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ON VAGUE FILTERS IN BE-ALGEBRAS

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Abstract. In this paper, we introduce the notion of a vague filter in BE-algebras, and investigate some properties of them. Also we give conditions for a vague set to be a vague filter, and we characterize vague filters in BE-algebras.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([6, 7]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [4, 5], Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([16]) introduced the notion of d-algebras which is another generalization of BCK-algebras, and also they introduced the notion of B-algebras ([17, 18]), i.e., (I) $x * x = 0$; (II) $x * 0 = x$; (III) $(x * y) * z = x * (z * (0 * y))$, for any $x, y, z \in X$, which is equivalent in some sense to the groups. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim ([9]) introduced a new notion, called an BH-*algebra*, which is a generalization of $BCH/BCI/BCK$-algebras, i.e., (I); (II) and (IV) $x * y = 0$ and $y * x = 0$ imply $x = y$ for any $x, y \in X$. A. Walendziak obtained the another equivalent axioms for B-algebra ([20]). H. S. Kim, Y. H. Kim and J. Neggers ([12]) introduced the notion a (pre-) Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. C. B. Kim and H. S. Kim ([10]) introduced the notion of a BM-algebra which is a specialization of B-algebras. They proved that the class of BM-algebras is a proper subclass of B-algebras and also showed that a BM-algebra is equivalent to a 0-commutative B-algebra. In [11], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a generalization of a BCK-algebra. Using the notion of upper sets they gave an equivalent condition of the filter in BE-algebras.

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2. Preliminaries

We recall some definitions and results discussed in [11].

Definition 2.1. An algebra $(X; *, 1)$ of type (2, 0) is called a *BE-algebra* if

(BE1) $x * x = 1$ for all $x \in X$;
(BE2) $x * 1 = 1$ for all $x \in X$;
(BE3) $1 * x = x$ for all $x \in X$;
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*).

We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$. A non-empty subset S of X is said to be a *subalgebra* of a BE-algebra X if it is closed under the operation "$*$". Noticing that $x * x = 1$ for all $x \in X$, it is clear that $1 \in S$.

Proposition 2.2. *If $(X; *, 1)$ is a BE-algebra, then $x * (y * x) = 1$ for any $x, y \in X$.*

Example 2.3. Let $X := \{1, a, b, c, d, 0\}$ be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra.

Definition 2.4. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is said to be a *filter* of X if

(F1) $1 \in F$;
(F2) $x * y \in F$ and $x \in F$ imply $y \in F$.

In Example 2.3, $F_1 := \{1, a, b\}$ is a filter of X, but $F_2 := \{1, a\}$ is not a filter of X, since $a * b \in F_2$ and $a \in F_2$, but $b \notin F_2$.

Proposition 2.5. *Let $(X; *, 1)$ be a BE-algebra and let F be a filter of X. If $x \leq y$ and $x \in F$ for any $y \in X$, then $y \in F$.*

3. Basic results on vague sets

Definition 3.1 ([3])**.** A *vague set A* in the universe of discourse U is characterized by two membership functions given by:

(1) A truth membership function

$$t_A : U \to [0,1],$$

and

(2) A false membership function

$$f_A : U \to [0,1],$$

where $t_A(u)$ is a lower bound of the grade of membership of u derived from the "evidence for u", and $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and

$$t_A(u) + f_A(u) \leq 1.$$

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0,1]$. This indicates that if the actual grade of membership is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{\langle u, [t_A(u), f_A(u)]\rangle | u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A and is denoted by $V_A(u)$.

Definition 3.2 ([3])**.** A vague set A of a set U is called

(1) the *zero vague set* of U if $t_A(u) = 0$ and $f_A(u) = 1$ for all $u \in U$,
(2) the *unit vague set* of U if $t_A(u) = 1$ and $f_A(u) = 0$ for all $u \in U$.
(3) the *α-vague set* of U if $t_A(u) = \alpha$ and $f_A(u) = 1 - \alpha$ where $\alpha \in (0,1)$.

For $\alpha, \beta \in [0,1]$ we now define (α, β)-cut and α-cut of a vague set.

Definition 3.3 ([3])**.** Let A be a vague set of a universe X with the true-membership function t_A and the false-membership function f_A. The (α, β)-cut of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by

$$A_{(\alpha,\beta)} = \{x \in X | V_A(x) \geq [\alpha, \beta]\}.$$

Clearly $A_{(0,0)} = X$. The (α, β)-cuts are also called *vague-cuts* of the vague set A.

Definition 3.4 ([3])**.** The α-cut of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$, and if $\alpha \leq \beta$, then $A_\beta \subseteq A_\alpha$ and $A_{(\alpha,\beta)} = A_\alpha$. Equivalently, we can define the α-cut as

$$A_\alpha = \{x \in X | t_A(x) \geq \alpha\}.$$

For our discussion, we shall use the following notations, which are given in [3], on interval arithmetic.

Notation. Let $I[0,1]$ denote the family of all closed subintervals of $[0,1]$. If $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are two elements of $I[0,1]$, we call $I_1 \geq I_2$ if $a_1 \geq a_2$ and $b_1 \geq b_2$. Similarly, we understand the relations $I_1 \leq I_2$ and $I_1 = I_2$. Clearly the relation $I_1 \geq I_2$ does not necessarily imply that $I_1 \supseteq I_2$ and conversely. We define the term "imax" to mean the maximum of two intervals as

$$\mathrm{imax}(I_1, I_2) = [\max(a_1, a_2), \max(b_1, b_2)].$$

Similarly, we define "imin". The concept of "imax" and "imin" could be extended to define "isup" and "iinf" of infinite number of elements of $I[0,1]$. It is obvious that $L = \{I[0,1], \mathrm{isup}, \mathrm{iinf}, \leq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$.

4. Vague filters

In what follows let X be a BE-algebra unless otherwise specified.

Definition 4.1. A vague set A of X is called a *vague filter* of X if the following conditions are true:

(c1) $(\forall x \in X)\ (V_A(1) \geq V_A(x))$,
(c2) $(\forall x, y \in X)\ (V_A(y) \geq \mathrm{imin}\{V_A(x * y), V_A(x)\})$,

that is,

$$t_A(1) \geq t_A(x), 1 - f_A(1) \geq 1 - f_A(x) \tag{4.1}$$

and

$$\begin{aligned} t_A(y) &\geq \min\{t_A(x * y), t_A(x)\}, \\ 1 - f_A(y) &\geq \min\{1 - f_A(x * y), 1 - f_A(x)\} \end{aligned} \tag{4.2}$$

for all $x, y \in X$.

Let us illustrate this definition using the following examples.

Example 4.2. Let $X := \{0, a, b, c\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Let A be a vague set in X defined as follows:

$$A := \{\langle 1, [0.7, 0.2]\rangle, \langle a, [0.5, 0.3]\rangle, \langle b, [0.5, 0.3]\rangle, \langle c, [0.7, 0.2]\rangle\}.$$

It is routine to verify that A is a vague filter of X.

Proposition 4.3. *Every vague filter A of X satisfies*:

$$(\forall x, y \in X)(x \leq y \Rightarrow V_A(x) \leq V_A(y)). \tag{4.3}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$ and so

$$\begin{aligned} t_A(y) \geq & \min\{t_A(x * y), t_A(x)\} = \min\{t_A(1), t_A(x)\} = t_A(x), \\ 1 - f_A(y) \geq & \min\{1 - f_A(x * y), 1 - f_A(x)\} = 1 - f_A(x). \end{aligned}$$

This shows that $V_A(y) \geq V_A(x)$. □

Proposition 4.4. *Every vague filter A of X satisfies*:

$$(\forall x, y, z \in X)(V_A(x * z) \geq \mathrm{imin}\{V_A(x * (y * z)), V_A(y)\}). \tag{4.4}$$

Proof. Using (c2) and (BE4), we have

$$\begin{aligned} V_A(x * z) &\geq \mathrm{imin}\{V_A(y * (x * z)), V_A(y)\} \\ &= \mathrm{imin}\{V_A(x * (y * z)), V_A(y)\} \end{aligned}$$

for all $x, y, z \in X$. □

Theorem 4.5. *If A is a vague set in X satisfying* (c1) *and* (4.4)*, then A is a vague filter of X.*

Proof. Taking $x := 1$ in (4.4) and using (BE3), we have

$$\begin{aligned} V_A(z) &= V_A(1 * z) \\ &\geq \mathrm{imin}\{V_A(1 * (y * z)), V_A(y)\} \\ &= \mathrm{imin}\{V_A(y * z), V_A(y)\} \end{aligned}$$

for all $y, z \in X$. Hence A is a vague filter of X. □

Corollary 4.6. *Let A be a vague set in X. Then A is a vague filter of X if and only if it satisfies* (c1) *and* (4.4).

Theorem 4.7. *Let A be a vague set in X. Then A is a vague filter of X if and only if it satisfies the following conditions*:

$$(\forall x, y \in X)(V_A(y * x) \geq V_A(x)), \tag{4.5}$$

$$(\forall x, a, b \in X)(V_A((a * (b * x)) * x) \geq \mathrm{imin}\{V_A(a), V_A(b)\}). \tag{4.6}$$

Proof. Assume that A is a vague filter of X. Using (c2), Proposition 2.2, and (c1), we get

$$\begin{aligned} V_A(y * x) &\geq \mathrm{imin}\{V_A(x * (y * x)), V_A(x)\} \\ &= \mathrm{imin}\{V_A(1), V_A(x)\} = V_A(x) \end{aligned}$$

for all $x, y \in X$.

$$\begin{aligned} V_A((a * (b * x)) * x) &\geq \mathrm{imin}\{V_A((a * (b * x)) * (b * x)), V_A(b)\} \\ &\geq \mathrm{imin}\{V_A(a), V_A(b)\}. \end{aligned}$$

Conversely, let A be a vague set in X satisfying conditions (4.5) and (4.6). If we take $y := x$ in (4.5), then $V_A(1) = V_A(x * x) \geq V_A(x)$ for all $x \in X$. Using (4.6), we obtain

$$\begin{aligned} V_A(y) &= V_A(1 * y) \\ &= V_A(((x * y) * (x * y)) * y) \\ &\geq \operatorname{imin}\{V_A(x * y), V_A(x)\} \end{aligned}$$

for all $x, y \in X$. Hence A is a vague filter of X. □

Proposition 4.8. *Let A be a vague set in X. Then A is a vague filter of X if and only if it satisfies*:

$$(4.7) \qquad (\forall x, y, z \in X)(z \leq x * y \Rightarrow V_A(y) \geq \operatorname{imin}\{V_A(x), V_A(z)\}).$$

Proof. Assume that A is a vague filter of X. Let $x, y, z \in X$ be such that $z \leq x * y$. By Proposition 4.3 and (c2), we have

$$\begin{aligned} V_A(y) &\geq \operatorname{imin}\{V_A(x * y), V_A(x)\} \\ &\geq \operatorname{imin}\{V_A(z), V_A(x)\}. \end{aligned}$$

Conversely, suppose that A satisfies (4.7). By (BE2), we have $x \leq x * 1 = 1$. Hence $V_A(1) \geq \operatorname{imin}\{V_A(x), V_A(x)\} = V_A(x)$ by (4.7). Thus (c1) is valid. Using (BE1) and (BE4), we obtain $x \leq (x * y) * y$ for all $x, y \in X$. It follows from (4.7) that $V_A(y) \geq \operatorname{imin}\{V_A(x * y), V_A(x)\}$. Hence (c2) holds. Therefore A is a vague filter of X. □

As a generalization of Proposition 4.8, we have the following results.

Theorem 4.9. *If a vague set A in X is a vague filter of X, then*

$$(4.8) \qquad \prod_{i=1}^{n} w_i * x = 1 \Rightarrow V_A(x) \geq \operatorname{imin}\{V_A(w_i) | i = 1, \ldots, n\}$$

for all $x, w_1, \ldots, w_n \in X$, where $\prod_{i=1}^{n} w_i * x = w_n * (w_{n-1} * (\cdots * (w_1 * x) \cdots))$.

Proof. The proof is by induction on n. Let A be a vague filter of X. By Proposition 4.3 and (4.7), we know that the condition (4.8) is valid for $n = 1, 2$. Assume that A satisfies the condition (4.8) for $n = k$, i.e.,

$$\prod_{i=1}^{k} w_i * x = 1 \Rightarrow V_A(x) \geq \operatorname{imin}\{V_A(w_i) | i = 1, \ldots, k\}$$

for all $x, w_1, \ldots, w_k \in X$. Let $x, w_1, \ldots, w_k, w_{k+1} \in X$ be such that $\prod_{i=1}^{k+1} w_i * x = 1$. Then

$$V_A(w_1 * x) \geq \operatorname{imin}\{V_A(w_j) | j = 2, \ldots, k+1\}.$$

Since A is a vague filter of X, it follows from (c2) that

$$\begin{aligned} V_A(x) &\geq \operatorname{imin}\{V_A(w_1 * x), V_A(w_1))\} \\ &\geq \operatorname{imin}\{V_A(w_1), \{V_A(w_j) | j = 2, \ldots, k+1\}\} \\ &= \operatorname{imin}\{V_A(w_j) | j = 1, \ldots, k+1\}. \end{aligned}$$

This completes the proof. □

Now we consider the converse of Theorem 4.9.

Theorem 4.10. *Let A be a vague set in X satisfying the condition* (4.8)*. Then A is a vague filter of X.*

Proof. Note that $\underbrace{1 * (1 * (1 * \cdots (1}_{n \text{ times}} * x)) \cdots) = x$. By (BE2), we have $x \leq x*1 = 1$. Hence $V_A(1) \geq V_A(x)$ for all $x \in X$. Thus (c1) is valid. Let $x, y, z \in X$ be such that $z \leq x * y$. Then

$$1 = z * (x * y) = z * (\underbrace{1 * \cdots (1 * (1}_{n-2 \text{ times}} * (x * y))) \cdots))$$

and so

$$\begin{aligned} V_A(y) &\geq \operatorname{imin}\{V_A(z), V_A(x), V_A(1)\} \\ &= \operatorname{imin}\{V_A(z), V_A(x)\}. \end{aligned}$$

Hence by Proposition 4.8, we conclude that A is a vague filter of X. □

Theorem 4.11. *Let A be a vague filter of X. Then for any $\alpha, \beta \in [0, 1]$, the vague-cut $A_{(\alpha,\beta)}$ is a crisp filter of X.*

Proof. Obviously, $1 \in A_{(\alpha,\beta)}$. Let $x, y \in X$ be such that $x \in A_{(\alpha,\beta)}$ and $x * y \in A_{(\alpha,\beta)}$. Then $V_A(x) \geq [\alpha, \beta]$, i.e., $t_A(x) \geq \alpha$ and $1 - f_A(x) \geq \beta$; and $V_A(x * y) \geq [\alpha, \beta]$, i.e., $t_A(x * y) \geq \alpha$ and $1 - f_A(x * y) \geq \beta$. It follows from (4.2) that

$$\begin{aligned} t_A(y) &\geq \min\{t_A(x * y), t_A(x)\} \geq \alpha, \\ 1 - f_A(y) &\geq \min\{1 - f_A(x * y), 1 - f_A(y)\} \geq \beta \end{aligned}$$

so that $V_A(y) \geq [\alpha, \beta]$. Hence $y \in A_{(\alpha,\beta)}$ and so $A_{(\alpha,\beta)}$ is a filter of X. □

The filter like $A_{(\alpha,\beta)}$ are also called *vague-cut filters* of X. Clearly we have the following results.

Proposition 4.12. *Let A be a vague filter of X. Two vague-cut filters $A_{(\alpha,\beta)}$ and $A_{(\omega,\gamma)}$ with $[\alpha, \beta] < [\omega, \gamma]$ are equal if and only if there is no $x \in X$ such that*

$$[\alpha, \beta] \leq V_A(x) \leq [\omega, \gamma].$$

Theorem 4.13. *Let X be a finite BE-algebra and let A be a vague filter of X. Consider the set $V(A)$ given by*

$$V(A) := \{V_A(x) | x \in X\}.$$

Then A_i are the only vague-cut filters of X, where $A_i \in V(A)$.

Proof. Consider $[a_1, a_2] \in I[0,1]$ where $[a_1, a_2] \notin V(A)$. If $[\alpha, \beta] < [a_1, a_2] < [\omega, \gamma]$ where $[\alpha, \beta], [\omega, \gamma] \in V(A)$, then $A_{(\alpha,\beta)} = A_{(a_1,a_2)} = A_{(\omega,\gamma)}$. If $[a_1, a_2] < [a_1, a_3]$ where $[a_1, a_3] = \text{imin}\{V_A(x) | x \in X\}$, then $A_{(a_1,a_3)} = X = A_{(a_1,a_2)}$. Hence for any $[a_1, a_2] \in I[0,1]$, the vague-cut filter $A_{(a_1,a_2)}$ is one of $A_i \in V(A)$. This competes the proof. □

Theorem 4.14. *Any filter F of X is a vague-cut filter of some vague filter of X.*

Proof. Consider the vague set A of X given by

$$V_A = \begin{cases} [\alpha, \alpha] & \text{if } x \in F \\ [0, 0] & \text{if } x \notin F, \end{cases}$$

where $\alpha \in (0,1)$. Since $1 \in F$, we have $V_A(1) = [\alpha, \alpha] \geq V_A(x)$ for all $x \in X$. Let $x, y \in X$. If $y \in F$, then

$$V_A(y) = [\alpha, \alpha] \geq \text{imin}\{V_A(x * y), V_A(x)\}.$$

Assume that $y \notin F$. Then $x \notin F$ or $x * y \notin F$. It follows that

$$V_A(y) = [0, 0] = \text{imin}\{V_A(x * y), V_A(x)\}.$$

Thus A is a vague filter of X. Clearly $F = A_{(\alpha,\alpha)}$. □

Theorem 4.15. *Let A be a vague filter of X. Then the set*

$$F := \{x \in X | V_A(x) = V_A(1)\}$$

is a crisp filter of X.

Proof. Obviously $1 \in F$. Let $x, y \in X$ be such that $x * y \in F$ and $x \in F$. Then $V_A(x * y) = V_A(1) = V_A(x)$, and so

$$V_A(y) \geq \text{imin}\{V_A(x * y), V_A(x)\} = V_A(1)$$

by (c2). Since $V_A(1) \geq V_A(y)$ for all $y \in X$, it follows that $V_A(y) = V_A(1)$ and so that $y \in F$. Therefore F is a crisp filter of X. □

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