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HYPER K-SUBALGEBRAS BASED ON FUZZY POINTS

Min Su Kang

Abstract. Generalizations of the notion of fuzzy hyper K-subalgebras are considered. The concept of fuzzy hyper K-subalgebras of type (α, β) where α, $\beta \in \{\in, \mathrm{q}, \in \vee\, \mathrm{q}, \in \wedge\, \mathrm{q}\}$ and $\alpha \neq\, \in \wedge\, \mathrm{q}$. Relations between each types are investigated, and many related properties are discussed. In particular, the notion of $(\in, \in \vee\, \mathrm{q})$-fuzzy hyper K-subalgebras is dealt with, and characterizations of $(\in, \in \vee\, \mathrm{q})$-fuzzy hyper K-subalgebras are established. Conditions for an $(\in, \in \vee\, \mathrm{q})$-fuzzy hyper K-subalgebra to be an $(\in, \in)$-fuzzy hyper K-subalgebra are provided. An $(\in, \in \vee\, \mathrm{q})$-fuzzy hyper K-subalgebra by using a collection of hyper K-subalgebras is established. Finally the implication-based fuzzy hyper K-subalgebras are discussed.

1. Introduction

The hyperstructure theory (called also multialgebras) is introduced in 1934 by F. Marty [13] at the 8th Congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia, Japan and Iran. Hyperstructures have many applications to several sectors of both pure and applied sciences. Y. B. Jun et al. [12] introduced and studied hyper BCK-algebra which is a generalization of a BCK-algebra. In [3] and [5], R. A. Borzooei et al. constructed the hyper K-algebras, and studied (weak) implicative hyper K-ideals in hyper K-algebras. In [9] and [10] Jun and Shim studied the fuzzy (implicative) hyper K-ideals in hyper K-algebras. Jun [6] introduced the concept of fuzzy hyper K-subalgebras and investigated some related properties. Murali [14] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [15], played a vital role to generate some different types of fuzzy subsets. It is worth pointing out that Bhakat and Das [1, 2] initiated the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point

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and a fuzzy subgroup, and introduced the concept of an $(\in, \in\vee\mathrm{q})$-fuzzy subgroup. In particular, an $(\in, \in\vee\mathrm{q})$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, Jun and Song [11] discussed general forms of fuzzy interior ideals in semigroups. Also, Jun [7, 8] introduced the concept of (α, β)-fuzzy subalgebra of a BCK/BCI-algebra and investigated related results. As a generalization of the notion of fuzzy hyper K-subalgebras, in this chapter, we introduce the concept of fuzzy hyper K-subalgebras of type (α, β) where $\alpha,\ \beta \in \{\in,\ \mathrm{q},\ \in\vee\mathrm{q},\ \in\wedge\mathrm{q}\}$ and $\alpha \neq\ \in\wedge\mathrm{q}$. We investigate relations between each types, and discuss many related properties. In particular, we deal with the notion of $(\in,\ \in\vee\mathrm{q})$-fuzzy hyper K-subalgebras. We consider characterizations of $(\in,\ \in\vee\mathrm{q})$-fuzzy hyper K-subalgebras. We provide conditions for an $(\in,\ \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra to be an $(\in,\ \in)$-fuzzy hyper K-subalgebra. We make an $(\in,\ \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra by using a collection of hyper K-subalgebras. We finally discuss the implication-based fuzzy hyper K-subalgebras.

2. Preliminaries

In [5], Borzoei et al. established the notion of hyper I-algebras/hyper K-algebras as follows: By a *hyper I-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HI1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(HI2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HI3) $x < x$,
(HI4) $x < y$ and $y < x$ imply $x = y$

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$ and $\exists b \in B$ such that $a < b$. If a hyper I-algebra $(H, \circ, 0)$ satisfies

(HI5) $0 < x$ for all $x \in H$,

then $(H, \circ, 0)$ is called a *hyper K-algebra*.

Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets A and B of H the following hold (see [4] and [5, Proposition 3.6]):

(a1) $x \in x \circ 0$,
(a2) $x \circ y < x$,
(a3) $A \circ B < A$,
(a4) $A \circ A < A$,
(a5) $0 \in x \circ (x \circ 0)$,
(a6) $x < x \circ 0$,
(a7) $A < A \circ 0$,
(a8) $A < A \circ B$ if $0 \in B$.

Definition 2.1 ([5])**.** Let $(H, \circ, 0)$ be a hyper K-algebra and let S be a subset of H containing 0. If S is a hyper K-algebra with respect to the hyperoperation "$\circ$" on H, we say that S is a *hyper K-subalgebra* of H.

Lemma 2.2 ([5])**.** *Let S be a nonempty subset of a hyper K-algebra* $(H, \circ, 0)$. *Then S is a hyper K-subalgebra of H if and only if $x \circ y \subseteq S$ for all $x, y \in S$.*

A fuzzy subset μ of a set X of the form

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [15] introduced the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$. To say that $x_t \in \mu$ (resp. $x_t \mathrm{q} \mu$), we mean $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ. To say that $x_t \in\vee\mathrm{q}\, \mu$ (resp. $x_t \in\wedge\mathrm{q}\, \mu$), we mean $x_t \in \mu$ or $x_t \mathrm{q} \mu$ (resp. $x_t \in \mu$ and $x_t \mathrm{q} \mu$). For $\alpha \in \{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$, we say that $x_t \overline{\alpha} \mu$ if $x_t \alpha \mu$ does not hold.

3. Fuzzy hyper K-subalgebras of type (α, β)

Definition 3.1 ([6])**.** A fuzzy set μ in a hyper K-algebra H is called a *fuzzy hyper K-subalgebra* of H if it satisfies:

$$(\forall x, y \in H) \left(\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\} \right). \tag{3.1}$$

Theorem 3.2. *For a fuzzy set μ in a hyper K-algebra H, the condition* (3.1) *is equivalent to the following condition, for any $x, y \in H$ and $t_1, t_2 \in (0,1]$,*

$$x_{t_1} \in \mu,\ y_{t_2} \in \mu \implies z_{\min\{t_1, t_2\}} \in \mu \text{ for all } z \in x \circ y. \tag{3.2}$$

Proof. Assume that (3.1) is valid and let $x, y \in H$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$, which imply from (3.1) that

$$\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\} \geq \min\{t_1, t_2\}.$$

Thus $\mu(z) \geq \min\{t_1, t_2\}$ for all $z \in x \circ y$. This shows that $z_{\min\{t_1, t_2\}} \in \mu$ for all $z \in x \circ y$.

Conversely suppose that (3.2) holds. Note that $x_{\mu(x)} \in \mu$ and $y_{\mu(y)} \in \mu$ for all $x, y \in H$. It follows from (3.2) that $z_{\min\{\mu(x), \mu(y)\}} \in \mu$ for all $z \in x \circ y$, i.e., $\mu(z) \geq \min\{\mu(x), \mu(y)\}$ for all $z \in x \circ y$. This induces the condition (3.1). $\square$

Definition 3.3. A fuzzy set μ in a hyper K-algebra H is called a *fuzzy hyper K-subalgebra in H of type* (α, β), or briefly, an (α, β)*-fuzzy hyper K-subalgebra* of H, where $\alpha \neq {\in\wedge\mathrm{q}}$, if for all $x, y \in H$ and $t_1, t_2 \in (0,1]$,

$$x_{t_1} \alpha \mu,\ y_{t_2} \alpha \mu \implies z_{\min\{t_1, t_2\}} \beta \mu \text{ for all } z \in x \circ y. \tag{3.3}$$

Table 1. Hyper operation for H

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

Let μ be a fuzzy set in a hyper K-algebra H such that $\mu(x) \leq 0.5$ for all $x \in H$. Let $x \in H$ and $t \in (0,1]$ be such that $x_t \in\!\wedge\, \mathrm{q}\, \mu$. Then $\mu(x) \geq t$ and $\mu(x) + t > 1$. It follows that

$$1 < \mu(x) + t \leq \mu(x) + \mu(x) = 2\,\mu(x)$$

so that $\mu(x) > 0.5$. This means that $\{x_t \mid x_t \in\!\wedge\, \mathrm{q}\, \mu\} = \emptyset$. Therefore the case $\alpha = \in\!\wedge\, \mathrm{q}$ in Definition 3.3 will be omitted.

To consider the notion of fuzzy hyper K-subalgebra of type $(\in\!\wedge\, \mathrm{q}\,, \beta)$ where $\beta \in \{\in, \mathrm{q}, \in\!\vee\, \mathrm{q}\,, \in\!\wedge\, \mathrm{q}\,\}$, we should take a fuzzy set μ in H satisfying $\mu(x) > 0.5$ for some $x \in H$.

Definition 3.4. Let μ be a fuzzy set in H such that $\mu(x) > 0.5$ for some $x \in H$. Then μ is called a *fuzzy hyper K-subalgebra in H of type* $(\in\!\wedge\, \mathrm{q}\,, \beta)$, or briefly, an $(\in\!\wedge\, \mathrm{q}\,, \beta)$*-fuzzy hyper K-subalgebra* of H if for all $x, y \in H$ and $t_1, t_2 \in (0,1]$,

$$x_{t_1} \in\!\wedge\, \mathrm{q}\, \mu,\ y_{t_2} \in\!\wedge\, \mathrm{q}\, \mu \implies z_{\min\{t_1,t_2\}} \beta \mu \text{ for all } z \in x \circ y. \tag{3.4}$$

According to Theorem 3.2, we know that the notion of fuzzy hyper K-subalgebras coincide with the notion of $(\in, \in)$-fuzzy hyper K-subalgebras.

Lemma 3.5 ([6])**.** *Let μ be a fuzzy set in a hyper K-algebra H. Then μ is an $(\in, \in)$-fuzzy hyper K-subalgebra of H if and only if its nonempty t-level set $U(\mu; t)$ is a hyper K-subalgebra of H for all $t \in [0,1]$.*

Theorem 3.6. *Every $(\in, \in)$-fuzzy hyper K-subalgebra is an $(\in, \in\!\vee\, \mathrm{q}\,)$-fuzzy hyper K-subalgebra.*

Proof. Straightforward. □

The following example shows that the converse of Theorem 3.6 may not be true in general.

Example 3.7. Consider a hyper K-algebra $H = \{0, a, b\}$ with a hyper operation "$\circ$" which is given by Table 1.

Let μ be a fuzzy set in H defined by

$$\mu = \begin{pmatrix} 0 & a & b \\ 0.5 & 0.7 & 0.2 \end{pmatrix}.$$

It is easy to check that μ is an $(\in, \in\!\vee\, \mathrm{q}\,)$-fuzzy hyper K-subalgebra of H. But it is not an $(\in, \in)$-fuzzy hyper K-subalgebra of H since $U(\mu; 0.6) = \{a\}$ is not a hyper K-subalgebra of H.

TABLE 2. Hyper operation for H

$\circ$	0	a	b	x	y
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{0,b\}$	$\{0,b\}$	$\{b\}$	$\{b\}$
x	$\{x\}$	$\{x\}$	$\{x\}$	$\{0,x\}$	$\{x\}$
y	$\{y\}$	$\{y\}$	$\{y\}$	$\{y\}$	$\{0\}$

Theorem 3.8. *Every* $(\in\vee\mathrm{q}\,,\,\in\vee\mathrm{q}\,)$*-fuzzy hyper* K*-subalgebra of a hyper* K*-algebra* H *is an* $(\in,\,\in\vee\mathrm{q}\,)$*-fuzzy hyper* K*-subalgebra of* H.

Proof. Let μ be an $(\in\vee\mathrm{q}\,,\in\vee\mathrm{q}\,)$-fuzzy hyper K-subalgebra of a hyper K-algebra H. Let $x,y\in H$ and $t_1,t_2\in(0,1]$ be such that $x_{t_1}\in\mu$ and $y_{t_2}\in\mu$. Then $x_{t_1}\in\vee\mathrm{q}\,\mu$ and $y_{t_2}\in\vee\mathrm{q}\,\mu$, which imply that $z_{\min\{t_1,t_2\}}\in\vee\mathrm{q}\,\mu$ for all $z\in x\circ y$ by (3.3). Hence μ is an $(\in,\in\vee\mathrm{q}\,)$-fuzzy hyper K-subalgebra of H. $\square$

The following example shows that the converse of Theorem 3.8 may not be true in general.

Example 3.9. Let $H=\{0,a,b,x,y\}$ and define a hyper operation $\circ$ on H by Table 2. Then $(H,\circ,0)$ is a hyper K-algebra. Define a fuzzy set μ in H as follows:

$$\mu : H\to[0,1],\ h\mapsto\begin{cases}0.6 & \text{if } h=0,\\ 0.3 & \text{if } h=a,\\ 0.7 & \text{if } h=b,\\ 0.4 & \text{if } h=x,\\ 0.1 & \text{if } h=y.\end{cases}$$

Then μ is an $(\in,\,\in\vee\mathrm{q}\,)$-fuzzy hyper K-subalgebra of H. Note that $a\circ 0=\{a\}$, $a_{0.72}\in\vee\mathrm{q}\,\mu$ and $0_{0.5}\in\vee\mathrm{q}\,\mu$. But $a_{\min\{0.72,0.5\}}=a_{0.5}\,\overline{\in\vee\mathrm{q}}\,\mu$ since $\mu(a)=0.3<0.5$ and $\mu(a)+0.5=0.8<1$. Hence μ is not an $(\in\vee\mathrm{q}\,,\,\in\vee\mathrm{q}\,)$-fuzzy hyper K-subalgebra of H.

Theorem 3.10. *Every* $(\in\vee\mathrm{q}\,,\,\in\vee\mathrm{q}\,)$*-fuzzy hyper* K*-subalgebra of a hyper* K*-algebra* H *is a* $(\mathrm{q},\,\in\vee\mathrm{q}\,)$*-fuzzy hyper* K*-subalgebra of* H.

Proof. Let μ be an $(\in\vee\mathrm{q}\,,\in\vee\mathrm{q}\,)$-fuzzy hyper K-subalgebra of a hyper K-algebra H. Let $x,y\in H$ and $t_1,t_2\in(0,1]$ be such that $x_{t_1}\,\mathrm{q}\,\mu$ and $y_{t_2}\,\mathrm{q}\,\mu$. Then $x_{t_1}\in\vee\mathrm{q}\,\mu$ and $y_{t_2}\in\vee\mathrm{q}\,\mu$, which imply that $z_{\min\{t_1,t_2\}}\in\vee\mathrm{q}\,\mu$ for all $z\in x\circ y$ by (3.3). Hence μ is a $(\mathrm{q},\in\vee\mathrm{q}\,)$-fuzzy hyper K-subalgebra of H. $\square$

For a fuzzy set μ in a hyper K-algebra H, the *support* of μ is defined to be the set

$$\mathrm{Supp}(\mu):=\{x\in H\mid\mu(x)>0\}.$$

Proposition 3.11. *If* μ *is a non-zero* $(\in,\,\in\vee\mathrm{q}\,)$*-fuzzy hyper* K*-subalgebra of a hyper* K*-algebra* H, *then the support of* μ *contains the zero element* 0.

Proof. Assume that $0 \notin \mathrm{Supp}(\mu)$. Then $\mu(0) = 0$. Since μ is non-zero, there exists $x \in H$ such that $0 \neq \mu(x) := t$. Thus $x_t \in \mu$. It follows from (3.3) that $z_t \in\vee\,\mathrm{q}\,\mu$ for all $z \in x \circ x$. In particular, since $0 \in x \circ x$ for all $x \in H$, we also have $0_t \in\vee\,\mathrm{q}\,\mu$, i.e., $0_t \in \mu$ or $0_t\,\mathrm{q}\,\mu$. This is a contradiction, and so $0 \in \mathrm{Supp}(\mu)$. □

Corollary 3.12. *For any $\beta \in \{\in, \mathrm{q}, \in\wedge\,\mathrm{q}\}$, if μ is a non-zero $(\in,\ \beta)$-fuzzy hyper K-subalgebra of a hyper K-algebra H, then the support of μ contains the zero element* 0.

Proof. Straightforward. □

Corollary 3.13. *Let μ be a non-zero $(\alpha,\ \beta)$-fuzzy hyper K-subalgebra of a hyper K-algebra H. If*

$$(\alpha, \beta) \in \{(\in\vee\,\mathrm{q}\,, \in\vee\,\mathrm{q}\,), (\in\vee\,\mathrm{q}\,, \mathrm{q}), (\in\vee\,\mathrm{q}\,, \in), (\in\vee\,\mathrm{q}\,, \in\wedge\,\mathrm{q}\,)\},$$

then the support of μ contains the zero element 0.

Proof. Straightforward. □

Proposition 3.14. *Let μ be a fuzzy set in a hyper K-algebra H such that $\mu(x) > 0.5$ for some $x \in H$. If μ is a $(\mathrm{q},\ \in\vee\,\mathrm{q}\,)$-fuzzy hyper K-subalgebra of H, then the support of μ contains the zero element* 0.

Proof. Let μ be a $(\mathrm{q},\ \in\vee\,\mathrm{q}\,)$-fuzzy hyper K-subalgebra of H such that $t := \mu(x) > 0.5$ for some $x \in H$. Then $x_t\,\mathrm{q}\,\mu$, and so $z_t \in\vee\,\mathrm{q}\,\mu$ for all $z \in x \circ x$ by (3.3). Assume that $0 \notin \mathrm{Supp}(\mu)$. Then $\mu(0) = 0$. In particular, since $0 \in x \circ x$ for all $x \in H$, we also have $0_t \in\vee\,\mathrm{q}\,\mu$, i.e., $0_t \in \mu$ or $0_t\,\mathrm{q}\,\mu$. This is a contradiction, and so $0 \in \mathrm{Supp}(\mu)$. □

Corollary 3.15. *Let μ be a fuzzy set in a hyper K-algebra H such that $\mu(x) > 0.5$ for some $x \in H$. For any $\beta \in \{\in, \mathrm{q}, \in\wedge\,\mathrm{q}\}$, if μ is a $(\mathrm{q},\ \beta)$-fuzzy hyper K-subalgebra of a hyper K-algebra H, then the support of μ contains the zero element* 0.

Proof. Straightforward. □

Proposition 3.16. *Let μ be a fuzzy set in a hyper K-algebra H such that $\mu(x) > 0.5$ for some $x \in H$. If μ is an $(\in\wedge\,\mathrm{q}\,,\ \in\vee\,\mathrm{q}\,)$-fuzzy hyper K-subalgebra of H, then the support of μ contains the zero element* 0.

Proof. Let μ be an $(\in\wedge\,\mathrm{q}\,,\ \in\vee\,\mathrm{q}\,)$-fuzzy hyper K-subalgebra of H such that $t := \mu(x) > 0.5$ for some $x \in H$. Then $\mu(x) \geq t$ and $\mu(x) + t = 2t > 1$, and so $x_t \in\wedge\,\mathrm{q}\,\mu$. Using (3.3), we have $z_t \in\vee\,\mathrm{q}\,\mu$ for all $z \in x \circ x$. Suppose that $0 \notin \mathrm{Supp}(\mu)$. Then $\mu(0) = 0$. Note that $0 \in x \circ x$ for all $x \in H$. Thus $0_t \in\vee\,\mathrm{q}\,\mu$, that is, $0_t \in \mu$ or $0_t\,\mathrm{q}\,\mu$. This is impossible, and hence $0 \in \mathrm{Supp}(\mu)$. □

Corollary 3.17. *Let μ be a fuzzy set in a hyper K-algebra H such that $\mu(x) > 0.5$ for some $x \in H$. For any $\beta \in \{\in, \mathrm{q}, \in\wedge\,\mathrm{q}\}$, if μ is an $(\in\wedge\,\mathrm{q}\,,\ \beta)$-fuzzy hyper K-subalgebra of H, then the support of μ contains the zero element* 0.

Proof. Straightforward. □

Theorem 3.18. *If μ is an $(\in, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of a hyper K-algebra H, then the support of μ is a hyper K-subalgebra of H.*

Proof. Let $x, y \in \mathrm{Supp}(\mu)$. Then $t_x := \mu(x) > 0$ and $t_y := \mu(y) > 0$. It follows that $x_{t_x} \in \mu$ and $y_{t_y} \in \mu$ so that $z_{\min\{t_x,t_y\}} \in\vee\mathrm{q}\,\mu$ for all $z \in x \circ y$ since μ is an $(\in, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of H. Suppose that there exists $a \in x \circ y$ such that $\mu(a) = 0$. Then $\mu(a) < \min\{t_x, t_y\}$ and $\mu(a) + \min\{t_x, t_y\} = \min\{t_x, t_y\} \leq 1$, which shows that $a_{\min\{t_x,t_y\}} \overline{\in\vee\mathrm{q}}\,\mu$. This is a contradiction, and so $\mu(z) > 0$ for all $z \in x \circ y$, i.e., $z \in \mathrm{Supp}(\mu)$ for all $z \in x \circ y$. Therefore $x \circ y \subseteq \mathrm{Supp}(\mu)$, and so $\mathrm{Supp}(\mu)$ is a hyper K-subalgebra of H by Lemma 2.2. □

Corollary 3.19. *For any $\beta \in \{\in, \mathrm{q}, \in\wedge\mathrm{q}\}$, if μ is an $(\in, \beta)$-fuzzy hyper K-subalgebra of a hyper K-algebra H, then the support of μ is a hyper K-subalgebra of H.*

Proof. Straightforward. □

Theorem 3.20. *If μ is a $(\mathrm{q}, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of a hyper K-algebra H, then the support of μ is a hyper K-subalgebra of H.*

Proof. Let $x, y \in \mathrm{Supp}(\mu)$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Thus $\mu(x) + 1 > 1$ and $\mu(y) + 1 > 1$, i.e., $x_1\,\mathrm{q}\,\mu$ and $y_1\,\mathrm{q}\,\mu$. Since μ is an $(\mathrm{q}, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of H, it follows that $z_1 = z_{\min\{1,1\}} \in\vee\mathrm{q}\,\mu$ for all $z \in x \circ y$. Assume that $x \circ y \nsubseteq \mathrm{Supp}(\mu)$. Then there exists $b \in x \circ y$ such that $\mu(b) = 0$. Thus $\mu(b) < 1 = \min\{1, 1\}$, i.e., $b_{\min\{1,1\}} \overline{\in}\,\mu$, and $\mu(b) + \min\{1, 1\} = 1$, i.e., $b_{\min\{1,1\}} \overline{\mathrm{q}}\,\mu$. Hence $b_{\min\{1,1\}} \overline{\in\vee\mathrm{q}}\,\mu$, a contradiction. Therefore $x \circ y \subseteq \mathrm{Supp}(\mu)$, and so $\mathrm{Supp}(\mu)$ is a hyper K-subalgebra of H by Lemma 2.2. □

Corollary 3.21. *For any $\beta \in \{\in, \mathrm{q}, \in\wedge\mathrm{q}\}$, if μ is a (q, β)-fuzzy hyper K-subalgebra of a hyper K-algebra H, then the support of μ is a hyper K-subalgebra of H.*

Proof. Straightforward. □

Corollary 3.22. *For any $\beta \in \{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$, if μ is an $(\in\vee\mathrm{q}, \beta)$-fuzzy hyper K-subalgebra of a hyper K-algebra H, then the support of μ is a hyper K-subalgebra of H.*

Proof. Straightforward. □

Theorem 3.23. *Let μ be a fuzzy set in a hyper K-algebra H such that $\mu(x) > 0.5$ for some $x \in H$. If μ is an $(\in\wedge\mathrm{q}, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of H, then the support of μ is a hyper K-subalgebra of H.*

Proof. Let $x, y \in \mathrm{Supp}(\mu)$. Then $t_x := \mu(x) > 0$ and $t_y := \mu(y) > 0$. If either $t_x \leq 0.5$ or $t_y \leq 0.5$, then it is clear. Assume that $t_x > 0.5$ and $t_y > 0.5$. Then $x_{0.5} \in \mu$, $y_{0.5} \in \mu$, $\mu(x) + 0.5 > 1$ and $\mu(y) + 0.5 > 1$. Hence $x_{0.5} \in\wedge\mathrm{q}\,\mu$ and

$y_{0.5} \in\!\wedge\, \mathrm{q}\, \mu$. It follows from (3.4) that $z_{\min\{0.5,0.5\}} \in\!\vee\, \mathrm{q}\, \mu$, i.e., $z_{\min\{0.5,0.5\}} \in \mu$ or $z_{\min\{0.5,0.5\}}\, \mathrm{q}\, \mu$ for all $z \in x \circ y$. From $z_{\min\{0.5,0.5\}} \in \mu$, we have $\mu(z) \geq \min\{0.5, 0.5\} > 0$. From $z_{\min\{0.5,0.5\}}\, \mathrm{q}\, \mu$, we obtain $\mu(z) + \min\{0.5, 0.5\} > 1$ and so $\mu(z) > 1 - \min\{0.5, 0.5\} \geq 0$. Thus $\mu(z) > 0$ for all $z \in x \circ y$, and hence $x \circ y \subseteq \mathrm{Supp}(\mu)$. Therefore $\mathrm{Supp}(\mu)$ is a hyper K-subalgebra of H. □

Corollary 3.24. *Let μ be a fuzzy set in a hyper K-algebra H such that $\mu(x) > 0.5$ for some $x \in H$. For any $\beta \in \{\in, \mathrm{q}, \in\!\wedge\, \mathrm{q}\}$, if μ is a $(\in\!\wedge\, \mathrm{q}\,, \beta)$-fuzzy hyper K-subalgebra of H, then the support of μ is a hyper K-subalgebra of H.*

Proof. Straightforward. □

Theorem 3.25. *Every (q, q)-fuzzy hyper K-subalgebra of a hyper K-algebra H is constant on the support of μ.*

Proof. Let μ be a (q, q)-fuzzy hyper K-subalgebra of a hyper K-algebra H. Assume that μ is not constant on $\mathrm{Supp}(\mu)$. Then there exists $y \in H$ such that $t_y = \mu(y) \neq \mu(0) = t_0$. Suppose that $t_y < t_0$ and choose $t_1, t_2 \in (0, 1]$ such that $1 - t_0 < t_1 < 1 - t_y < t_2$. Then $\mu(0) + t_1 = t_0 + t_1 > 1$ and $\mu(y) + t_2 = t_y + t_2 > 1$, i.e., $0_{t_1}\, \mathrm{q}\, \mu$ and $y_{t_2}\, \mathrm{q}\, \mu$. Note that $y \in y \circ 0$ and $\mu(y) + \min\{t_1, t_2\} = t_y + t_1 < 1$, i.e., $y_{\min\{t_1,t_2\}}\, \overline{\mathrm{q}}\, \mu$. This is a contradiction. Next if $t_y > t_0$, then $\mu(y) + (1 - t_0) = t_y + 1 - t_0 > 1$ and so $y_{1-t_0}\, \mathrm{q}\, \mu$. Note that $0 \in y \circ y$ and $\mu(0) + (1 - t_0) = t_0 + 1 - t_0 = 1$, i.e., $0_{1-t_0}\, \overline{\mathrm{q}}\, \mu$. This is also a contradiction. Therefore μ is constant on the support of μ. □

Theorem 3.26. *Let μ be a $(\mathrm{q}, \in\!\vee\, \mathrm{q})$-fuzzy hyper K-subalgebra of a hyper K-algebra H such that μ is not constant on the support of μ. If*

$$\mu(0) = \bigvee\{\mu(x) \mid x \in H\},$$

then $\mu(x) \geq 0.5$ for all $x \in \mathrm{Supp}(\mu)$.

Proof. Assume that $\mu(x) < 0.5$ for all $x \in H$. Since μ is not constant on the support of μ, there exists $y \in \mathrm{Supp}(\mu)$ such that $t_y = \mu(y) \neq \mu(0) = t_0$. Then $t_y < t_0 < 0.5$. Choose $t \in (0.5, 1]$ such that $t_y < 1 - t < t_0$. Then $0_t\, \mathrm{q}\, \mu$ and $y_1\, \mathrm{q}\, \mu$. It follows that $z_t = z_{\min\{t,1\}} \in\!\vee\, \mathrm{q}\, \mu$ for all $z \in y \circ 0$, in particular, $y_t \in\!\vee\, \mathrm{q}\, \mu$ because $y \in y \circ 0$. But, $\mu(y) = t_y < 0.5 < t$, i.e., $y_t\, \overline{\in}\, \mu$, and $\mu(y) + t = t_y + t < 1$, i.e., $y_t\, \overline{\mathrm{q}}\, \mu$. This shows that $y_t\, \overline{\in\!\vee\, \mathrm{q}}\, \mu$, and it is a contradiction. Therefore $\mu(x) \geq 0.5$ for some $x \in H$. Now if possible, let $t_0 = \mu(0) < 0.5$. Then there exists $x \in H$ such that $t_x = \mu(x) \geq 0.5$, and so $t_0 < t_x$. Take $t \in (0, 1)$ such that $t > t_0$ and $t_0 + t < 1 < t_x + t$. Then $x_t\, \mathrm{q}\, \mu$ and $0_1\, \mathrm{q}\, \mu$. It follows that $z_t = z_{\min\{t,1\}} \in\!\vee\, \mathrm{q}\, \mu$ for all $z \in 0 \circ x$, in particular, $0_t \in\!\vee\, \mathrm{q}\, \mu$ since $0 \in 0 \circ x$. But, $\mu(0) = t_0 < t$ and $\mu(0) + t = t_0 + t < 1$, i.e., $0_t\, \overline{\in\!\vee\, \mathrm{q}}\, \mu$ which is a contradiction. Hence $\mu(0) \geq 0.5$. Finally, let $t_x = \mu(x) < 0.5$ for some $x \in \mathrm{Supp}(\mu)$. We can take $t \in (0, 0.5)$ such that $t_x + t < 0.5$. Then $x_1\, \mathrm{q}\, \mu$ and $0_{0.5+t}\, \mathrm{q}\, \mu$ since $\mu(x) = t_x > 0$ and $\mu(0) \geq 0.5$. Thus $z_{0.5+t} = z_{\min\{1,0.5+t\}} \in\!\vee\, \mathrm{q}\, \mu$ for all $z \in x \circ 0$, in particular, $x_{0.5+t} \in\!\vee\, \mathrm{q}\, \mu$ since $x \in x \circ 0$. But, $\mu(x) = t_x < 0.5 + t$

and $\mu(x) + 0.5 + t = t_x + 0.5 + t < 1$, i.e., $x_{0.5+t} \overline{\in\vee\mathrm{q}}\, \mu$, a contradiction. Consequently, $\mu(x) \geq 0.5$ for all $x \in \mathrm{Supp}(\mu)$. □

Theorem 3.27. *Let S be a hyper K-subalgebra of a hyper K-algebra H and let μ be a fuzzy set in H such that*

(1) *$\mu(x) = 0$ for all $x \in H \setminus S$,*
(2) *$\mu(x) = t \geq 0.5$ for all $x \in S$.*

Then μ is an $(\alpha, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of H for $\alpha \in \{\in, \mathrm{q}\}$.

Proof. Obviously, μ is a fuzzy hyper K-subalgebra of H, that is, an $(\in, \in)$-fuzzy hyper K-subalgebra of H, and so μ is an $(\in, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of H. Let $x, y \in H$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1}\mathrm{q}\,\mu$ and $y_{t_2}\mathrm{q}\,\mu$. Then $\mu(x) + t_1 > 1$ and $\mu(y) + t_2 > 1$, which imply that $\mu(x) \neq 0 \neq \mu(y)$ so that $\mu(x) \geq 0.5$ and $\mu(y) \geq 0.5$ by the construction of μ. Thus $x, y \in S$ and so $x \circ y \subseteq S$. It follows that $\mu(z) \geq 0.5$ for all $z \in x \circ y$. If $\min\{t_1, t_2\} > 0.5$, then $\mu(z) + \min\{t_1, t_2\} > 1$ and so $z_{\min\{t_1,t_2\}}\mathrm{q}\,\mu$. If $\min\{t_1, t_2\} \leq 0.5$, then $\mu(z) \geq 0.5 \geq \min\{t_1, t_2\}$ and thus $z_{\min\{t_1,t_2\}} \in \mu$. Therefore $z_{\min\{t_1,t_2\}} \in\vee\mathrm{q}\,\mu$ for all $z \in x \circ y$. Consequently, μ is a $(\mathrm{q}, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of H. □

Theorem 3.28. *Let μ be a $(\mathrm{q}, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of a hyper K-algebra H. If μ is not constant on the support of μ, then there exists $x \in H$ such that $\mu(x) \geq 0.5$. Moreover, $\mu(x) \geq 0.5$ for all $x \in \mathrm{Supp}(\mu)$.*

Proof. Assume that $\mu(x) < 0.5$ for all $x \in H$. Since μ is not constant on $\mathrm{Supp}(\mu)$, there exists $x \in \mathrm{Supp}(\mu)$ such that $t_x = \mu(x) \neq \mu(0) = t_0$. Then either $t_0 < t_x$ or $t_0 > t_x$. For the first case, choose $\delta > 0.5$ such that $t_0 + \delta < 1 < t_x + \delta$. Then $x_\delta \mathrm{q}\,\mu$. Note that $0 \in x \circ x$, $\mu(0) = t_0 < \delta = \min\{\delta, \delta\}$ and $\mu(0) + \min\{\delta, \delta\} = t_0 + \delta < 1$. Hence $0_\delta \overline{\in\vee\mathrm{q}}\, \mu$, a contradiction. Now, if $t_0 > t_x$, we can choose $\delta > 0.5$ such that $t_x + \delta < 1 < t_0 + \delta$. Then $0_\delta\, \mathrm{q}\,\mu$ and $x_1\, \mathrm{q}\,\mu$. Note that $x \in x \circ 0$, $\mu(x) = t_x < 0.5 < \delta$, i.e., $x_\delta \,\overline{\in}\, \mu$, and $\mu(x) + \delta = t_x + \delta < 1$, i.e., $x_\delta\, \overline{\mathrm{q}}\, \mu$. Therefore $x_{\min\{1,\delta\}} = x_\delta \overline{\in\vee\mathrm{q}}\, \mu$, a contradiction. Consequently, $\mu(x) \geq 0.5$ for some $x \in H$. We now show that $\mu(0) \geq 0.5$. Assume $\mu(0) = t_0 < 0.5$. Since there exists $x \in H$ such that $\mu(x) = t_x \geq 0.5$, it follows that $t_0 < t_x$. Choose $t_1 > t_0$ such that $t_0 + t_1 < 1 < t_x + t_1$. Then $\mu(x) + t_1 = t_x + t_1 > 1$, and so $x_{t_1}\mathrm{q}\,\mu$. Next, $\mu(0) = t_0 < t_1 = \min\{t_1, t_1\}$ and $\mu(0) + \min\{t_1, t_1\} = t_0 + t_1 < 1$. This shows that $0_{\min\{t_1,t_1\}} \overline{\in\vee\mathrm{q}}\, \mu$, which induces a contradiction since $0 \in x \circ x$. Thus $\mu(0) \geq 0.5$. Finally suppose that $t_x = \mu(x) < 0.5$ for some $x \in \mathrm{Supp}(\mu)$. Take $t > 0$ such that $t_x + t < 0.5$. Then $\mu(x) + 1 = t_x + 1 > 1$ and $\mu(0) + 0.5 + t > 1$, which imply that $x_1\, \mathrm{q}\,\mu$ and $0_{0.5+t}\, \mathrm{q}\,\mu$. But $x_{\min\{1,0.5+t\}} = x_{0.5+t} \overline{\in\vee\mathrm{q}}\, \mu$ which is a contradiction since $x \in x \circ 0$. Therefore $\mu(x) \geq 0.5$ for all $x \in \mathrm{Supp}(\mu)$. □

Note that every fuzzy hyper K-subalgebras has relations between a part of types as seen in Figure 1 where $\alpha, \beta \in \{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$.

$$
\begin{tikzcd}
(\alpha,\, \in\vee\mathrm{q}\,) & (\alpha,\, \mathrm{q}) \arrow[l] & (\in\wedge\mathrm{q}\,,\, \beta) & (\mathrm{q},\, \beta) \arrow[l] \\
(\alpha,\, \in) \arrow[u] & (\alpha,\, \in\wedge\mathrm{q}\,) \arrow[l] \arrow[u] & (\in,\, \beta) \arrow[u] & (\in\vee\mathrm{q}\,,\, \beta) \arrow[l] \arrow[u]
\end{tikzcd}
$$

FIGURE 1. Relation between a part of types of fuzzy hyper K-subalgebra

4. $(\in, \in\vee\mathrm{q}\,)$-fuzzy hyper K-subalgebras

In this section, we consider a special case of (α, β)-fuzzy hyper K-subalgebras. An $(\in, \in\vee\mathrm{q}\,)$-fuzzy hyper K-subalgebra is important because it is a useful generalization of ordinary fuzzy hyper K-subalgebra.

Theorem 4.1. *A fuzzy set μ in a hyper K-algebra H is an $(\in, \in\vee\mathrm{q}\,)$-fuzzy hyper K-subalgebra of H if and only if it satisfies*:

$$
(\forall x, y \in H) \left(\inf_{z\in x\circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\} \right). \tag{4.1}
$$

Proof. Assume that μ is an $(\in, \in\vee\mathrm{q}\,)$-fuzzy hyper K-subalgebra of H. For any $x, y \in H$, we consider the following two cases:

(i) $\min\{\mu(x), \mu(y)\} < 0.5$,

(ii) $\min\{\mu(x), \mu(y)\} \geq 0.5$.

For the first case, assume that

$$
\inf_{z\in x\circ y} \mu(z) < \min\{\mu(x), \mu(y)\}.
$$

Choose a number t such that

$$
\inf_{z\in x\circ y} \mu(z) < t \leq \min\{\mu(x), \mu(y)\}.
$$

Then $x_t \in \mu$ and $y_t \in \mu$. But for $z \in x \circ y$, $z_{\min\{t,t\}} = z_t \overline{\in\vee\mathrm{q}}\,\mu$. This is a contradiction, and so (4.1) holds. Case (ii) implies $x_{0.5} \in \mu$ and $y_{0.5} \in \mu$. Suppose that $\mu(z) < 0.5$ for some $z \in x \circ y$. Then $z_{0.5} \overline{\in\vee\mathrm{q}}\,\mu$, a contradiction. Therefore $\inf_{z\in x\circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in H$.

Conversely, let μ be a fuzzy set in H satisfying the condition (4.1). Let $x, y \in H$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$. For every $z \in x \circ y$, we have

$$
\mu(z) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t_1, t_2, 0.5\}.
$$

If $\min\{t_1, t_2\} > 0.5$, then $\mu(z) \geq 0.5$ and so $\mu(z) + \min\{t_1, t_2\} > 1$. If $\min\{t_1, t_2\} \leq 0.5$, then $\mu(z) \geq \min\{t_1, t_2\}$. Therefore $z_{\min\{t_1,t_2\}} \in\vee\mathrm{q}\,\mu$ for all $z \in x \circ y$, and so μ is an $(\in, \in\vee\mathrm{q}\,)$-fuzzy hyper K-subalgebra of H. $\square$

Corollary 4.2. *For any* $\beta \in \{\in, \mathrm{q}, \in \wedge\, \mathrm{q}\}$, *every* $(\in,\ \beta)$*-fuzzy hyper* K*-subalgebra of* H *satisfies the inequality* (4.1).

Proof. Straightforward. $\square$

Theorem 4.3. *For any subset* S *of a hyper* K*-algebra* H, *the characteristic function* χ_S *of* S *is an* $(\in, \in\vee\, \mathrm{q})$*-fuzzy hyper* K*-subalgebra of* H *if and only if* S *is a hyper* K*-subalgebra of* H.

Proof. Assume that χ_S of S is an $(\in, \in\vee\, \mathrm{q})$-fuzzy hyper K-subalgebra of H. Let $x, y \in S$. Then $\chi_S(x) = 1 = \chi_S(y)$, and so $x_1 \in \chi_S$ and $y_1 \in \chi_S$. It follows from (3.3) that $z_1 \in\vee\, \mathrm{q}\, \chi_S$ for all $z \in x \circ y$ so that $\chi_S(z) > 0$ for all $z \in x \circ y$. Hence $x \circ y \subseteq S$, and so S is a hyper K-subalgebra of H. The converse is straightforward. $\square$

Theorem 4.4. *Let* S *be a hyper* K*-subalgebra of a hyper* K*-algebra* H. *For any* $t \in (0, 0.5]$, *there exists an* $(\in,\ \in\vee\, \mathrm{q})$*-fuzzy hyper* K*-subalgebra* μ *of* H *such that* S *is the* t*-level set of* μ, *that is,* $S = U(\mu; t)$.

Proof. Let μ be a fuzzy set in H defined by

$$\mu(x) := \begin{cases} t & \text{if } x \in S, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in H$ where $t \in (0, 0.5]$. Obviously, $U(\mu; t) = S$. Assume that $\inf_{z \in x\circ y} \mu(z) < \min\{\mu(x), \mu(y), 0.5\}$ for some $x, y \in H$. Since $|\mathrm{Im}(\mu)| = 2$, it follows that $\inf_{z\in x\circ y} \mu(z) = 0$ and $\min\{\mu(x), \mu(y), 0.5\} = t$, and so $\mu(x) = t = \mu(y)$, so that $x, y \in S$. But $\inf_{z\in x\circ y} \mu(z) = 0$ implies that $\mu(z) = 0$ for some $z \in x \circ y$. Hence $z \notin S$, and thus $x \circ y \nsubseteq S$. This is a contradiction. Therefore $\inf_{z\in x\circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in H$. Using Theorem 4.1, we know that μ is an $(\in,\ \in\vee\, \mathrm{q})$-fuzzy hyper K-subalgebra of H. $\square$

Now, we characterize $(\in,\ \in\vee\, \mathrm{q})$-fuzzy hyper K-subalgebras by their level sets.

Theorem 4.5. *Let* μ *be a fuzzy set in a hyper* K*-algebra* H. *Then* μ *is an* $(\in,\ \in\vee\, \mathrm{q})$*-fuzzy hyper* K*-subalgebra of* H *if and only if its nonempty* t*-level set* $U(\mu; t)$ *is a hyper* K*-subalgebra of* H *for all* $t \in (0, 0.5]$.

Proof. Let μ be an $(\in,\ \in\vee\, \mathrm{q})$-fuzzy hyper K-subalgebra of H. Let $t \in (0, 0.5]$ and $x, y \in U(\mu; t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. For any $z \in x \circ y$, we have

$$\mu(z) \geq \inf_{u\in x\circ y} \mu(u) \geq \min\{\mu(x), \mu(y), 0.5\} \geq t$$

by Theorem 4.1. Thus $z \in U(\mu; t)$, and so $x \circ y \subseteq U(\mu; t)$. Hence $U(\mu; t)$ is a hyper K-subalgebra of H for all $t \in (0, 0.5]$.

Conversely suppose that the nonempty t-level set $U(\mu; t)$ of μ is a hyper K-subalgebra of H for all $t \in (0, 0.5]$. For any $x, y \in H$, let $t_0 := \min\{\mu(x), \mu(y), 0.5\}$. Then $t_0 \in (0, 0.5]$, $\mu(x) \geq \min\{\mu(x), \mu(y), 0.5\} = t_0$ and $\mu(y) \geq \min\{\mu(x), \mu(y), 0.5\} = t_0$. It follows that $x, y \in U(\mu; t_0)$ so that $x \circ y \subseteq U(\mu; t_0)$ since

$U(\mu; t_0)$ is a hyper K-subalgebra of H. Therefore $z \in U(\mu; t_0)$, i.e., $\mu(z) \geq t_0$ for all $z \in x \circ y$. Thus $\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}$, which implies from Theorem 4.1 that μ is an $(\in, \in\vee\, \mathrm{q}\,)$-fuzzy hyper K-subalgebra of H. □

Proposition 4.6. *Let μ be an $(\in, \in\vee\, \mathrm{q}\,)$-fuzzy hyper K-subalgebra of a hyper K-algebra H. If there exists $x \in H$ such that $\mu(x) \geq 0.5$, then $\mu(0) \geq 0.5$.*

Proof. Assume that $\mu(x) \geq 0.5$ for some $x \in H$. Then

$$\inf_{z \in x \circ x} \mu(z) \geq \min\{\mu(x), 0.5\} = 0.5.$$

Since $0 \in x \circ x$, it follows that $\mu(0) \geq 0.5$. □

Note that every $(\in, \in)$-fuzzy hyper K-subalgebra is an $(\in, \in\vee\, \mathrm{q}\,)$-fuzzy hyper K-subalgebra, but the converse may not be true (see Theorem 3.6 and Example 3.7).

Now, we provide a condition for an $(\in, \in\vee\, \mathrm{q}\,)$-fuzzy hyper K-subalgebra to be an $(\in, \in)$-fuzzy hyper K-subalgebra.

Theorem 4.7. *Let μ be an $(\in, \in\vee\, \mathrm{q}\,)$-fuzzy hyper K-subalgebra of a hyper K-algebra H such that $\mu(x) < 0.5$ for all $x \in H$. Then μ is an $(\in, \in)$-fuzzy hyper K-subalgebra of H.*

Proof. Let $x, y \in H$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$. It follows from Theorem 4.1 that

$$\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\} = \min\{\mu(x), \mu(y)\} \geq \min\{t_1, t_2\}$$

so that $\mu(z) \geq \min\{t_1, t_2\}$ for all $z \in x \circ y$. Hence $z_{\min\{t_1, t_2\}} \in \mu$ for all $z \in x \circ y$. Therefore μ is an $(\in, \in)$-fuzzy hyper K-subalgebra of H. □

Corollary 4.8. *Let μ be an $(\in, \in\vee\, \mathrm{q}\,)$-fuzzy hyper K-subalgebra of a hyper K-algebra H which is not of type $(\in, \in)$. Then there exists $x \in H$ such that $\mu(x) \geq 0.5$.*

Proof. Straightforward. □

Theorem 4.9. *For a fuzzy set μ in a hyper K-algebra H, the following assertions are equivalent*:

(1) *The nonempty level set $U(\mu; t)$ of μ is a hyper K-subalgebra of H for all $t \in (0.5, 1]$.*
(2) $\inf_{z \in x \circ y}\{\max\{\mu(z), 0.5\}\} \geq \min\{\mu(x), \mu(y)\}$ *for all $x, y \in H$.*

Proof. Suppose that the level set $U(\mu; t)$ of μ is a hyper K-subalgebra of H for all $t \in (0.5, 1]$. If there exist $a, b, c \in H$ with $c \in a \circ b$ such that $\max\{\mu(c), 0.5\} < \min\{\mu(a), \mu(b)\} = t$, then $t \in (0.5, 1]$, $\mu(c) < t$ and $a, b \in U(\mu; t)$. It follows from the hypothesis that $a \circ b \subseteq U(\mu; t)$ so that $z \in U(\mu; t)$, i.e., $\mu(z) \geq t$ for all $z \in a \circ b$. This is a contradiction, and so $\max\{\mu(z), 0.5\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, y, z \in H$ with $z \in x \circ y$. Hence $\inf_{z \in x \circ y}\{\max\{\mu(z), 0.5\}\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in H$.

Conversely, suppose that (2) is valid. Let $t \in (0.5, 1]$ such that $U(\mu; t) \neq \emptyset$ and let $x, y \in U(\mu; t)$. Using the condition (2), we have

$$\inf_{z \in x \circ y} \{\max\{\mu(z), 0.5\}\} \geq \min\{\mu(x), \mu(y)\} \geq t > 0.5.$$

Thus $\max\{\mu(z), 0.5\} \geq t > 0.5$ for all $z \in x \circ y$, which implies $\mu(z) \geq t$, i.e., $z \in U(\mu; t)$. Hence $x \circ y \subseteq U(\mu; t)$, which shows that $U(\mu; t)$ is a hyper K-subalgebra of H for all $t \in (0.5, 1]$. □

For any fuzzy subset μ of a hyper K-algebra H and any $t \in (0, 1]$, we consider two subsets:

$$Q(\mu; t) := \{x \in H \mid x_t \,\mathrm{q}\, \mu\} \text{ and } [\mu]_t := \{x \in H \mid x_t \in\!\vee\,\mathrm{q}\, \mu\}.$$

It is clear that $[\mu]_t = U(\mu; t) \cup Q(\mu; t)$.

Theorem 4.10. *If μ is an $(\in, \in\!\vee\,\mathrm{q})$-fuzzy hyper K-subalgebra of a hyper K-algebra H, then the set $Q(\mu; t)$ is a hyper K-subalgebra of H for all $t \in (0.5, 1]$.*

Proof. For any $t \in (0.5, 1]$, let $x, y \in Q(\mu; t)$. Then $\mu(x) + t > 1$ and $\mu(y) + t > 1$. Using Theorem 4.1, we have

$$\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}.$$

If $\min\{\mu(x), \mu(y)\} \geq 0.5$, then $\inf_{z \in x \circ y} \mu(z) \geq 0.5$, and so $\mu(z) \geq 0.5 > 1 - t$ for all $z \in x \circ y$. Thus $z_t \,\mathrm{q}\, \mu$, i.e., $z \in Q(\mu; t)$ for all $z \in x \circ y$. If $\min\{\mu(x), \mu(y)\} < 0.5$, then $\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\}$, and thus $\mu(z) \geq \min\{\mu(x), \mu(y)\} > 1 - t$ for all $z \in x \circ y$. This shows that $z_t \,\mathrm{q}\, \mu$, i.e., $z \in Q(\mu; t)$ for all $z \in x \circ y$. Consequently, $Q(\mu; t)$ is a hyper K-subalgebra of H for all $t \in (0.5, 1]$. □

Corollary 4.11. *If μ is an $(\in, \in)$-fuzzy hyper K-subalgebra of a hyper K-algebra H, then the set $Q(\mu; t)$ is a hyper K-subalgebra of H for all $t \in (0.5, 1]$.*

Theorem 4.12. *For any fuzzy set μ in a hyper K-algebra H, the following are equivalent:*

(1) *μ is an $(\in, \in\!\vee\,\mathrm{q})$-fuzzy hyper K-subalgebra of H.*
(2) *$[\mu]_t$ is a hyper K-subalgebra of H for all $t \in (0, 1]$.*

We say that $[\mu]_t$ is an $(\in\!\vee\,\mathrm{q})$-*level hyper K-subalgebra* of μ.

Proof. Suppose that μ is an $(\in, \in\!\vee\,\mathrm{q})$-fuzzy hyper K-subalgebra of H. Let $x, y \in [\mu]_t$ for $t \in (0, 1]$. Then $\mu(x) \geq t$ or $\mu(x) + t > 1$, and $\mu(y) \geq t$ or $\mu(y) + t > 1$. We can consider four cases:

(c1) $\mu(x) \geq t$ and $\mu(y) \geq t$,
(c2) $\mu(x) \geq t$ and $\mu(y) + t > 1$,
(c3) $\mu(x) + t > 1$ and $\mu(y) \geq t$,
(c4) $\mu(x) + t > 1$ and $\mu(y) + t > 1$.

For the first case, (4.1) implies that

$$\inf_{z\in x\circ y}\mu(z)\geq \min\{\mu(x),\mu(y),0.5\}\geq \min\{t,0.5\}=\begin{cases}0.5 & \text{if } t>0.5,\\ t & \text{if } t\leq 0.5.\end{cases}$$

Hence if $t>0.5$, then $\mu(z)\geq 0.5$ for all $z\in x\circ y$, which implies that $\mu(z)+t\geq 0.5+t>1$, i.e., $z\in Q(\mu;t)$ for all $z\in x\circ y$. If $t\leq 0.5$, then $\mu(z)\geq t$ for all $z\in x\circ y$, and so $z\in U(\mu;t)$ for all $z\in x\circ y$. Therefore $x\circ y\subseteq U(\mu;t)\cup Q(\mu;t)=[\mu]_t$. For the case (c2), assume that $t>0.5$. Then $1-t<0.5$. If $\min\{\mu(y),0.5\}\leq\mu(x)$, then $\inf_{z\in x\circ y}\mu(z)\geq\min\{\mu(y),0.5\}$ and so $\mu(z)\geq\min\{\mu(y),0.5\}>1-t$ for all $z\in x\circ y$. This shows that $z_t\,\mathrm{q}\,\mu$, i.e., $z\in Q(\mu;t)$ for all $z\in x\circ y$. If $\min\{\mu(y),0.5\}>\mu(x)$, then $\inf_{z\in x\circ y}\mu(z)\geq\mu(x)$ which implies that $\mu(z)\geq\mu(x)\geq t$ for all $z\in x\circ y$. Thus $z\in U(\mu;t)$ for all $z\in x\circ y$. Therefore $z\in U(\mu;t)\cup Q(\mu;t)=[\mu]_t$ for all $z\in x\circ y$, that is, $x\circ y\subseteq[\mu]_t$ for $t>0.5$. Suppose that $t\leq 0.5$. Then $1-t\geq 0.5$. If $\min\{\mu(x),0.5\}\leq\mu(y)$, then $\inf_{z\in x\circ y}\mu(z)\geq\min\{\mu(x),0.5\}$ and thus $\mu(z)\geq\min\{\mu(x),0.5\}\geq t$ for all $z\in x\circ y$. If $\min\{\mu(x),0.5\}\geq\mu(y)$, then $\inf_{z\in x\circ y}\mu(z)\geq\mu(y)$, and so $\mu(z)\geq\mu(y)>1-t$ for all $z\in x\circ y$. Hence $z\in U(\mu;t)\cup Q(\mu;t)=[\mu]_t$ for all $z\in x\circ y$ whenever $t\leq 0.5$, that is, $x\circ y\subseteq[\mu]_t$ for $t\leq 0.5$. We have similar result for the case (c3). Now we consider the final case. If $t>0.5$, then $1-t<0.5$.

$$\begin{aligned}\inf_{z\in x\circ y}\mu(z) &\geq \min\{\mu(x),\mu(y),0.5\}\\ &= \begin{cases}0.5>1-t & \text{if } \min\{\mu(x),\mu(y)\}\geq 0.5,\\ \min\{\mu(x),\mu(y)\}>1-t & \text{if } \min\{\mu(x),\mu(y)\}<0.5,\end{cases}\end{aligned}$$

and so $z\in Q(\mu;t)\subseteq[\mu]_t$ for all $z\in x\circ y$. If $t\leq 0.5$, then $1-t\geq 0.5$. Thus

$$\begin{aligned}\inf_{z\in x\circ y}\mu(z) &\geq \min\{\mu(x),\mu(y),0.5\}\\ &= \begin{cases}0.5\geq t & \text{if } \min\{\mu(x),\mu(y)\}\geq 0.5,\\ \min\{\mu(x),\mu(y)\}>1-t & \text{if } \min\{\mu(x),\mu(y)\}<0.5,\end{cases}\end{aligned}$$

which implies that $z\in U(\mu;t)\cup Q(\mu;t)=[\mu]_t$ for all $z\in x\circ y$. Consequently, $[\mu]_t$ is a hyper K-subalgebra of H for all $t\in(0,1]$.

Conversely, let μ be a fuzzy set in H such that $[\mu]_t$ is a hyper K-subalgebra of H for all $t\in(0,1]$. Assume that (4.1) is false. Then there exist $a,b\in H$ such that

$$\inf_{z\in a\circ b}<\min\{\mu(a),\mu(b),0.5\}.$$

It follows that $\mu(w)<\min\{\mu(a),\mu(b),0.5\}$ for some $w\in a\circ b$ so that there exists $t_w\in(0,1]$ such that

$$\mu(w)<t_w\leq\min\{\mu(a),\mu(b),0.5\}.$$

Then $t_w\in(0,0.5]$ and $a,b\in U(\mu;t_w)\subseteq[\mu]_{t_w}$. Since $[\mu]_{t_w}$ is a hyper K-subalgebra of H, we have $a\circ b\subseteq[\mu]_{t_w}$ and so $w\in[\mu]_{t_w}$ for all $w\in a\circ b$. This is a contradiction since $w\notin U(\mu;t_w)$ and $\mu(w)+t_w<2t_w\leq 1$, i.e., $w\notin Q(\mu;t_w)$. Therefore

$$\inf_{z\in x\circ y}\mu(z)\geq\min\{\mu(x),\mu(y),0.5\}$$

for all $x, y \in H$. Using Theorem 4.1, we conclude that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of H. $\square$

Corollary 4.13. *If μ is an $(\in, \in)$-fuzzy hyper K-subalgebra of a hyper K-algebra H, then the set $[\mu]_t$ is a hyper K-subalgebra of H for all $t \in (0, 1]$.*

Theorem 4.14. *Let $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = H$ be a chain of hyper K-subalgebras of a hyper K-algebra H. Then there exists an $(\in, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of H whose $(\in\vee\mathrm{q})$-level hyper K-subalgebras are precisely the members of the chain.*

Proof. Let $\{t_i \in (0, 0.5] \mid i = 1, 2, \ldots, r\}$ be such that $t_1 > t_2 > \cdots > t_r$. Define a fuzzy set μ in H by

$$\mu : H \to [0,1],\ x \mapsto \begin{cases} t_0\ (> 0.5) & \text{if } x = 0, \\ t\ (> t_0) & \text{if } x \in G_0 \setminus \{0\}, \\ t_1 & \text{if } x \in G_1 \setminus G_0, \\ t_2 & \text{if } x \in G_2 \setminus G_1, \\ \cdots & \\ t_r & \text{if } x \in G_r \setminus G_{r-1}. \end{cases}$$

Then

$$U(\mu; s) = \begin{cases} G_0 & \text{if } s \in (t_1, 0.5], \\ G_1 & \text{if } s \in (t_2, t_1], \\ G_2 & \text{if } s \in (t_3, t_2], \\ \cdots & \\ G_r = H & \text{if } s \in (0, t_r]. \end{cases}$$

Using Theorem 4.5, we know that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of H, and clearly whose $(\in\vee\mathrm{q})$-level hyper K-subalgerbas are precisely the members of the chain. $\square$

A fuzzy set μ in H is said to be *proper* if $\mathrm{Im}(\mu)$ has at least two elements. Two fuzzy sets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be *non-equivalent.*

Theorem 4.15. *Let μ be a proper $(\in, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of a hyper K-algebra H such that $|\{\mu(x) \mid \mu(x) < 0.5\}| \geq 2$. Then there exist two proper non-equivalent $(\in, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebras of H such that μ can be expressed as the union of them.*

Proof. Let $\{\mu(x) \mid \mu(x) < 0.5\} = \{t_1, t_2, \ldots, t_r\}$, where $t_1 > t_2 > \cdots > t_r$ and $r \geq 2$. Then the chain of $(\in\vee\mathrm{q})$-level hyper K-subalgebras of μ is

$$[\mu]_{0.5} \subseteq [\mu]_{t_1} \subseteq [\mu]_{t_2} \subseteq \cdots \subseteq [\mu]_{t_r} = H.$$

Let ν and λ be fuzzy sets in H defined by

$$\nu(x) = \begin{cases} t_1 & \text{if } x \in [\mu]_{t_1}, \\ t_2 & \text{if } x \in [\mu]_{t_2} \setminus [\mu]_{t_1}, \\ \cdots & \\ t_r & \text{if } x \in [\mu]_{t_r} \setminus [\mu]_{t_{r-1}}, \end{cases}$$

and

$$\lambda(x) = \begin{cases} \mu(x) & \text{if } x \in [\mu]_{0.5}, \\ k & \text{if } x \in [\mu]_{t_2} \setminus [\mu]_{0.5}, \\ t_3 & \text{if } x \in [\mu]_{t_3} \setminus [\mu]_{t_2}, \\ \cdots & \\ t_r & \text{if } x \in [\mu]_{t_r} \setminus [\mu]_{t_{r-1}}, \end{cases}$$

respectively, where $t_3 < k < t_2$. Then ν and λ are $(\in, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebras of H, and $\nu, \lambda \leq \mu$. The chains of $(\in\vee\mathrm{q})$-level hyper K-subalgebras of ν and λ are, respectively, given by

$$[\mu]_{t_1} \subseteq [\mu]_{t_2} \subseteq \cdots \subseteq [\mu]_{t_r}$$

and

$$[\mu]_{0.5} \subseteq [\mu]_{t_2} \subseteq \cdots \subseteq [\mu]_{t_r}.$$

Therefore ν and λ are non-equivalent and clearly $\mu = \nu \cup \lambda$. This completes the proof. □

5. Implication-based fuzzy hyper *K*-subalgebras

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge$, $\vee$, $\neg$, $\rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe U of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper

$$[x \in \mu] = \mu(x), \tag{5.1}$$

$$[\Phi \wedge \Psi] = \min\{[\Phi], [\Psi]\}, \tag{5.2}$$

$$[\Phi \rightarrow \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\}, \tag{5.3}$$

$$[\forall x \Phi(x)] = \inf_{x \in U} [\Phi(x)], \tag{5.4}$$

$$\models \Phi \text{ if and only if } [\Phi] = 1 \text{ for all valuations.} \tag{5.5}$$

The truth valuation rules given in (5.3) are those in the Łukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{\mathrm{GR}}(a, b) = \begin{cases} 1 & \text{if } a \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Gödel implication operator (I_{G}):

$$I_{\mathrm{G}}(a, b) = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

(c) The contraposition of Gödel implication operator (I_{cG}):

$$I_{\mathrm{cG}}(a,b) = \begin{cases} 1 & \text{if } \ a \le b, \\ 1-a & \text{otherwise.} \end{cases}$$

Ying [16] introduced the concept of fuzzifying topology. We can expand his/her idea to hemirings, and we define a fuzzifying hyper K-subalgebra as follows.

Definition 5.1. A fuzzy set μ in a hyper K-algebra H is called a *fuzzifying hyper K-subalgerba* of H if for all $x, y \in H$,

$$\models \ [x \in \mu] \wedge [y \in \mu] \to [(\forall z \in x \circ y)\ (z \in \mu)]. \tag{5.6}$$

Obviously, the condition (5.6) is equivalent to the condition (3.1). Therefore a fuzzifying hyper K-subalgebra is an ordinary fuzzy hyper K-subalgebra.

In [17], the concept of t-tautology is introduced, i.e.,

$$\models_t \Phi \text{ if and only if } [\Phi] \ge t \text{ for all valuations.} \tag{5.7}$$

Definition 5.2. Let μ be a fuzzy set in a hyper K-algebra H and $t \in (0,1]$. μ is called a *t-implication-based fuzzy hyper K-subalgerba* of H if for all $x, y \in H$, we have

$$\models_t [x \in \mu] \wedge [y \in \mu] \to [(\forall z \in x \circ y)\ (z \in \mu)]. \tag{5.8}$$

Let I be an implication operator. Clearly, μ is a t-implication-based fuzzy hyper K-subalgebra of H if and only if it satisfies:

$$(\forall x, y \in H)\ (I(\min\{\mu(x), \mu(y)\}, \inf_{z \in x \circ y} \mu(z)) \ge t). \tag{5.9}$$

Theorem 5.3. *For any fuzzy subset μ of H, we have*

(1) *If $I = I_{\mathrm{GR}}$, then μ is a 0.5-implication-based fuzzy hyper K-subalgebra of H if and only if μ is a fuzzy hyper K-subalgebra of H.*

(2) *If $I = I_{\mathrm{G}}$, then μ is a 0.5-implication-based fuzzy hyper K-subalgebra of H if and only if μ is an $(\in, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of H.*

(3) *If $I = I_{\mathrm{cG}}$, then μ is a 0.5-implication-based fuzzy hyper K-subalgebra of H if and only if μ satisfies the following conditions:*

$$(\forall x, y \in H)\ \left(\max\left\{\inf_{z \in x \circ y} \mu(z), 0.5\right\} \ge \min\{\mu(x), \mu(y), 1\}\right). \tag{5.10}$$

Proof. (1) Straightforward.

(2) Assume that μ is a 0.5-implication-based fuzzy hyper K-subalgebra of H. Then

$$(\forall x, y \in H)\ \left(I_{\mathrm{G}}\left(\min\{\mu(x), \mu(y)\}, \inf_{z \in x \circ y} \mu(z)\right) \ge 0.5\right).$$

Thus $\inf_{z \in x \circ y} \mu(z) \ge \min\{\mu(x), \mu(y)\}$ or

$$\min\{\mu(x), \mu(y)\} > \inf_{z \in x \circ y} \mu(z) \ge 0.5.$$

It follows that $\inf_{z\in x\circ y}\mu(z) \geq \min\{\mu(x),\mu(y),0.5\}$. Using Theorem 4.1, we know that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of H.

Conversely, suppose that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy hyper K-subalgebra of H. Then

$$(\forall x, y \in H)\ \left(\inf_{z\in x\circ y}\mu(z) \geq \min\{\mu(x),\mu(y),0.5\}\right).$$

Hence if $\min\{\mu(x),\mu(y),0.5\} = \min\{\mu(x),\mu(y)\}$, then

$$\inf_{z\in x\circ y}\mu(z) \geq \min\{\mu(x),\mu(y)\},$$

and thus $I_{\mathrm{G}}(\min\{\mu(x),\mu(y)\}, \inf_{z\in x\circ y}\mu(z)) = 1 \geq 0.5$. If $\min\{\mu(x),\mu(y),0.5\} = 0.5$, then $\inf_{z\in x\circ y}\mu(z) \geq 0.5$ and so

$$I_{\mathrm{G}}\left(\min\{\mu(x),\mu(y)\}, \inf_{z\in x\circ y}\mu(z)\right) \geq 0.5.$$

Therefore μ is a 0.5-implication-based fuzzy hyper K-subalgebra of H.

(3) Suppose that μ satisfies the condition (5.10). If $\min\{\mu(x),\mu(y),1\} = 1$, then $\max\{\inf_{z\in x\circ y}\mu(z), 0.5\} = 1$ and so

$$\inf_{z\in x\circ y}\mu(z) = 1 \geq \min\{\mu(x),\mu(y)\}.$$

Hence $I_{\mathrm{cG}}(\min\{\mu(x),\mu(y)\}, \inf_{z\in x\circ y}\mu(z)) = 1 \geq 0.5$. If $\min\{\mu(x),\mu(y),1\} = \min\{\mu(x),\mu(y)\}$, then

$$\max\left\{\inf_{z\in x\circ y}\mu(z), 0.5\right\} \geq \min\{\mu(x),\mu(y)\}. \tag{5.11}$$

Thus, if $\max\{\inf_{z\in x\circ y}\mu(z), 0.5\} = 0.5$ in (5.11), then

$$\inf_{z\in x\circ y}\mu(z) \leq 0.5 \text{ and } \min\{\mu(x),\mu(y)\} \leq 0.5.$$

Therefore

$$\begin{aligned} & I_{\mathrm{cG}}(\min\{\mu(x),\mu(y)\}, \inf_{z\in x\circ y}\mu(z)) \\ = & \begin{cases} 1 \geq 0.5 & \text{if } \inf_{z\in x\circ y}\mu(z) \geq \min\{\mu(x),\mu(y)\}, \\ 1-\min\{\mu(x),\mu(y)\} \geq 0.5 & \text{otherwise.} \end{cases} \end{aligned}$$

If $\max\{\inf_{z\in x\circ y}\mu(z), 0.5\} = \inf_{z\in x\circ y}\mu(z)$ in (5.11), then $\inf_{z\in x\circ y}\mu(z) \geq \min\{\mu(x),\mu(y)\}$ and so $I_{\mathrm{cG}}(\min\{\mu(x),\mu(y)\}, \inf_{z\in x\circ y}\mu(z)) = 1 \geq 0.5$. Consequently μ is a 0.5-implication-based fuzzy hyper K-subalgebra of H.

Conversely, assume that μ is a 0.5-implication-based fuzzy hyper K-subalgebra of H. Then

$$(\forall x, y \in H)\ \left(I_{\mathrm{cG}}(\min\{\mu(x),\mu(y)\}, \inf_{z\in x\circ y}\mu(z)\right) \geq 0.5).$$

Thus $I_{\mathrm{cG}}(\min\{\mu(x), \mu(y)\}, \inf_{z\in x\circ y} \mu(z)) = 1$, i.e., $\min\{\mu(x), \mu(y)\} \leq \inf_{z\in x\circ y} \mu(z)$, or $1 - \min\{\mu(x), \mu(y)\} \geq 0.5$. Hence

$$\max\left\{\inf_{z\in x\circ y} \mu(z), 0.5\right\} \geq \min\{\mu(x), \mu(y)\} = \min\{\mu(x), \mu(y), 1\}$$

for all $x, y \in H$. □

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Department of Mathematics
Hanyang University
Seoul 133-791, Korea
E-mail address: sinchangmyun@hanmail.net