OSCILLATION BEHAVIOR OF SOLUTIONS OF THIRD-ORDER NONLINEAR DELAY DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. By using the Riccati transformation technique, we study the oscillation and asymptotic behavior for the third-order nonlinear delay dynamic equations

$$\left(c(t)\left(p(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta} + q(t)f(x(\tau(t))) = 0$$

on a time scale \mathbb{T} , where c(t), p(t) and q(t) are real-valued positive rdcontinuous functions defined on \mathbb{T} . We establish some new sufficient conditions which ensure that every solution oscillates or converges to zero. Our oscillation results are essentially new. Some examples are considered to illustrate the main results.

1. Introduction

Following Hilger's landmark paper [16], a rapidly expanding body of literature has sought to unify, extend and generalize ideas from discrete calculus, quantum calculus and continuous calculus to arbitrary time scale calculus, where a time scale \mathbb{T} is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models which are discrete in season (and may follow a difference scheme with variable step-size or often modeled by continuous dynamic systems), die out, say in winter, while their eggs are incubating

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Received April 27, 2009.

²⁰¹⁰ Mathematics Subject Classification. 39A21, 34C10, 34K11, 34N05.

Key words and phrases. oscillation behavior, third order delay dynamic equations, time scales.

This research is supported by the Natural Science Foundation of China (11071143, 60904024, 11026112), China Postdoctoral Science Foundation funded project (200902564) and supported by Shandong Provincial Natural Science Foundation (ZR2010AL002, ZR2009AL003, Y2008A28), also supported by University of Jinan Research Funds for Doctors (XBS0843) and Natural Science Foundation of Educational Department of Shandong Province (J11LA01).

or dormant, and then in season again, hatching gives rise to a nonoverlapping population (see Bohner and Peterson [3]). Not only does the new theory of the so-called "dynamic equations" unify the theories of differential equations and difference equations, but also extends these classical cases to cases "in between", e.g., to the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0, q > 1\}$ (which has important applications in quantum theory) and can be applied on different types of time scales like $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$ the space of the harmonic numbers. Several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal *et al.* [1] and references cited therein. A book on the subject of time scales, by Bohner and Peterson [3] summarizes and organizes much of the time scale calculus. We refer also to the last book by Bohner and Peterson [3] for advances in dynamic equations on time scales. For the notation used below we refer to the next section that provides some basic facts on time scales extracted from Bohner and Peterson [3].

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to Bohner and Saker [5], Erbe [6], Saker [18], Hassan [14]. And there are some results dealing with the oscillation of the solutions of secondorder delay dynamic equations on time scales [2, 7, 11, 12, 13, 17, 19, 20, 22]. However there are little results on the oscillation and asymptotic behavior of solutions for third-order dynamic equations on time scales, we refer the reader to the papers [8, 9, 10, 15, 21].

Erbe et al. [8] considered the general third-order nonlinear dynamic equation

(1.1)
$$(c(t)(a(t)x^{\Delta}(t))^{\Delta})^{\Delta} + q(t)f(x(t)) = 0, \quad t \in \mathbb{T},$$

where c(t), a(t) and q(t) are positive, real-valued rd-continuous functions defined on \mathbb{T} , and $f \in C(\mathbb{R}, \mathbb{R})$ such that satisfies for each positive constant k > 0, there exists $M = M_k > 0$, such that $\frac{f(u)}{u} \ge M$, $|u| \ge k$, xf(x) > 0 for all nonzero x. By employing the generalized Riccati transformation techniques, they obtained if

(1.2)
$$\int_{t_0}^{\infty} \frac{1}{c(t)} \Delta t = \int_{t_0}^{\infty} \frac{1}{a(t)} \Delta t = \infty,$$

and there exists a positive Δ -differentiable function $\delta(t)$, for all M > 0 and all sufficiently large t_1 , such that

(1.3)
$$\limsup_{t \to \infty} \int_{t_1}^t [M\delta(s)q(s) - \frac{(\delta^{\Delta}(s))^2 a(s)}{4\delta(s) \int_{t_1}^s \frac{\Delta u}{c(u)}}] \Delta s = \infty.$$

Then every solution of (1.1) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite). Furthermore, if

(1.4)
$$\int_{t_0}^{\infty} q(t)\Delta t = \infty$$

holds, then every solution x(t) of (1.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$. Erbe et al. [9] studied the third-order linear dynamic equation

(1.5)
$$x^{\Delta\Delta\Delta}(t) + p(t)x(t) = 0, \quad t \in \mathbb{T},$$

where p(t) is a positive, real-valued rd-continuous function defined on \mathbb{T} , and they established Hille and Nehari type oscillation criteria for the equation (1.5).

Yu and Wang [21] considered the third order nonlinear dynamic equations

(1.6)
$$\left(\frac{1}{a_2(t)}\left((\frac{1}{a_1(t)}(x^{\Delta}(t))^{\alpha_1})^{\Delta}\right)^{\alpha_2}\right)^{\Delta} + q(t)f(x(t)) = 0, \quad t \in \mathbb{T},$$

where $a_i(t)$ and q(t) are positive, real-valued rd-continuous functions defined on \mathbb{T} , α_i is a quotient of odd positive integers, $i = 1, 2, f \in C(\mathbb{R}, \mathbb{R})$ which satisfies for each positive constant k > 0, there exists $M = M_k > 0$, such that $\frac{f(u)}{u} \ge M, |u| \ge k, xf(x) > 0$ for all nonzero x. By employing the generalized Riccati transformation techniques, they obtained if

(1.7)
$$\alpha_1 \alpha_2 = 1, \quad \int_{t_0}^{\infty} [a_i(s)]^{\frac{1}{\alpha_i}} \Delta s = \infty, \quad i = 1, 2,$$

and there exists a positive Δ -differentiable function $\delta(t)$ on \mathbb{T} , for all M > 0and all sufficiently large t_1, t_2 , with $t_2 > t_1$,

(1.8)
$$\limsup_{t \to \infty} \int_{t_2}^t \left[M\delta(s)q(s) - \frac{(\delta^{\Delta}(s))^2}{4Q(s)} \right] \Delta s = \infty,$$

where

$$Q(t) = \delta(t)[a_1(t)\delta(t,t_1)]^{\frac{1}{\alpha_1}}, \quad \delta(t,t_1) = \int_{t_1}^t [a_2(s)]^{\frac{1}{\alpha_2}} \Delta s,$$

then every solution x(t) of (1.6) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite). In addition to the condition (1.4), they obtained that every solution x(t) of (1.6) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Hassan [15] investigated the third order delay dynamic equation

(1.9)
$$\left(a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma}\right)^{\Delta} + f(t,x(\tau(t))) = 0, \quad t \in \mathbb{T},$$

and established some results when the condition $\tau(\sigma(t)) = \sigma(\tau(t))$ holds.

Following this trend, we consider oscillatory and asymptotic behavior of the third-order nonlinear delay dynamic equations

(1.10)
$$\left(c(t) \left(p(t)x^{\Delta}(t) \right)^{\Delta} \right)^{\Delta} + q(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T},$$

where the functions c(t), p(t) and q(t) are positive, real-valued rd-continuous functions defined on \mathbb{T} , the so-called delay function $\tau : \mathbb{T} \to \mathbb{T}$ is a function such that $\tau(t) \leq t$, $\tau(t) \to \infty$ as $t \to \infty$, and $f \in C(\mathbb{R}, \mathbb{R})$, $|f(x)| \geq L|x|$ for some positive constant L, xf(x) > 0 for all nonzero x.

Since we are interested in oscillatory and asymptotic behavior, we assume throughout this paper that the given time scale \mathbb{T} is unbounded above. We

assume $t_0 \in \mathbb{T}$ and it is convenient to assume $t_0 > 0$. We define the time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$.

The paper is organized as follows: In Section 2, we intend to use the Riccati transformation technique to obtain some sufficient conditions which guarantee that every solution x(t) of (1.10) is either oscillatory or converges as $t \to \infty$. In Section 3, some applications and examples are considered to illustrate the main results.

2. Main results

In this section, we establish some sufficient conditions which guarantee that every solution x(t) of (1.10) either oscillates on $[t_0, \infty)$ or converges as $t \to \infty$.

Throughout this paper, we denote

$$d_+(t) := \max\{0, d(t)\}, \quad d_-(t) := \max\{0, -d(t)\}.$$

Before we state and prove our main results, we give the following lemmas which will play an important role in the proof of our main results.

Lemma 2.1. Assume that

(2.1)
$$\int_{t_0}^{\infty} \frac{1}{c(t)} \Delta t = \int_{t_0}^{\infty} \frac{1}{p(t)} \Delta t = \infty$$

holds and assume further that x(t) is an eventually positive solution of (1.10). Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that either

- (1) x(t) > 0, $x^{\Delta}(t) > 0$, $(p(t)x^{\Delta}(t))^{\Delta} > 0$, $t \in [t_1, \infty)_{\mathbb{T}}$; or
- (2) $x(t) > 0, \quad x^{\Delta}(t) < 0, \quad (p(t)x^{\Delta}(t))^{\Delta} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}.$

The proof is similar to that of Erbe et al. [8, Lemma 1].

Lemma 2.2. Assume that x(t) is a solution of (1.10) which satisfies the case (1) in Lemma 2.1. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that

(2.2)
$$x^{\Delta}(t) \ge \frac{\int_{t_1}^t \frac{1}{c(s)} \Delta s}{p(t)} c(t) (p(t)x^{\Delta}(t))^{\Delta}, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

The proof is similar to that of Erbe et al. [8, Lemma 3].

Lemma 2.3. Assume that x(t) is a solution of (1.10) which satisfies the case (1) in Lemma 2.1. Furthermore, assume that $p^{\Delta}(t) \leq 0$ and

(2.3)
$$\int_{t_0}^{\infty} \tau(t)q(t)\Delta t = \infty.$$

Then there exists $t_* \in [t_0, \infty)_{\mathbb{T}}$ such that

(2.4)
$$x(t) > tx^{\Delta}(t), \quad \frac{x(t)}{t} \text{ is strictly decreasing on } [t_*, \infty)_{\mathbb{T}}.$$

Proof. Assume x(t) is a solution of (1.10) which satisfies the case (1) in Lemma 2.1. In view of $(p(t)x^{\Delta}(t))^{\Delta} = p^{\Delta}(t)x^{\Delta}(t) + p^{\sigma}(t)x^{\Delta\Delta}(t) > 0, \ p^{\Delta}(t) \le 0, x^{\Delta}(t) > 0, \ t \in [t_1, \infty)_{\mathbb{T}}$, we obtain $x^{\Delta\Delta}(t) > 0, \ t \in [t_1, \infty)_{\mathbb{T}}$. Let

$$X(t) := x(t) - tx^{\Delta}(t).$$

Thus (2.5)

$$X^{\Delta}(t) = x^{\Delta}(t) - (x^{\Delta}(t) + \sigma(t)x^{\Delta\Delta}(t)) = -\sigma(t)x^{\Delta\Delta}(t) < 0, \quad t \in [t_1, \infty)_{\mathbb{T}},$$

hence, we claim that there is $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that X(t) > 0 on $[t_2, \infty)_{\mathbb{T}}$. Assume not. Then X(t) < 0 on $[t_2, \infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{X(t)}{t\sigma(t)} > 0, \quad t \in [t_2, \infty)_{\mathbb{T}},$$

which implies that $\frac{x(t)}{t}$ is strictly increasing on $[t_2, \infty)_{\mathbb{T}}$. Pick $t_3 \in [t_2, \infty)_{\mathbb{T}}$ so that $\tau(t) \geq \tau(t_3)$ for $t \in [t_3, \infty)_{\mathbb{T}}$. Then

$$\frac{x(\tau(t))}{\tau(t)} \ge \frac{x(\tau(t_3))}{\tau(t_3)} := d > 0,$$

so that $x(\tau(t)) \ge d\tau(t)$ for $t \in [t_3, \infty)_{\mathbb{T}}$. Now by integrating both sides of (1.10) from t_3 to t, we get

$$c(t)(p(t)x^{\Delta}(t))^{\Delta} - c(t_3)(p(t_3)x^{\Delta}(t_3))^{\Delta} = -\int_{t_3}^t q(s)f(x(\tau(s)))\Delta s,$$

which yields

$$c(t_3)(p(t_3)x^{\Delta}(t_3))^{\Delta} = c(t)(p(t)x^{\Delta}(t))^{\Delta} + \int_{t_3}^t q(s)f(x(\tau(s)))\Delta s$$
$$\geq \int_{t_3}^t q(s)f(x(\tau(s)))\Delta s \ge Ld \int_{t_3}^t q(s)\tau(s)\Delta s,$$

which contradicts (2.3). Hence there is $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that X(t) > 0 on $[t_2, \infty)_{\mathbb{T}}$, and we have that $\frac{x(t)}{t}$ is eventually strictly decreasing. The proof is complete.

Lemma 2.4. Assume that x(t) is a solution of (1.10) which satisfies the case (2) in Lemma 2.1. Furthermore,

(2.6)
$$\int_{t_0}^{\infty} \frac{1}{p(v)} \int_{v}^{\infty} \frac{1}{c(u)} \int_{u}^{\infty} q(s) \Delta s \Delta u \Delta v = \infty.$$

Then $\lim_{t\to\infty} x(t) = 0.$

Proof. Assume x(t) is a solution of (1.10) which satisfies the case (2) in Lemma 2.1. Then x(t) is decreasing and $\lim_{t\to\infty} x(t) = b \ge 0$. We assert that b = 0.

If not, then $x(\tau(t)) \ge x(t) \ge b > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Integrating (1.10) from sufficiently large t to ∞ , we get

$$(p(t)x^{\Delta}(t))^{\Delta} \ge \frac{L}{c(t)} \int_{t}^{\infty} q(s)x(\tau(s))\Delta s,$$

integrating this inequality from sufficiently large t to ∞ , we have

$$-x^{\Delta}(t) \ge \frac{1}{p(t)} \int_{t}^{\infty} \frac{L}{c(u)} \int_{u}^{\infty} q(s) x(\tau(s)) \Delta s \Delta u,$$

integrating the last inequality again from t_0 to ∞ , we can obtain

$$\begin{aligned} x(t_0) &\geq \int_{t_0}^{\infty} \frac{1}{p(v)} \int_{v}^{\infty} \frac{L}{c(u)} \int_{u}^{\infty} q(s) x(\tau(s)) \Delta s \Delta u \Delta v \\ &\geq b L \int_{t_0}^{\infty} \frac{1}{p(v)} \int_{v}^{\infty} \frac{1}{c(u)} \int_{u}^{\infty} q(s) \Delta s \Delta u \Delta v = \infty, \end{aligned}$$

which contradicts (2.6). This completes the proof.

Lemma 2.5 ([2]). If $\tau(t) < t$ and the inequality $x^{\Delta}(t) + p(t)x(\tau(t)) \leq 0$ has an eventually positive solution, then $\alpha \geq 1$, where

$$\alpha = \lim_{t_0 \to \infty} \sup_{t > t_0} \sup_{\lambda \in E} \left\{ \lambda \exp\{\int_{\tau(t)}^t \zeta_{\mu(s)}(-\lambda p(s))\Delta s\} \right\},$$
$$E = \{\lambda : \lambda > 0, \quad 1 - \lambda p(t)\mu(t) > 0\}.$$

Theorem 2.1. Assume (2.1), (2.3) and $p^{\Delta}(t) \leq 0$ hold. Furthermore, assume that there exists a positive function $\delta(t) \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, and for all sufficiently large t_1 , there exists $T > t_1$, we have

(2.7)
$$\limsup_{t \to \infty} \int_T^t \left[\delta(\sigma(s))q(s)\frac{\tau(s)}{\sigma(s)} - \frac{p(s)\sigma(s)(\delta^{\Delta}(s))^2}{4Ls\delta(\sigma(s))\int_{t_1}^s \frac{1}{c(u)}\Delta u} \right] \Delta s = \infty.$$

Then every solution x(t) of (1.10) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).

Proof. Assume (1.10) has a nonoscillatory solution x(t) on $[t_0, \infty)_{\mathbb{T}}$. We may assume without loss of generality that x(t) > 0 and $x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$, $t_1 \in [t_0, \infty)_{\mathbb{T}}$. Since (2.1) holds, by Lemma 2.1 we see that x(t) satisfies either case (1) or case (2).

We claim that case (1) of Lemma 2.1 is not true. Define the function $\omega(t)$ by

(2.8)
$$\omega(t) = \frac{\delta(t)c(t)(p(t)x^{\Delta}(t))^{\Delta}}{x(t)}, \quad t \in [t_1, \infty)_{\mathbb{T}},$$

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then $\omega(t) > 0$. By the product and then the quotient rule

$$\begin{split} \omega^{\Delta}(t) &= \delta^{\Delta}(t) \frac{c(t)(px^{\Delta})^{\Delta}(t)}{x(t)} + \delta(\sigma(t)) \frac{(c(t)(p(t)x^{\Delta}(t))^{\Delta})^{\Delta}}{x(\sigma(t))} \\ &- \delta(\sigma(t)) \frac{c(t)(p(t)x^{\Delta}(t))^{\Delta}x^{\Delta}(t)}{x(t)x(\sigma(t))}. \end{split}$$

In view of (1.10) and (2.8), we get

$$\begin{split} \omega^{\Delta}(t) &\leq -L\delta(\sigma(t))q(t)\frac{x(\tau(t))}{x(\sigma(t))} + \frac{\delta^{\Delta}(t)}{\delta(t)}\omega(t) \\ &- \frac{\delta(\sigma(t))}{\delta^{2}(t)}\frac{x(t)}{x(\sigma(t))}\frac{x^{\Delta}(t)}{c(t)(p(t)x^{\Delta}(t))^{\Delta}}\omega^{2}(t), \end{split}$$

from Lemma 2.2, there exists $t_2 \ge t_1$ such that

$$x^{\Delta}(t) \ge \frac{\int_{t_1}^t \frac{1}{c(s)} \Delta s}{p(t)} c(t) (p(t) x^{\Delta}(t))^{\Delta}, \ t \ge t_2,$$

also from Lemma 2.3, we have $\frac{x(\tau(t))}{x(\sigma(t))} \ge \frac{\tau(t)}{\sigma(t)}, \frac{x(t)}{x(\sigma(t))} \ge \frac{t}{\sigma(t)}$, so we obtain (2.9)

$$\omega^{\Delta}(t) \leq -L\delta(\sigma(t))q(t)\frac{\tau(t)}{\sigma(t)} + \frac{(\delta^{\Delta}(t))_{+}}{\delta(t)}\omega(t) - \frac{\delta(\sigma(t))}{\delta^{2}(t)}\frac{t}{\sigma(t)}\frac{\int_{t_{1}}^{t}\frac{1}{c(s)}\Delta s}{p(t)}\omega^{2}(t).$$

By the averaging technique, we find that

$$\frac{(\delta^{\Delta}(t))_{+}}{\delta(t)}\omega(t) - \frac{\delta(\sigma(t))}{\delta^{2}(t)}\frac{t}{\sigma(t)}\frac{\int_{t_{1}}^{t}\frac{1}{c(s)}\Delta s}{p(t)}\omega^{2}(t) \leq \frac{p(t)\sigma(t)(\delta^{\Delta}(t))^{2}}{4t\delta(\sigma(t))\int_{t_{1}}^{t}\frac{1}{c(s)}\Delta s},$$

hence, we get

(2.10)
$$\omega^{\Delta}(t) \leq -\left[L\delta(\sigma(t))q(t)\frac{\tau(t)}{\sigma(t)} - \frac{p(t)\sigma(t)(\delta^{\Delta}(t))^2}{4t\delta(\sigma(t))\int_{t_1}^t \frac{1}{c(s)}\Delta s}\right],$$

integrating (2.10) from t_2 to t, we obtain

$$-\omega(t_2) \le \omega(t) - \omega(t_2) \le -\int_{t_2}^t \left[L\delta(\sigma(s))q(s)\frac{\tau(s)}{\sigma(s)} - \frac{p(s)\sigma(s)(\delta^{\Delta}(s))^2}{4s\delta(\sigma(s))\int_{t_1}^s \frac{1}{c(u)}\Delta u} \right] \Delta s.$$

Which yields

$$\int_{t_2}^t \left[\delta(\sigma(s))q(s) \frac{\tau(s)}{\sigma(s)} - \frac{p(s)\sigma(s)(\delta^{\Delta}(s))^2}{4Ls\delta(\sigma(s))\int_{t_1}^s \frac{1}{c(u)}\Delta u} \right] \Delta s \le \omega(t_2).$$

This is contrary to (2.7). Hence, case (1) is not possible. If case (2) in Lemma 2.1 holds, then clearly $\lim_{t\to\infty} x(t)$ exists (finite). This completes the proof. \Box

Corollary 2.1. Assume (2.1), (2.3) and $p^{\Delta}(t) \leq 0$ hold, and for all sufficiently large t_1 , there exists $T > t_1$,

(2.11)
$$\limsup_{t \to \infty} \int_T^t (\tau(s)q(s) - \frac{p(s)}{4Ls \int_{t_1}^s \frac{\Delta u}{c(u)}}) \Delta s = \infty.$$

Then every solution x(t) of (1.10) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).

Proof. This follows from Theorem 2.1 by taking $\delta(t) = t$. This completes the proof.

Using Theorem 2.1 and Lemma 2.4, we get the following result.

Corollary 2.2. Assume (2.1), (2.3), (2.6) and $p^{\Delta}(t) \leq 0$ hold, and for all sufficiently large t_1 , there exists $T > t_1$ such that (2.7) holds. Then every solution x(t) of (1.10) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Theorem 2.2. Assume (2.1), (2.3) and $p^{\Delta}(t) \leq 0$ hold. Furthermore, assume that there exist a function H, $h \in C_{rd}(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} \equiv \{(t, s) : t \geq s \geq t_0\}$ such that

(2.12)
$$H(t,t) = 0, \quad t \ge t_0, \quad H(t,s) > 0, \quad t > s \ge t_0,$$

and H has a nonpositive continuous Δ -partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable and satisfies

(2.13)
$$H^{\Delta_s}(\sigma(t), \sigma(s)) + H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} = -\frac{h(t, s)}{\delta(s)} (H(\sigma(t), \sigma(s)))^{\frac{1}{2}},$$

and for all sufficiently large t_1 , there exists $T > t_1$, we have

(2.14)
$$\limsup_{t \to \infty} \frac{1}{H(\sigma(t), T)} \int_{T}^{\sigma(t)} K(t, s) \Delta s = \infty$$

where $\delta(t)$ is a positive Δ -differentiable function and

$$K(t,s) = H(\sigma(t),\sigma(s))L\delta(\sigma(s))q(s)\frac{\tau(s)}{\sigma(s)} - \frac{p(s)\sigma(s)\left(h_{-}\left(t,s\right)\right)^{2}}{4s\delta(\sigma(s))\int_{t_{1}}^{s}\frac{\Delta u}{c(u)}}.$$

Then every solution x(t) of (1.10) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).

Proof. Suppose that (1.10) has a nonoscillatory solution x(t) on $[t_0, \infty)_{\mathbb{T}}$. We may assume without loss of generality that x(t) > 0 and $x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}, t_1 \in [t_0, \infty)_{\mathbb{T}}$. By Lemma 2.1 there are two possible cases. If case (1) holds, we proceed as in the proof of Theorem 2.1 and we get (2.9). Then from (2.9) with $(\delta^{\Delta}(t))_{\perp}$ replaced by $\delta^{\Delta}(t)$, we have

$$(2.15) \quad L\delta(\sigma(t))q(t)\frac{\tau(t)}{\sigma(t)} \le -\omega^{\Delta}(t) + \frac{\delta^{\Delta}(t)}{\delta(t)}\omega(t) - \frac{\delta(\sigma(t))}{\delta^{2}(t)}\frac{t}{\sigma(t)}\frac{\int_{t_{1}}^{t}\frac{\Delta s}{c(s)}}{p(t)}\omega^{2}(t).$$

Multiplying both sides of (2.15), with t replaced by s, by $H(\sigma(t), \sigma(s))$, integrating with respect to s from t_2 to $\sigma(t)$, $t \ge t_2$, we get

$$\begin{split} &\int_{t_2}^{\sigma(t)} H(\sigma(t), \sigma(s)) L\delta(\sigma(s)) q(s) \frac{\tau(s)}{\sigma(s)} \Delta s \\ &\leq -\int_{t_2}^{\sigma(t)} H(\sigma(t), \sigma(s)) \omega^{\Delta}(s) \Delta s + \int_{t_2}^{\sigma(t)} H(\sigma(t), \sigma(s)) \frac{\delta^{\Delta}(s)}{\delta(s)} \omega(s) \Delta s \\ &- \int_{t_2}^{\sigma(t)} \left[H(\sigma(t), \sigma(s)) \frac{\delta(\sigma(s))}{\delta^2(s)} \frac{s}{\sigma(s)} \frac{\int_{t_1}^s \frac{\Delta u}{c(u)}}{p(s)} \omega^2(s) \right] \Delta s. \end{split}$$

Integrating by parts and using (2.12) and (2.13), we obtain

$$\begin{split} &\int_{t_2}^{\sigma(t)} H(\sigma(t),\sigma(s))L\delta(\sigma(s))q(s)\frac{\tau(s)}{\sigma(s)}\Delta s\\ &\leq H(\sigma(t),t_2)\omega(t_2) + \int_{t_2}^{\sigma(t)} H^{\Delta_s}(\sigma(t),s)\omega(s)\Delta s\\ &\quad + \int_{t_2}^{\sigma(t)} H(\sigma(t),\sigma(s))\frac{\delta^{\Delta}(s)}{\delta(s)}\omega(s)\Delta s\\ &\quad - \int_{t_2}^{\sigma(t)} \left[H(\sigma(t),\sigma(s))\frac{\delta(\sigma(s))}{\delta^2(s)}\frac{s}{\sigma(s)}\frac{\int_{t_1}^s\frac{\Delta u}{c(u)}}{p(s)}\omega^2(s) \right]\Delta s\\ &\leq H(\sigma(t),t_2)\omega(t_2) + \int_{t_2}^{\sigma(t)} \left[-\frac{h(t,s)}{\delta(s)}(H(\sigma(t),\sigma(s)))^{\frac{1}{2}}\omega(s) \right. \\ &\quad - H(\sigma(t),\sigma(s))\frac{\delta(\sigma(s))}{\delta^2(s)}\frac{s}{\sigma(s)}\frac{\int_{t_1}^s\frac{\Delta u}{c(u)}}{p(s)}\omega^2(s) \right]\Delta s\\ &\leq H(\sigma(t),t_2)\omega(t_2) + \int_{t_2}^{\sigma(t)} \left[\frac{h_-(t,s)}{\delta(s)}(H(\sigma(t),\sigma(s)))^{\frac{1}{2}}\omega(s) \right. \\ &\quad - H(\sigma(t),\sigma(s))\frac{\delta(\sigma(s))}{\delta^2(s)}\frac{s}{\sigma(s)}\frac{\int_{t_1}^s\frac{\Delta u}{c(u)}}{p(s)}\omega^2(s) \right]\Delta s. \end{split}$$

Using averaging technique, we have

$$\begin{split} &\frac{h_{-}(t,s)}{\delta(s)}(H(\sigma(t),\sigma(s)))^{\frac{1}{2}}\omega(s) - H(\sigma(t),\sigma(s))\frac{\delta(\sigma(s))}{\delta^{2}(s)}\frac{s}{\sigma(s)}\frac{\int_{t_{1}}^{s}\frac{\Delta u}{c(u)}}{p(s)}\omega^{2}(s)\\ &\leq \frac{p(s)\sigma(s)(h_{-}(t,s))^{2}}{4s\delta(\sigma(s))\int_{t_{1}}^{s}\frac{1}{c(u)}\Delta u}. \end{split}$$

From this inequality and the definition of K(t,s), we get

$$\int_{t_2}^t K(t,s)\Delta s \le H(\sigma(t),t_2)\omega(t_2),$$

consequently

$$\frac{1}{H(\sigma(t), t_2)} \int_{t_2}^{\sigma(t)} K(t, s) \Delta s \le \omega(t_2).$$

which contradicts assumption (2.14). Hence, case (1) is not possible. If case (2) in Lemma 2.1 holds, then clearly $\lim_{t\to\infty} x(t)$ exists (finite). This completes the proof.

Using Theorem 2.2 and Lemma 2.4, we get the following result.

Corollary 2.3. Assume (2.1), (2.3), (2.6) and $p^{\Delta}(t) \leq 0$ hold, and for all sufficiently large t_1 , there exists $T > t_1$ such that (2.14) holds. Then every solution x(t) of (1.10) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Theorem 2.3. Assume (2.1), (2.3) hold, $p^{\Delta}(t) \leq 0$, and

(2.16)
$$\limsup_{t \to \infty} \int_{t_0}^t q(s) \frac{\tau(s)}{s} \Delta s = \infty.$$

Then every solution x(t) of (1.10) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).

Proof. Suppose that (1.10) has a nonoscillatory solution x(t) on $[t_0, \infty)_{\mathbb{T}}$. We may assume without loss of generality that x(t) > 0 and $x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}, t_1 \in [t_0, \infty)_{\mathbb{T}}$. By Lemma 2.1 there are two possible cases. If case (1) holds, then there exists $\alpha > 0$ such that $x(t) \ge \alpha, t \in [t_1, \infty)_{\mathbb{T}}$.

Let $y(t) = c(t)(p(t)x^{\Delta}(t))^{\Delta}$. Then $y(t) > 0, t \in [t_1, \infty)_{\mathbb{T}}$. Hence

$$y(t) = y(t_1) + \int_{t_1}^t y^{\Delta}(s) \Delta s \le y(t_1) - L \int_{t_1}^t q(s) x(\tau(s)) \Delta s,$$

from Lemma 2.3, we have

$$L\alpha \int_{t_1}^t q(s) \frac{\tau(s)}{s} \Delta s \le y(t_1),$$

which is a contradiction of (2.16). Hence, case (1) is not possible. If case (2) holds, then clearly $\lim_{t\to\infty} x(t)$ exists (finite). This completes the proof. \Box

Using Theorem 2.3 and Lemma 2.4, we get the following result.

Corollary 2.4. Assume (2.1), (2.3), (2.6), (2.16) and $p^{\Delta}(t) \leq 0$ hold. Then every solution x(t) of (1.10) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Theorem 2.4. Assume (2.1), (2.3) and $p^{\Delta}(t) \leq 0$ hold. Furthermore, assume that for all sufficiently large t_1 ,

(2.17)
$$\limsup_{t \to \infty} \left[\frac{t \int_{t_1}^t \frac{\Delta s}{c(s)}}{p(t)} \int_t^\infty q(s) \frac{\tau(s)}{s} \Delta s \right] > \frac{1}{L}.$$

Then every solution x(t) of (1.10) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).

Proof. Suppose that (1.10) has a nonoscillatory solution x(t) on $[t_0, \infty)_{\mathbb{T}}$. We may assume without loss of generality that x(t) > 0 and $x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}, t_1 \in [t_0, \infty)_{\mathbb{T}}$. By Lemma 2.1 there are two possible cases. If case (1) holds, then there exists $\alpha > 0$ such that $x(t) \ge \alpha, t \in [t_1, \infty)_{\mathbb{T}}$.

Let
$$y(t) = c(t)(p(t)(x^{\Delta}(t)))^{\Delta}$$
. Then $y(t) > 0, t \in [t_1, \infty)_{\mathbb{T}}$. For $T > t$,
 $y(T) = y(t) + \int_t^T y^{\Delta}(s)\Delta s \le y(t) - L \int_t^T q(s)x(\tau(s))\Delta s$,

and

$$L\int_t^T q(s)x(\tau(s))\Delta s \le y(t) - y(T) \le y(t) = c(t)(p(t)x^{\Delta}(t))^{\Delta}.$$

In view of Lemma 2.2 and Lemma 2.3, we have

$$\frac{p(t)}{t\int_{t_1}^t \frac{\Delta s}{c(s)}} x(t) \ge x^{\Delta}(t) \frac{p(t)}{\int_{t_1}^t \frac{\Delta s}{c(s)}} \ge c(t)(p(t)x^{\Delta}(t))^{\Delta} \ge Lx(t) \int_t^{\infty} q(s)\frac{\tau(s)}{s} \Delta s,$$

thus

$$\limsup_{t \to \infty} \left[\frac{t \int_{t_1}^t \frac{\Delta s}{c(s)}}{p(t)} \int_t^\infty q(s) \frac{\tau(s)}{s} \Delta s \right] \le \frac{1}{L},$$

which is a contradiction of (2.17). Hence, case (1) is impossible. If case (2) holds, then as before, $\lim_{t\to\infty} x(t)$ exists (finite). This completes the proof. \Box

From Theorem 2.4 and Lemma 2.4, we get the following result.

Corollary 2.5. Assume (2.1), (2.3), (2.6) and $p^{\Delta}(t) \leq 0$ hold, and (2.17) holds for all sufficiently large t_1 . Then every solution x(t) of (1.10) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Theorem 2.5. Assume (2.1) and (2.3) hold. $\tau(t) < t$, $p^{\Delta}(t) \leq 0$, furthermore, x(t) is a solution of (1.10) which satisfies case (1) in Lemma 2.1. Then for sufficiently large t_1 , there is $\beta \geq 1$, where

(2.18)
$$\beta = \lim_{t_0 \to \infty} \sup_{t > t_0} \sup_{\lambda \in E} \left\{ \lambda \exp\{\int_{\tau(t)}^t \zeta_{\mu(s)}(-\lambda Lq(s)\tau(s)\frac{\int_{t_1}^{\tau(s)} \frac{\Delta u}{c(u)}}{p(\tau(s))})\Delta s\} \right\},$$

(2.19)
$$E = \{\lambda : \lambda > 0, 1 - \lambda Lq(t)\tau(t) \frac{\int_{t_1}^{\tau(t)} \frac{\Delta s}{c(s)}}{p(\tau(t))} \mu(t) > 0\}.$$

Proof. We assume that x(t) > 0 and $x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}, t_1 \in [t_0, \infty)_{\mathbb{T}}$. In view of Lemma 2.2 and Lemma 2.3, there exists $t_2 \ge t_1$, we have that for $t \ge t_2$,

$$\begin{aligned} (c(t)(p(t)x^{\Delta}(t))^{\Delta})^{\Delta} &\leq -Lq(t)x(\tau(t)) \leq -Lq(t)\tau(t)x^{\Delta}(\tau(t)) \\ &\leq -Lq(t)\tau(t)\frac{\int_{t_1}^{\tau(t)}\frac{\Delta s}{c(s)}}{p(\tau(t))}c(\tau(t))(p(\tau(t))x^{\Delta}(\tau(t)))^{\Delta}. \end{aligned}$$

Let $y(t) = c(t)(p(t)x^{\Delta}(t))^{\Delta}$. Then $y^{\Delta}(t) \leq -Lq(t)\tau(t)\frac{\int_{t_1}^{\tau(t)}\frac{\Delta s}{c(s)}}{p(\tau(t))}y(\tau(t))$. Hence, y(t) is an eventually positive solution for the inequality

(2.20)
$$y^{\Delta}(t) + Lq(t)\tau(t)\frac{\int_{t_1}^{\tau(t)}\frac{\Delta s}{c(s)}}{p(\tau(t))}y(\tau(t)) \le 0,$$

by Lemma 2.5, we obtain the desired contradiction. This completes the proof. $\hfill \Box$

From Theorem 2.5, we have the following results.

Theorem 2.6. Assume (2.1) and (2.3) hold. $\tau(t) < t$, $p^{\Delta}(t) \leq 0$, and for some sufficiently large t_1 , $\beta < 1$, then every solution x(t) of (1.10) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).

Theorem 2.7. Assume (2.1) and (2.3) hold. $\tau(t) < t, p^{\Delta}(t) \leq 0$, and for all sufficiently large t_1 , such that

(2.21)
$$\int_{\tau(t)}^{t} q(s)\tau(s) \frac{\int_{t_1}^{\tau(s)} \frac{\Delta u}{c(u)}}{p(\tau(s))} \Delta s > \frac{1}{L}.$$

Then every solution x(t) of (1.10) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).

Proof. Suppose that (1.10) has a nonoscillatory solution x(t) on $[t_0, \infty)_{\mathbb{T}}$. We may assume without loss of generality that x(t) > 0 and $x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}, t_1 \in [t_0, \infty)_{\mathbb{T}}$. By Lemma 2.1, there are two possible cases. If case (1) holds, then from Theorem 2.5, we have (2.20) holds. By the definition of y(t) in Theorem 2.5, we obtain that for sufficiently large t,

(2.22)
$$0 \ge y(t) + y(\tau(t)) \left[\int_{\tau(t)}^{t} Lq(s)\tau(s) \frac{\int_{t_1}^{\tau(s)} \frac{\Delta u}{c(u)}}{p(\tau(s))} \Delta s - 1 \right] > 0,$$

this is a contradiction. Hence, case (1) is not possible. If case (2) holds, then clearly $\lim_{t\to\infty} x(t)$ exists (finite). This completes the proof.

3. Applications and examples

In this section, we give some examples to illustrate our main results.

Example 3.1 Consider the third order delay dynamic equation

(3.1)
$$\left(\frac{1}{t}x^{\Delta}(t)\right)^{\Delta\Delta} + \frac{\beta}{t\tau(t)}x(\tau(t)) = 0, \quad t \in [1,\infty)_{\mathbb{T}},$$

where

$$c(t) = 1, \ p(t) = \frac{1}{t}, \ q(t) = \frac{\beta}{t\tau(t)}, \ L = 1, \ \tau(t) < t, \ \beta > 0.$$

Clearly (2.1) holds. $p^{\Delta}(t) < 0$, $\int_{t_0}^{\infty} \tau(t)q(t)\Delta t = \int_{t_0}^{\infty} \frac{\beta}{t}\Delta t = \infty$, so (2.3) holds. Let $\delta(t) = t$. From Corollary 2.1, we have

$$\limsup_{t \to \infty} \int_{t_1}^t \left(\tau(s)q(s) - \frac{p(s)}{4Ls \int_{t_1}^s \frac{\Delta u}{c(u)}} \right) \Delta s$$
$$= \limsup_{t \to \infty} \int_{t_1}^t \left[\frac{\beta}{s} - \frac{1}{4s^2(s - t_1)} \right] \Delta s = \infty.$$

It is easy to find that

$$\int_{t_0}^{\infty} \frac{1}{p(v)} \int_{v}^{\infty} \frac{1}{c(u)} \int_{u}^{\infty} q(s) \Delta s \Delta u \Delta v = \infty.$$

Hence, by corollary 2.2, every solution x(t) of equation (3.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Remark 3.1. The important point to note here is that the recent results due to Hassan [15] do not apply to Eq.(3.1) for the condition $\tau(\sigma(t)) = \sigma(\tau(t))$ can be a restrictive condition.

Example 3.2 Consider the third order delay dynamic equation

(3.2)
$$(tx^{\Delta\Delta}(t))^{\Delta} + \frac{1}{t}x(\tau(t)) = 0,$$

where

$$\mathbb{T} = q^{\mathbb{N}}, \ c(t) = t, \ p(t) = 1, \ q(t) = \frac{1}{t}, \ \tau(t) = t - q^{n_0}, \ p^{\Delta}(t) = 0.$$

Clearly (2.1), (2.3) and (2.6) hold. By Theorem 2.3, we have

$$\limsup_{t \to \infty} \int_{t_0}^t q(s) \frac{\tau(s)}{s} \Delta s = \infty,$$

then by Corollary 2.4, every solution x(t) of the equation (3.2) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Remark 3.2. The important point to note here is that the recent results due to Hassan [15] do not apply to Eq. (3.2) for the condition $\tau(\sigma(t)) = \sigma(\tau(t))$ does not hold (see Saker et al. [19]).

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