrow y \in F)$.

Proposition 2.3. ([2]) *Let* L *be an MTL-algebra,* F *is a filter of* L*. Then*

(f3) $(\forall x, y \in F)$ $(x \wedge y \in F)$

Proposition 2.4. ([12]) *A nonempty subset* F *of an MTL-algebra* L *is a filter of* L *if and only if it satisfies:*

(f4) $1 \in F$.
(f5) $(\forall x \in F)$ $(\forall y \in L)$ $(x \to y \in F \Rightarrow y \in F)$.

Definition 4. ([1]) A *vague set* A in the universe of discourse U is characterized by two membership functions given by:

(1) A true membership function

$$t_A : U \to [0, 1],$$

and

(2) A false membership function

$$f_A : U \to [0, 1],$$

where $t_A(u)$ is a lower bound on the grade of membership of u derived from the "evidence for u", $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and

$$t_A(u) + f_A(u) \leq 1.$$

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0,1]$. This indicates that if the actual grade of membership of u is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{\langle u, [t_A(u), f_A(u)]\rangle \mid u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A, denoted by $V_A(u)$.

For our discussion, we shall use the following notation.

Notations[1]. (1) If $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are two subintervals of $[0,1]$, we can define a relation between I_1 and I_2 by $I_1 \succeq I_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.

(2) Let $I[0,1]$ denote the family of all closed subintervals of $[0,1]$. We define the term "imax" to mean the maximum of two intervals as

$$\text{imax}(I_1, I_2) := [\max(a_1, a_2), \max(b_1, b_2)],$$

where $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2] \in I[0,1]$. Similarly we define "imin". The concepts of "imax" and "imin" could be extended to define "isup" and "iinf" of infinite number of elements of $I[0,1]$.

For $\alpha, \beta \in [0,1]$ we now define (α, β)-cut and α-cut of a vague set.

Definition 5. ([1]) Let A be a vague set of a universe X with the true-membership function t_A and the false-membership function f_A. The (α, β)*-cut* of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by

$$A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \succeq [\alpha, \beta]\}.$$

Clearly $A_{(0,0)} = X$. The (α, β)-cuts of the vague set A are also called *vague-cuts* of A.

Definition 6. ([1]) The α*-cut* of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$, and if $\alpha \geq \beta$ then $A_\alpha \subseteq A_\beta$ and $A_{(\alpha,\beta)} = A_\alpha$. Equivalently, we can define the α-cut as $A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}$.

3. Vague filters

In what follows let L denote an MTL-algebra unless otherwise specified.

We first define the notion of vague filter of MTL-algebra.

Definition 7. A vague set A of L is called a *vague filter* of L if it satisfies

(vf1) $(\forall x, y \in L)\ (V_A(x \odot y) \succeq \text{imin}\{V_A(x), V_A(y)\})$,

(vf2) $(\forall x, y \in L)\ (x \leq y \Rightarrow V_A(x) \preceq V_A(y)$.

that is,

$$t_A(x \odot y) \geq \min\{t_A(x), t_A(y)\},\ 1 - f_A(x \odot y) \geq \min\{1 - f_A(x), 1 - f_A(y)\},$$

and

$$x \leq y \Rightarrow t_A(x) \leq t_A(y),$$
$$x \leq y \Rightarrow 1 - f_A(x) \leq 1 - f_A(y)$$

for all $x, y \in L$.

We now offer an example of vague filter of L.

Example 8. Let $L = \{0, a, b, 1\}$,where $0 < a < b < 1$ be aset with the Caley tables:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Define $\vee$ and $\wedge$-operation on L as follows:

$$(\forall x, y \in L)(x \vee y = \max\{x, y\} \,\text{and}\, x \wedge y = \min\{x, y\})$$

Then $L = (L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ is a MTL-algebra.[11] Let A be the vague set in L defined as follows:

$$A = \{\langle 0, [0.2, 0.7]\rangle, \langle a, [0.4, 0.5]\rangle, \langle b, [0.4, 0.5]\rangle, \langle 1, [0.8, 0.1]\rangle\}.$$

It is routine to verify that A is a vague filter of L.

We give characterizations of a vague filter.

Theorem 3.1. *A vage set A in L is an vague filter of L if and only if it satisfies*

(vf3) $(\forall x \in L)\ (V_A(1) \geq V_A(x))$,

(vf4) $(\forall x, y \in L)\ (V_A(y) \succeq imin\{V_A(x), V_A(x \to y)\}$.

Proof. Suppose that A is a vague filter of L. Since $x \leq 1$ for all $x \in L$, it follows from (vf2) that $t_A(1) \geq t_A(x)$ and $1 - f_A(1) \geq 1 - f_A(x)$ for all $x \in L$. This prove (vf3) hold. Let $x, y \in L$. Since $x \leq (x \to y) \to y$, we have $x \odot (x \to y) \leq y$ by the Galois correspondence. Hence

$$t_A(y) \geq t_A(x \odot (x \to y)) \geq \min\{t_A(x),\, t_A(x \to y)\},$$
$$1 - f_A(y) \geq 1 - f_A(x \odot (x \to y)) \geq \min\{1 - f_A(x),\, 1 - f_A(x \to y)\}$$

by (vf2) and (vf1). This provs (vf4) hold. Conversely,assume that A satisfies conditions (vf3) and (vf4). Using (a4), we can prove $x \to (y \to (x \odot y)) = (x \odot y) \to (x \odot y) = 1$. So,

$$\begin{aligned} t_A(x \odot y) &\geq \min\{t_A(y), t_A(y \to (x \odot y))\} \\ &\geq \min\{t_A(y), \min\{t_A(x), t_A(x \to (y \to (x \odot y)))\}\} \\ &= \min\{t_A(y), \min\{t_A(x), t_A(1)\}\} \\ &= \min\{t_A(x), t_A(y)\}, \end{aligned}$$

$$\begin{aligned} 1 - f_A(x \odot y) &\geq \min\{1 - f_A(y), 1 - f_A(y \to (x \odot y))\} \\ &\geq \min\{1 - f_A(y), \min\{1 - f_A(x), 1 - f_A(x \to (y \to (x \odot y)))\}\} \\ &= \min\{1 - f_A(y), \min\{1 - f_A(x), 1 - f_A(1)\}\} \\ &= \min\{1 - f_A(x), 1 - f_A(y)\}. \end{aligned}$$

by (vf3) and (vf4). This proves (vf1) hold. Let $x, y \in L$ be such that $x \leq y$. Then $x \to y = 1$. Then by (vf3) and (vf4), we get

$$t_A(y) \geq \min\{t_A(x), t_A(x \to y)\} = \min\{t_A(x), t_A(1)\} = t_A(x),$$

$$1-f_A(y) \geq \min\{1-f_A(x), 1-f_A(x \to y)\} = \min\{1-f_A(x), 1-f_A(1)\} = 1-f_A(x)$$

This proves (vf2) hold.

□

Theorem 3.2. *Let A be a vague filter of L. Then the following are equivalent:*

(i) $(\forall x, y, z \in L)$ $(V_A(x \to z) \succeq imin\{V_A(x \to (y \to z)), V_A(x \to y)\}$,
(ii) $(\forall x, y \in L)$ $(V_A(x \to y) \succeq V_A(x \to (x \to y))$,
(iii) $(\forall x, y, z \in L)$ $(V_A((x \to y) \to (x \to z)) \succeq V_A(x \to (y \to z))$.

Proof. (i) ⇒ (ii) Suppose that A satisfies the condition (i). Taking $z = y$ and $y = x$ in (i) and using (vf3), we have

$$\begin{aligned} t_A(x \to y) &\geq \min\{t_A(x \to (x \to y)), t_A(x \to x)\} \\ &= \min\{t_A(x \to (x \to y)), t_A(1)\} \\ &= t_A(x \to (x \to y)), \\ 1 - f_A(x \to y) &\geq \min\{1 - f_A(x \to (x \to y)), 1 - f_A(x \to x)\} \\ &= \min\{1 - f_A(x \to (x \to y)), 1 - f_A(1)\} \\ &= 1 - f_A(x \to (x \to y)) \end{aligned}$$

for all $x, y, z \in L$.

(ii) ⇒ (iii) Suppose that A satisfies the condition (ii) and let $x, y, z \in L$. Since $x \to (y \to z) \leq x \to ((x \to y) \to (x \to z))$, it follows that

$$\begin{aligned} &t_A((x \to y) \to (x \to z) = t_A(x \to ((x \to y) \to z)) \\ &\geq t_A(x \to (x \to ((x \to y) \to z))) \\ &= t_A(x \to ((x \to y) \to (x \to z))) \\ &\geq t_A(x \to (y \to z)), \end{aligned}$$

$$\begin{aligned}1 - f_A((x \to y) \to (x \to z) &= 1 - f_A(x \to ((x \to y) \to z)) \\ &\geq 1 - f_A(x \to (x \to ((x \to y) \to z))) \\ &= 1 - f_A(x \to ((x \to y) \to (x \to z))) \\ &\geq 1 - f_A(x \to (y \to z)).\end{aligned}$$

(iii) $\Rightarrow$ (i) If A satisfies the condition (iii), then

$$\begin{aligned} t_A(x \to y) &\geq \min\{t_A((x \to y) \to (x \to z)),\, t_A(x \to y)\} \\ &\geq \min\{t_A(x \to (y \to z)),\, t_A(x \to y)\}, \\ 1 - f_A(x \to y) &\geq \min\{1 - f_A((x \to y) \to (x \to z)),\, 1 - f_A(x \to y)\} \\ &\geq \min\{1 - f_A(x \to (y \to z)),\, 1 - f_A(x \to y)\}. \end{aligned}$$

This completes the proof. □

Theorem 3.3. *A vague set A in L is a vague filter of L if and only if for every $a, b, c \in L$ with $a \leq b \to c$, we have*

$$V_A(c) \succeq imin\{V_A(a), V_A(b)\}.$$

Proof. Suppose that A is a vague filter of L. Let $a, b, c \in L$ be such that $a \leq b \to c$. Since $a \leq b \to c$, we have $t_A(a) \leq t_A(b \to c)$ and $1 - f_A(a) \geq 1 - f_A(b \to c)$, and so

$$t_A(c) \geq \min\{t_A(b), t_A(b \to c)\} \geq \min\{t_A(b), t_A(a)\},$$

$$1 - f_A(c) \geq \min\{1 - f_A(b), 1 - f_A(b \to c)\} \geq \min\{1 - f_A(b), 1 - f_A(a)\}$$

Therefore $V_A(c) \succeq \mathrm{imin}\{V_A(a), V_A(b)\}$. Conversely, suppose that $V_A(c) \geq \min\{V_A(a), V_A(b)\}$. Since $x \leq x \to 1$ for all $x \in L$, we get

$$t_A(1) \geq \min\{t_A(x), t_A(x)\} = t_A(x),$$

$$1 - f_A(1) \geq \min\{1 - f_A(x), 1 - f_A(x)\} = 1 - f_A(x)$$

for all $x \in L$. This proves $V_A(1) \geq V_A(x)$ hold. Since $x \to y \leq x \to y$ for all $x, y \in L$, we get

$$t_A(y) \geq \min\{t_A(x), t_A(x \to y)\},$$

$$1 - f_A(y) \geq \min\{1 - f_A(x), 1 - f_A(x \to y)\}$$

for all $x, y \in L$. This proves $V_A(y) \succeq \mathrm{imin}\{V_A(x), V_A(x \to y)\}$ hold. Therefore A is a vague filter of L. □

Theorem 3.4. *Let A be a vague filter of L. Then for any $\alpha, \beta \in [0, 1]$, the vague-cut $A_{(\alpha,\beta)}$ of L is a crisp filter of L.*

Proof. Assume that A is a vague filter. Obviously, $1 \in A_{(\alpha,\beta)}$. Let $x, y \in L$ be such that $x \in A_{(\alpha,\beta)}$ and $x \to y \in A_{(\alpha,\beta)}$. Then $V_A(x) \succeq [\alpha, \beta]$, i.e., $t_A(x) \geq \alpha$ and $1 - f_A(x) \geq \beta$; and $V_A(x \to y) \succeq [\alpha, \beta]$, i.e., $t_A(x \to y) \geq \alpha$ and $1 - f_A(x \to y) \geq \beta$. It follows from (vf4) that

$$t_A(y) \geq \min\{t_A(x), t_A(x \to y)\} \geq \alpha$$

and

$$1 - f_A(y) \geq \min\{1 - f_A(y), 1 - f_A(x \to y)\} \geq \beta$$

so that $V_A(y) \succeq [\alpha, \beta]$. Hence $y \in A_{(\alpha,\beta)}$. Therefore $A_{(\alpha,\beta)}$ is a filter of L. □

The filters like $A_{(\alpha,\beta)}$ are also called *vague-cut filters* of X.

Definition 9. ([1, 3]) If A is a vague set of L and θ is a map from L into itself, we define a maps ${t_A}^\theta : L \to [0,1]$ and ${f_A}^\theta : L \to [0,1]$ given by, respectively,

(1) $(\forall x \in L)\ {t_A}^\theta(x) = t_A(\theta(x))$ and
(2) $(\forall x \in L)\ {f_A}^\theta(x) = f_A(\theta(x))$.

In such case we write ${V_A}^\theta(x) = V_A(\theta(x))$ for all $x \in L$.

Theorem 3.5. *If A is a vague filter of L and θ is a homomorphism of L, then the vague set A^θ of X given by*

$$A^\theta = \{\langle x, [{t_A}^\theta(x), {t_A}^\theta(x)]\rangle \mid x \in L\},$$

is also a vague filter of L.

Proof. For every $x, y \in L$ we have

$$\begin{aligned} {t_A}^\theta(x \odot y) &= t_A(\theta(x \odot y)) = t_A(\theta(x) \odot \theta(y)) \\ &\geq \min\{t_A(\theta(x)), t_A(\theta(y))\} \\ &= \min\{{t_A}^\theta(x)), {t_A}^\theta(y))\} \end{aligned}$$

and

$$\begin{aligned} 1 - {f_A}^\theta(x \odot y) &= 1 - f_A(\theta(x \odot y)) = 1 - f_A(\theta(x) \odot \theta(y)) \\ &\geq \min\{1 - f_A(\theta(x)), 1 - f_A(\theta(y))\} \\ &= \min\{1 - {f_A}^\theta(x)), 1 - {f_A}^\theta(y))\} \end{aligned}$$

Also, let $x \leq y$ we have

$${t_A}^\theta(x) = t_A(\theta(x) \leq t_A(\theta(y) = {t_A}^\theta(y)$$

and

$$1 - {f_A}^\theta(x) = 1 - f_A(\theta(x) \leq 1 - f_A(\theta(y) = 1 - {f_A}^\theta(y)$$

Therefore A^θ is an vague filter of L. □

Theorem 3.6. *If A is a vague filter of L, then the set*

$$\Omega_a := \{x \in L \mid V_A(x) \geq V_A(a)\}$$

is a filter of L for every $a \in L$.

Proof. Since $V_A(1) \geq V_A(x)$ for all $x \in L$. we have $1 \in \Omega_a$. Let $x, y \in L$ be such that $x \in \Omega_a$ and $x \to y \in \Omega_a$. Then $t_A(x) \geq t_A(a)$, $1 - f_A(x) \geq 1 - f_A(a)$, $t_A(x \to y) \geq t_A(a)$ and $1 - f_A(x \to y) \geq 1 - f_A(a)$. Since A is a vague filter of L, it follows from (vf4) that

$$t_A(y) \geq \min\{t_A(x), t_A(x \to y)\} \geq t_A(a),$$

$$1 - f_A(y) \geq \min\{1 - f_A(x), 1 - f_A(x \to y)\} \geq 1 - f_A(a)$$

so that $y \in \Omega_a$. Hence Ω_a is a filter of L. □

Theorem 3.7. *Let $a \in L$ and let A be a vague set in L. Then*

(i) *If Ω_a is a filter of L, then A satisfies the following implications:*

$V_A(a) \preceq \text{imin}\{V_A(x \to y), V_A(x)\} \Rightarrow V_A(a) \preceq V_A(y) - - - (*)$

for all $x, y \in L$.

(ii) *If A satisfies* ((vf3) *and* (*)*, then Ω_a is a filter of L.*

Proof. (i) Assume that Ω_a is a filter of L. Let $x, y \in L$ be such that $V_A(a) \preceq \text{imin}\{V_A(x \to y), V_A(x)\}$ Then $x \to y \in \Omega_a$ and $x \in \Omega_a$. Using (f4), we get $y \in \Omega_a$. Threfore $V_A(y) \succeq V_A(a)$.

(ii) Suppose that A satisfies (vf3) and (*). From (vf3) it follows that $1 \in \Omega_a$. Let $x, y \in L$ be such that $x \in \Omega_a$ and $x \to y \in \Omega_a$. Then $V_A(a) \preceq V_A(x)$ and $V_A(a) \preceq V_A(x \to y)$. This means that $V_A(a) \preceq \text{imin}\{V_A(x), V_A(x \to y)\}$. Thus $V_A(a) \preceq V_A(y)$ by (*). So $y \in \Omega_a$. Therefore Ω_a is a filter of L. □

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Young Sik Park
Department of Mathematics, University of Ulsan, Ulsan 680-749, Korea
E-mail address: yspark@ulsan.ac.kr

Chul Hwan Park
School of Digital, Mechanics, Ulsan Colleage, Namgu, Ulsan 680-749, Korea
E-mail address: skyrosemary@gmail.com