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GENERALIZED IDEAL ELEMENTS IN le-Γ-SEMIGROUPS

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Abstract. In this paper we introduce and give some characterizations of (m,n)-regular le-Γ-semigroup in terms of (m,n)-ideal elements and (m,n)-quasi-ideal elements. Also, we give some characterizations of subidempotent (m,n)-ideal elements in terms of r_α- and l_α- closed elements.

1. Introduction and preliminaries

In 1981, Sen [19] introduced the concept and notion of the Γ-semigroup as a generalization of semigroup and of ternary semigroup. Many classical notions and results of the theory of semigroups have been extended and generalized to Γ-semigroups. We [1, 2, 5] introduced and gave several other properties and characterizations in le-Γ-semigroups in general and regular le-Γ-semigroups in particular. The concept of generalized ideals and (m,n)-ideals elements in semigroups have been introduced by Lajos in [14] as a generalization of one-sided (left or right) ideals in semigroups and it was studied by several authors such as [7], [10], [11], [15], [16], [17], [18] and others. Kehayopulu [8, 9] generalized this concept and several results related in *poe*-and *le*-semigroups. In this paper we extend these concepts and results in le-Γ-semigroups. During this paper we introduce and give some characterizations of (m,n)-regular le-Γ-semigroup in terms of (m,n)-ideal elements and (m,n)-quasi-ideal elements. Also, we give some characterizations of subidempotent (m,n)-ideal elements in terms of r_α- and l_α- closed elements with respect to appropriate elements.

We introduce below necessary notions and present a few auxiliary results that will be used throughout the paper.

In 1986, Sen and Saha [20] defined Γ-semigroup as a generalization of semigroup and ternary semigroup as follows:

Definition 1.1. Let M and Γ be two non-empty sets. Denote by the letters of the English alphabet the elements of M and with the letters of the Greek alphabet the elements of Γ. Then M is called a Γ-semigroup if

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(1) $a\gamma b \in M$ *for all* $a, b \in M$ and $\gamma \in \Gamma$.
(2) $(a\alpha b)\beta c = a\alpha(b\beta c)$ *for all* $a, b, c \in M$ *and* $\alpha, \beta \in \Gamma$.
(3) *If* $m_1, m_2, m_3, m_4 \in M, \gamma_1, \gamma_2 \in \Gamma$ *such that* $m_1 = m_3, \gamma_1 = \gamma_2$ *and* $m_2 = m_4$, *then* $m_1\gamma_1 m_2 = m_3\gamma_2 m_4$.

Example 1.2. Let M be a semigroup and Γ be any non-empty set. If we define $a\gamma b = ab$ for all $a, b \in M$ and $\gamma \in \Gamma$. Then M is a Γ-semigroup.

Example 1.3. Let M be a set of all negative rational numbers. Obviously M is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} : p$ is prime$\}$. Let $a, b, c \in M$ and $\alpha \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in M$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence M is a Γ-semigroup.

Example 1.4. Let $M = \{-i, 0, i\}$ and $\Gamma = M$. Then M is a Γ-semigroup under the multiplication over complex numbers while M is not a semigroup under complex number multiplication.

These examples show that every semigroup is a Γ-semigroup and Γ-semigroups are a generalization of semigroups.

A Γ-semigroup M is called a *commutative* Γ*-semigroup* if for all $a, b \in M$ and $\gamma \in \Gamma$, $a\gamma b = b\gamma a$. A nonempty subset K of a Γ-semigroup M is called a *sub-*Γ*-semigroup* of M if for all $a, b \in K$ and $\gamma \in \Gamma$, $a\gamma b \in K$.

Example 1.5. Let $M = [0, 1]$ and $\Gamma = \{\frac{1}{n} | n$ is a positive integer$\}$. Then M is a Γ-semigroup under usual multiplication. Let $K = [0, 1/2]$. We have that K is a nonemtpy subset of M and $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then K is a sub-Γ-semigroup of M.

Different examples can be found in [1, 3, 4, 19, 20].

Definition 1.6. A *po-*Γ*-semigroup* (: *ordered* Γ*-semigroup*) is an ordered set M at the same time a Γ-semigroup such that for all $a, b, c \in M$ and for all $\gamma \in \Gamma$

$$a \leq b \Rightarrow a\gamma c \leq b\gamma c, c\gamma a \leq c\gamma b.$$

A *poe*-Γ-semigroup is a *po*-Γ-semigroup M with a greatest element "e" (i.e., $e \geq a, \forall a \in M$).

In a *po*-Γ-semigroup M, a is called a *right* (resp. *left*) *ideal element* if $a\alpha b \leq a$ (resp. $b\alpha a \leq a$) for all $b \in M$ and for all $\alpha \in \Gamma$. And a is called an *ideal element* if it is both a right and left ideal element. In a *poe*-Γ-semigroup M, a is called *right* (resp. *left*) *ideal element* if $a\alpha e \leq a$ (resp. $e\alpha a \leq a$) for all $\alpha \in \Gamma$.

Examples of ordered Γ-semigroups can be found in [3, 4, 6].

For nonempty subsets A and B of M and a nonempty subset Γ' of Γ, let $A\Gamma' B = \{a\gamma b : a \in A, b \in B$ and $\gamma \in \Gamma'\}$. If $A = \{a\}$, then we also write $\{a\}\Gamma' B$ as $a\Gamma' B$, and similarly if $B = \{b\}$ or $\Gamma' = \{\gamma\}$.

Let T be a sub-Γ-semigroup of M. For $A \subseteq T$ we denote

$$(A]_T = \{t \in T | t \leq a \text{ for some } a \in A\},$$
$$[A)_T = \{t \in T | t \geq a \text{ for some } a \in A\}.$$

An element a of a *po*-Γ-semigroup M is called *regular* if there exists $b \in M$ such that $a \leq a\alpha b\beta a$ for some $\alpha, \beta \in \Gamma$. A *po*-Γ-semigroup M is called *regular* if every element of M is regular. The following are equivalent:

(1) For every $A \subseteq M, A \subseteq (A\Gamma M\Gamma A]$,
(2) For every element $a \in M, a \in (a\Gamma M\Gamma a]$.

An element a of a *poe*-Γ-semigroup is called a *quasi-ideal element* if $e\alpha a \wedge a\alpha e$ exists and $a\alpha e \wedge e\alpha a \leq a$ for all $\alpha \in \Gamma$. We denote by $q(a)$ the quasi-ideal element of M generated by a, i.e., the least quasi-ideal element of M containing a. We say that $a \in M$ is a *bi-ideal element* of M if and only if $a\alpha e\beta a \leq a$, $\forall \alpha, \beta \in \Gamma$.

Definition 1.7. Let M be a semilattice under $\vee$ with a greatest element e and at the same time a *po*-Γ-semigroup such that for all $a, b, c \in M$ and for all $\gamma \in \Gamma$

$$a\gamma(b \vee c) = a\gamma b \vee a\gamma c$$

and

$$(a \vee b)\gamma c = a\gamma c \vee b\gamma c.$$

Then M is called a $\vee e$-Γ-*semigroup*.

A $\vee e$-Γ-semigroup which is also a lattice is called an le-Γ-*semigroup*.

The usual order relation $\leq$ on M is defined in the following way

$$a \leq b \Leftrightarrow a \vee b = b.$$

Then we can show that for any $a, b, c \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

Example 1.8 ([1])**.** Let $(X, \leq)$ and $(Y, \leq)$ be two finite chains. Let M be the set of all isotone mappings from X into Y and Γ be the set of all isotone mappings from Y into X. Let $f, g \in M$ and $\alpha \in \Gamma$. We define $f\alpha g$ to denote the usual mapping composition of f, α and g. Then M is a Γ-semigroup. For $f, g \in M$, the mappings $f \vee g$ and $f \wedge g$ are defined by letting, for each $a \in X$

$$(f \vee g)(a) = \max\{f(a), g(a)\},\ (f \wedge g)(a) = \min\{f(a), g(a)\}$$

(the maximum and minimum are considered with respect to the order $\leq$ in X and Y). The greatest element e is the mapping that sends every $a \in X$ to the greatest element of finite chains $(Y, \leq)$. Then M is an le-Γ-semigroup.

Example 1.9 ([1])**.** Let M be a *po*-Γ-semigroup. Let M_1 be the set of all ideals of M. Then $(M_1, \subseteq, \cap, \cup)$ is an le-Γ-semigroup.

Example 1.10 ([1])**.** Let M be a *po*-Γ-semigroup. Let $M_1 = P(M)$ be the set of all subsets of M and $\Gamma_1 = P(\Gamma)$ the set of all subsets of Γ. Then M_1 is a

po-Γ_1-semigroup if

$$A\Lambda B = \begin{cases} (A](\Lambda](B] = (A\Lambda B] & \text{if } A, B \in M_1\backslash\{\emptyset\}, \Lambda \in \Gamma_1\backslash\{\emptyset\} \\ \emptyset & \text{if } A = \emptyset \text{ or } B = \emptyset. \end{cases}$$

Then $(M_1, \subseteq, \cap, \cup)$ is an le-Γ_1-semigroup.

Example 1.11. Let G be a group, $I, \wedge$ two index sets and Γ the collection of some $\wedge \times I$ matrices over $G^o = G \cup \{0\}$, the group with zero. Let μ^o be the set of all elements $(a)_{i\lambda}$ where $i \in I, \lambda \in \wedge$ and $(a)_{i\lambda}$ the $I \times \wedge$ matrix over G^o having a in the i-th row and λ-th column, its remaining entries being zero. The expression $(0)_{i\lambda}$ will be used to denote the zero matrix. For any $(a)_{i\lambda}, (b)_{j\mu}, (c)_{k\nu} \in \mu^o$ and $\alpha = (p_{\lambda i}), \beta = (q_{\lambda i}) \in \Gamma$ we define $(a)_{i\lambda}\alpha(b)_{j\mu} = (ap_{\lambda j}b)_{i\mu}$. Then it is easy verified that $[(a)_{i\lambda}\alpha(b)_{j\mu}]\beta(c)_{k\nu} = (a)_{i\lambda}\alpha[(b)_{j\mu}\beta(c)_{k\nu}]$. Thus μ^o is a Γ-semigroup. We call Γ the sandwich matrix set and μ^o the Rees $I \times \wedge$ matrix Γ-semigroup over G^o with sandwich matrix set Γ and denote it by $\mu^o(G : I, \wedge, \Gamma)$. In [6] we deal with lattice-ordered Rees matrix Γ-semigroups.

If M is a $\vee e$-Γ-semigroup, then every map $\varphi : M \to M$ is called a topology on M [1]. A topology φ on M is said to be an:

(1) S-topology on M if and only if $a_1, a_2 \in M$, with $a_1 \leq a_2$ implies $\varphi(a_1) \leq \varphi(a_2)$.
(2) I-topology on M if and only if $a \in M$, implies $a \leq \varphi(a)$.
(3) U-topolgy on M if and only if $\varphi(\varphi(a)) = \varphi(a)$ for every $a \in M$.

An element $a \in M$ is called a closed element of M related to a topology φ (or φ-closed) if and only if $\varphi(a) = a$. The set of all closed element of M related to φ will be denoted by F_φ.

In a $\vee e$-Γ-semigroup M, we define two mappings r_α and l_β for each $\alpha, \beta \in \Gamma$ as follows:

$$r_\alpha : M \to M, r_\alpha(a) = a\alpha e \vee a,$$
$$l_\beta : M \to M, l_\beta(a) = e\beta a \vee a$$

for all $a \in M$.

It is clear that r_α and l_β, for $\alpha, \beta \in \Gamma$, are I- and S-topologies on M. If M is an le-Γ-semigroup, then r_α and l_β are U-topologies on M.

For other definitions and terminologies not given in this paper, the reader is refered to [1], [19], [20].

2. On (m, n)-ideal elements in le-Γ-semigroups

Let M be a $\vee e$-Γ-semigroup and $m, n \in \mathbb{Z}^+$.

Definition 2.1. An element $a \in M$ is called an (m, n)-*ideal element* of M if

$$(a\gamma_1 a\gamma_2 a \ldots \gamma_{m-1} a)\alpha e\beta(a\rho_1 a\rho_2 a \ldots \rho_{n-1} a) \leq a$$

for all $\alpha, \beta \in \Gamma$ and for some $\gamma_1, \gamma_2, \ldots, \gamma_{m-1}, \rho_1, \ldots, \rho_{n-1} \in \Gamma$.

a^0 is defined such that $a^0\gamma b = b\gamma a^0 = b$ $(b \in M, \gamma \in \Gamma)$.

For $m = 0, n = 1$ (resp. $m = 1, n = 0$) Definition 2.1 give us the trivial case of left (resp. right)-ideal elements. It is clear that the right (resp. left)-ideal elements are $(m, 0)$ (resp. $(0, n)$)-ideal elements for every $m \geq 1$ (resp. $n \geq 1$).

An le-Γ-semigroup is called *subidempotent* if $a\alpha a \leq a$ for all $a \in M, \alpha \in \Gamma$.

In the sequel of the paper, for the sake of simplicity, we denote $a^m = a\gamma_1 a\gamma_2 a \dots \gamma_{m-1} a$ for some $\gamma_1, \gamma_2, \dots, \gamma_{m-1} \in \Gamma$ and for some $m \in \mathbb{Z}^+$.

It can be easily proved the following statements:

(1) Let M be a poe-Γ-semigroup and a, b two (m, n)-ideal elements of M. Then the intersection $a \wedge b$ if exists is an (m, n)-ideal element of M.

(2) Let M be a poe-Γ-semigroup. Then the k-power $(k \geq 1)$ of an (m, n)-ideal element is also an (m, n)-ideal element.

(3) Let M be a poe-Γ-semigroup. If a is an (m, n)-ideal element of M $(m, n \geq 1)$, then for every $b \in M, \alpha \in \Gamma$ such that $a\alpha b \leq a$ (resp. $b\alpha a \leq a$), the products $a\alpha b$ and $b\alpha a$ are (m, n)-ideal elements of M. For $m = n = 1$ we have: If a is a bi-ideal element, then $a\gamma b$ and $b\gamma a$, $\gamma \in \Gamma$, are bi-ideal elements, for all $b \in M$. Clearly, $\forall \alpha, \beta, \gamma \in \Gamma$,

$$(a\gamma b)\alpha e\beta(a\gamma b) = (a\gamma b)\alpha(e\beta a)\gamma b \leq (a\gamma b)\alpha e\gamma b = a\gamma(b\alpha e\gamma b) \leq a\gamma b,$$

similarly $(b\gamma a)\alpha e\beta(b\gamma a) \leq b\gamma a$.

(4) Let M be a $\vee e$-Γ-semigroup and a, b two left (resp. right)-ideal elements of M. Then the union $a \vee b$ is a subidempotent (m, n)-ideal element, $\forall m \geq 0, \forall n \geq 1$ (resp. $n \geq 0, m \geq 1$).

In the following, $\langle a \rangle_{(m,n)}$ will be denoted the principal (m, n)-ideal element of M generated by a, i.e., the least $\langle m, n \rangle$-ideal element of M containing a and by $I_{(m,n)}$ the set of all (m, n)-ideal elements of M.

Definition 2.2. Let M be a $\vee e$-Γ-semigroup and $m, n \in \mathbb{Z}^+$. M is called (m, n)-*regular* if

$$a \leq (a\gamma_1 a\gamma_2 a \dots \gamma_{m-1} a)\alpha e\beta(a\rho_1 a\rho_2 a \dots \rho_{n-1} a)$$

for all $a \in M, \alpha, \beta \in \Gamma$ and for some $\gamma_1, \gamma_2, \dots, \gamma_{m-1}, \rho_1, \dots, \rho_{n-1} \in \Gamma$.

Lemma 2.3. *Let M be a $\vee e$-Γ-semigroup and $m, n, k \in \mathbb{Z}^+$. Then the following hold true*:

(1) $(a \vee a^m\alpha e\beta a^k)^m\gamma e = a^m\gamma e$, $\forall a \in M, \alpha, \beta, \gamma \in \Gamma$.

(2) $e\gamma(a \vee a^k\alpha e\beta a^n)^n = e\gamma a^n$, $\forall a \in M, \alpha, \beta, \gamma \in \Gamma$.

(3) $\langle a \rangle_{(m,n)} = a \vee a^m\alpha e\beta a^n$, $\forall a \in M, \alpha, \beta, \in \Gamma$.

Proof. We prove the first two conditions in case $m, n \geq 1$, since the case $m = 0$ or $n = 0$ is obvious. For $a \in M, \alpha, \beta, \rho \in \Gamma$, we have

(1) $(a \vee a^m\alpha e\beta a^k)^m\gamma e = ((a \vee a^m\alpha e\beta a^k)\rho)^{m-1}(a \vee a^m\alpha e\beta a^k)\gamma e = ((a \vee a^m\alpha e\beta a^k)\rho)^{m-1}((a\gamma e \vee a^m\alpha e\beta a^k\gamma e) = ((a \vee a^m\alpha e\beta a^k)\rho)^{m-1}a\gamma e = ((a \vee a^m\alpha e\beta a^k)\rho)^{m-2}(a \vee a^m\alpha e\beta a^k)\rho a\gamma e = ((a \vee a^m\alpha e\beta a^k)\rho)^{m-2}(a\rho a\gamma e \vee a^m\alpha e\beta a^{k+1}\gamma e) = ((a \vee a^m\alpha e\beta a^k)\rho)^{m-2}a^2\gamma e = \dots = a^m\gamma e$.

The proof of (2) is analogues to that of (1).

(3) Let $m, n \geq 0, \alpha, \beta \in \Gamma$. Then, $(a \vee a^m \alpha e \beta a^n)^m \alpha e \beta (a \vee a^m \alpha e \beta a^n)^n = a^m \alpha e \beta a^n$ (by (1) and (2)), so that $a \vee a^m \alpha e \beta a^n \in I_{(m,n)}$. Now, if b is an (m, n)-ideal element of M containing a, then $a \vee a^m \alpha e \beta a^n \leq b$. This finishes the proof. □

Theorem 2.4. *A $\vee e$-Γ-semigroup M is (m, n)-regular if and only if*

$$(*) \qquad a^m \alpha e \beta a^n = a, \ \forall a \in I_{(m,n)}, \forall \alpha, \beta \in \Gamma.$$

Proof. ($\Rightarrow$) This statement is obvious.

($\Leftarrow$) Let $a \in M, \alpha, \beta \in \Gamma$. Since $\langle a \rangle_{(m,n)} \in I_{(m,n)}$, we have:

$$(\langle a \rangle_{(m,n)})^m \alpha e \beta (\langle a \rangle_{(m,n)})^n = \langle a \rangle_{(m,n)}$$

But, $(\langle a \rangle_{(m,n)})^m \alpha e = a^m \alpha e$ (by Lemma 2.3(3),(1)), and $e \beta (\langle a \rangle_{(m,n)})^n = e \beta a^n$ (by Lemma 2.3(3),(2)).

Thus, $a \leq a^m \alpha e \beta a^n$ and M is (m, n)-regular. □

Theorem 2.5. *Let M be a subidempotent le-Γ-semigroup. Then M is (m, n)-regular if and only if*

$$(**) \qquad a \wedge b = a^m \alpha b \wedge a \beta b^n, \ \forall a \in I_{(m,0)}, b \in I_{(0,n)}, \alpha, \beta \in \Gamma.$$

Proof. ($\Rightarrow$) Let M be a (m, n)-regular le-Γ-semigroup. Let $a \in I_{(m,0)}$ and $b \in I_{(0,n)}$. Then, $a^m \alpha e \leq a$ and $e \beta b^n \leq b$, hence $a^m \alpha b \wedge a \beta b^n \leq a^m \alpha e \wedge e \beta b^n \leq a \wedge b$. On the other hand, $a \wedge b \leq (a \wedge b)^m \alpha e \beta (a \wedge b)^n \leq a^m \alpha e \beta b^n \leq a^m \alpha b$, similarly, $a \wedge b \leq a \beta b^n$, hence $a \wedge b \leq a^m \alpha b \wedge a \beta b^n$.

($\Leftarrow$) Since M is a (0,0)-regular le-Γ-semigroup, the the statement is true for $m = n = 0$.

Let $m \neq 0, n = 0$. If $a \in I_{(m,0)}$, then since e is a (0,0)-ideal element of M, we have by $(**)$, that $a = a^m \alpha e, \alpha \in \Gamma$, so that M is $(m, 0)$-regular (Theorem 2.4). The proof in the case $m = 0, n \neq 0$ is analogues.

Now, let $m \neq 0, n \neq 0$. Then, M has the property:

$$(***) \qquad a \wedge b \leq a \alpha b, \ \forall a \in I_{(m,0)}, \ \forall b \in I_{(0,n)}, \ \forall \alpha \in \Gamma.$$

Indeed: let $a \in I_{(m,0)}$. Then $\forall \alpha \in \Gamma$,

$$a \wedge b = a^m \alpha b \wedge a \alpha b^n \leq a \alpha b \ (m \geq 1, n \geq 1).$$

Now, let $a \in M$. Since $\langle a \rangle_{(m,0)} \in I_{(m,0)}$ and e is a $(0, n)$-ideal element of M, we have by $(**)$, that $\forall \alpha, \beta \in \Gamma$

$$\begin{aligned} \langle a \rangle_{(m,0)} &= (\langle a \rangle_{(m,0)})^m \alpha e \wedge \langle a \rangle_{(m,0)} \beta e^n \\ &\leq (\langle a \rangle_{(m,0)})^m \alpha e = a^m \alpha e \text{ (by Lemma 2.3(3),(1))}, \end{aligned}$$

thus $\langle a \rangle_{(m,0)} = a^m \alpha e$. Similarly, $\langle a \rangle_{(0,n)} = e \beta a^n$. On the other hand,

$$\begin{aligned} a &\leq \langle a \rangle_{(m,0)} \wedge \langle a \rangle_{(0,n)} \leq \langle a \rangle_{(m,0)} \gamma \langle a \rangle_{(0,n)}, \forall \gamma \in \Gamma (\text{by} (***)) \\ &= a^m \alpha e^2 \beta a^n \leq a^m \alpha e \beta a^n. \end{aligned}$$

Therefore, M is (m,n)-regular, and this finishes the proof. □

Theorem 2.6. *Let M be a poe-Γ-semigroup and $m,n \in \mathbb{Z}^+$ with $m+n \geq 1$. Let be the following statements for $a \in M, \rho \in \Gamma$:*

(1) $\exists a_i \in F_{r_\rho}^{(a_{i-1})}$, *$i = 1,2,\ldots,m$ and*
$\exists b_j \in F_{l_\rho}^{(b_{j-1})}$, *$j = 1,2,\ldots,n$ where*
$a_0 = e, b_0 = a_m$ *and* $b_n = a$ *(resp.* $b_0 = e, a_0 = b_n$ *and* $a_m = a$*).*

(2) *a is a subidempotent (m,n)-ideal element of M.*

Then, (1) ⇒ (2). In particular, if M is a $\vee e$-Γ-semigroup, then (1) ⇔ (2).

Proof. (1) ⇒ (2). In fact a is subidempotent, and for $\alpha,\beta,\rho \in \Gamma$, we have:

$$\begin{aligned}
b_n^m \alpha e \beta b_n^n &= (b_n\rho)^{m-1} b_n \alpha e \beta b_n^n \leq (b_n\rho)^{m-1} a_1 \alpha e \beta b_n^n \\
&\leq (b_n\rho)^{m-1} a_1 \beta b_n^n = (b_n\rho)^{m-2} b_n \rho a_1 \beta b_n^n \leq (b_n\rho)^{m-2} a_2 \rho a_1 \beta b_n^n \\
&\leq (b_n\rho)^{m-2} a_2 \beta b_n^n \leq \cdots \leq (b_n\rho)^{m-(m-1)} a_{m-1} \beta b_n^n \\
&= b_n \rho a_{m-1} \beta b_n^n \leq a_m \rho a_{m-1} \beta b_n^n \leq a_m \beta b_n^n \\
&= a_m \beta b_n \rho b_n^{n-1} \leq a_m \beta b_1 \rho b_n^{n-1} \leq b_1 \rho b_n^{n-1} \\
&= b_1 \rho b_n \rho b_n^{n-2} \leq b_1 \rho b_2 \rho b_n^{n-2} \leq b_2 \rho b_n^{n-2} \\
&\leq \cdots \leq b_{n-1} \rho b_n^{n-(n-1)} = b_{n-1} \rho b_n \leq b_n.
\end{aligned}$$

(2) ⇒ (1). Let M be a $\vee e$-Γ-semigroup and let a be a subidempotent $\langle m,n\rangle$-ideal element of M. We put:

$$a_i = \langle a \rangle_{(i,0)},\ i = 0,1,2,\ldots,m \text{ and } b_j = \langle a \rangle_{(m,j)},\ j = 0,1,2,\ldots,n.$$

Then, by Lemma 2.3(3), we have $\forall \alpha,\rho \in \Gamma$,

$$\begin{aligned}
a_i &= \langle a \rangle_{(i,0)} = a \vee a^i \alpha e = a \vee (a\rho)^{i-1} a \alpha e \leq a \vee a^{i-1} \alpha e \\
&= \langle a \rangle_{(i-1,0)} = a_{i-1}, i = 1,2,\ldots,m
\end{aligned}$$

and $\forall \delta,\gamma,\rho \in \Gamma$

$$\begin{aligned}
b_j &= \langle a \rangle_{(m,j)} = a \vee a^m \delta e \gamma a^j = a \vee a^m \delta e \gamma a \rho a^{j-1} \leq a \vee a^m \delta e \gamma a^{j-1} \\
&= \langle a \rangle_{(m,j-1)} = b_{j-1}, j = 1,2,\ldots,n.
\end{aligned}$$

Also, $a_0 = e, b_0 = a_m$ and $b_n = a$. Moreover,

$$\begin{aligned}
a_i \rho a_{i-1} &= \langle a \rangle_{(i,0)} \rho \langle a \rangle_{(i-1,0)} = (a \vee a^i \alpha e) \rho (a \vee a^{i-1} \alpha e) \\
&= a^2 \vee a^i \alpha e \rho a \vee a^i \alpha e \vee a^i \alpha e \rho a^{i-1} \alpha e \leq a \vee a^i \alpha e \\
&= \langle a \rangle_{(i,0)} = a_i, i = 1,2,\ldots,m
\end{aligned}$$

that is,

$$a_i \in F_{r_\rho}^{(a_{i-1})}, i = 1,2,\ldots,m.$$

Also, $\forall \delta, \gamma, \rho \in \Gamma$

$$\begin{aligned}
&\langle a\rangle_{(m,j-1)}\rho\langle a\rangle_{(m,j)}\\
&= (a \vee a^m\delta e\gamma a^{j-1})\rho(a \vee a^m\delta e\gamma a^j)\\
&= a^2 \vee a^m\delta e\gamma a^j \vee a^{m+1}\delta e\gamma a^j \vee a^m\delta e\gamma a^{j-1+m}\delta e\gamma a^j\\
&\le a \vee a^m\delta e\gamma a^j = \langle a\rangle_{(m,j)}, j = 1, 2, \ldots, n.
\end{aligned}$$

Therefore,

$$b_{j-1}\rho b_j \le b_j, j = 1, 2, \ldots, n$$

that is,

$$b_j \in F_{l_\rho}^{(b_{j-1})}, j = 1, 2, \ldots, n.$$

The other case can be proved similarly. In that case, for (2) $\Rightarrow$ (1) we put:

$$b_j = \langle a\rangle_{(0,j)}, j = 0, 1, 2, \ldots, n$$

and

$$a_i = \langle a\rangle_{(i,n)}, i = 0, 1, 2, \ldots, m.$$

Let M be a *poe*-Γ-semigroup. An element $a \in M$ is called $r_\alpha l_\alpha$-*closed*, $\alpha \in \Gamma$, if there exists a right-ideal element $b \in M$ such that a is l_α-closed with respect to b. Similarly, a is called $l_\alpha r_\alpha$-*closed*, if there exists a left-ideal element b with $a \in F_{r_\alpha}^{(b)}$. □

Corollary 2.7. *Let M be a poe-Γ-semigroup. Then all $r_\alpha l_\alpha$-closed and $l_\alpha r_\alpha$-closed elements are subidempotent bi-ideal elements. In particular, if M is a $\vee e$-Γ-semigroup, the preceding three classes of elements are the same.*

3. On (m, n)-quasi-ideal elements in le-Γ-semigroups

Definition 3.1. Let M be a *poe*-Γ-semigroup. An element q of M is called an (m, n)-*quasi-ideal element* of M if $q^m\alpha e \wedge e\beta q^n$ exists and $q^m\alpha e \wedge e\beta q^n \le q$, $\alpha, \beta \in \Gamma$.

Remark 3.2. Every quasi-ideal element q of a *poe*-Γ-semigroup M is an (m, n)-quasi-ideal element of M for all $m, n \in \mathbb{Z}^+$ such that $q^m\alpha e \wedge e\beta q^n$ exists. For $m, n \in \mathbb{Z}^+$, every (m, n)-quasi-ideal element is an (m, n)-ideal element of M. If $\{q_i; i \in I\}$ is a nonempty family of (m, n)-quasi-ideal elements of M, then $\bigwedge_{i\in I} q_i$ is an (m, n)-quasi-ideal element if $(\bigwedge_{i\in I} q_i)\alpha e \wedge e\alpha(\bigwedge_{i\in I} q_i)$, $\forall \alpha \in \Gamma$ exists.

Remark 3.3. Quasi-ideal elements are subidempotent. In *poe*-Γ-semigroups, quasi-ideal elements are subidempotent bi-ideal elements. In $\vee e\Gamma$-semigroups, quasi-ideal elements are $r_\alpha l_\alpha$ (resp. $l_\alpha r_\alpha$)-closed. In regular le-Γ-semigroups, the converse of the last statement also holds (by Corollary 2.7 above and Corollary 2.3 [1]).

Let we denote by $Q_{(m,n)}$ the set of all (m,n)-quasi-ideal elements of M.

In distributive le-Γ-semigroups [2], the quasi-ideal elements are exactly the intersections of the left- and right- ideal elements. The following theorem shows that the analogues property is true for the (m,n)-quasi-ideal elements, too.

Theorem 3.4. *Let M be a distributive le-Γ-semigroup. Then, an element q is an (m,n)-quasi-ideal element of M if and only if there exists an $(m,0)$-ideal element a and a $(0,n)$-ideal element b of M such that*

$$q = a \wedge b.$$

Proof. $(\Rightarrow)$ Let $a \in I_{(m,0)}$ and $b \in I_{(0,n)}$. Then, since $a, b \in Q_{(m,n)}$, we have $a \wedge b \in Q_{(m,n)}$.

$(\Leftarrow)$ Let $q \in Q_{(m,n)}$. Then, by Lemma 2.2(3), we have

$$\begin{aligned} q &= q \vee (q^m \alpha e \wedge e\beta q^n) = (q \vee q^m \alpha e) \wedge (q \vee e\beta q^n) \\ &= \langle q\rangle_{(m,0)} \wedge \langle q\rangle_{(0,n)} \end{aligned}$$

where $\langle q\rangle_{(m,0)} \in I_{(m,0)}$ and $\langle q\rangle_{(0,n)} \in I_{(0,n)}$. □

It is clear that $(m,0)$ (resp. $(0,n)$)-ideal elements and $(m,0)$ (resp. $(0,n)$)-quasi-ideal elements are the same. So, the following theorem is true:

Theorem 3.5. *Let M be a distributive le-Γ-semigroup. Then, an element q is an (m,n)-quasi-ideal element of M if and only if there exists an $(m,0)$-quasi-ideal element a and a $(0,n)$-quasi-ideal element b of M such that*

$$q = a \wedge b.$$

Since the poe-Γ-semigroups are semilattices under $\wedge$, we have the following theorem.

Theorem 3.6. *Let M be a poe-Γ-semigroup. Then the element*

$$q = a \wedge b$$

where a is an $(m,0)$-ideal element and b an $(0,n)$-ideal element of M, is a (m,n)-quasi-ideal element of M.

We denote by $(a)_{(m,n)}$ the (m,n)-quasi-ideal element of M generated by $a \in M$.

Lemma 3.7. *Let M be an le-Γ-semigoup, $a \in M, \alpha, \beta, \gamma, \rho \in \Gamma$ and $m, n \in \mathbb{Z}^+$. The following hold true*:

(1) $(a \vee (a^m \alpha e \wedge e\beta a^n))^m \gamma e \leq a^m \gamma e$.
(2) $e\gamma(a \vee (a^m \alpha e \wedge e\beta a^n))^m \leq e\gamma a^n$.
(3) $(a)_{(m,n)}$ *exists and* $(a)_{(m,n)} = a \vee (a^m \alpha e \wedge e\beta a^n)$.

Proof. (1) Since for $m=0$ it is clear. Let $m \geq 1$. Then we have

$$\begin{aligned}
& (a \vee (a^m \alpha e \wedge e\beta a^n))^m \gamma e \\
= & (a \vee (a^m \alpha e \wedge e\beta a^n))^{m-1} \rho (a \vee (a^m \alpha e \wedge e\beta a^n)) \gamma e \\
\leq & (a \vee (a^m \alpha e \wedge e\beta a^n))^{m-1} \rho (a\gamma e \vee (a^m \alpha e^2 \wedge e\beta a^n \gamma e))
\end{aligned}$$

$$\begin{aligned}
= & (a \vee (a^m \alpha e \wedge e\beta a^n))^{m-1} \rho a \gamma e \\
= & (a \vee (a^m \alpha e \wedge e\beta a^n))^{m-2} \rho (a \vee (a^m \alpha e \wedge e\beta a^n)) \rho a \gamma e \\
\leq & (a \vee (a^m \alpha e \wedge e\beta a^n))^{m-2} \rho (a^2 \gamma e \vee (a^m \alpha e \rho a \gamma e \wedge e\beta a^{n+1} \gamma e)) \\
= & (a \vee (a^m \alpha e \wedge e\beta a^n))^{m-2} \rho a^2 \gamma e \\
\vdots & \\
\leq & a^m \gamma e.
\end{aligned}$$

(2) It is proved similarly.

(3) Let $m, n \geq 0$. From (1) and (2) we have $\forall \alpha, \beta \in \Gamma$

$$(a \vee (a^m \alpha e \wedge e\beta a^n))^m \alpha e \wedge e\beta (a \vee (a^m \alpha e \wedge e\beta a^n))^n \leq a^m \alpha e \wedge e\beta a^n$$

hence $a \vee (a^m \alpha e \wedge e\beta a^n)$ is an (m,n)-quasi-ideal element of M containing a. Now, if b is an (m,n)-quasi-ideal element of M such that $b \geq a$, then $a \vee (a^m \alpha e \wedge e\beta a^n) \leq b$. □

Remark 3.8. In general in le-Γ-semigroups, $\langle a \rangle_{(m,n)} \leq (a)_{(m,n)}$. In particular, $\langle a \rangle_{(m,0)} = (a)_{(m,0)}$ and $\langle a \rangle_{(0,n)} = (a)_{(0,n)}$.

Theorem 3.9. *Let M be an le-Γ-semigroup and $m, n \in \mathbb{Z}^+$. Then the following are equivalent*:

(1) *M is (m,n)-regular.*
(2) $a^m \alpha e \beta a^n = a$, $\forall a \in I_{(m,n)}, \alpha, \beta \in \Gamma$.
(3) $q^m \alpha e \beta q^n = q$, $\forall q \in Q_{(m,n)}, \alpha, \beta \in \Gamma$.
(4) $(\langle a \rangle_{(m,n)})^m \alpha e \beta (\langle a \rangle_{(m,n)})^n = \langle a \rangle_{(m,n)}$, $\forall a \in M, \alpha, \beta \in \Gamma$.
(5) $((a)_{(m,n)})^m \alpha e \beta ((a)_{(m,n)})^n = (a)_{(m,n)}$, $\forall a \in M, \alpha, \beta \in \Gamma$.

Proof. (1) $\Rightarrow$ (2). It is obvious by Theorem 2.4.

(2) $\Rightarrow$ (3). It is obvious by Remark 3.2.

(3) $\Rightarrow$ (1). Let $a \in M$. By Theorem 3.5 it follows that the element $\langle a \rangle_{(m,0)} \wedge \langle a \rangle_{(0,n)}$ is an (m,n)-quasi-ideal element of M. Thus, by (3) and Lemma 2.3,

$$\begin{aligned}
a & \leq \langle a \rangle_{(m,0)} \wedge \langle a \rangle_{(0,n)} \\
& = (\langle a \rangle_{(m,0)} \wedge \langle a \rangle_{(0,n)})^m \alpha e \beta (\langle a \rangle_{(m,0)} \wedge \langle a \rangle_{(0,n)})^n \\
& \leq (\langle a \rangle_{(m,0)})^m \alpha e \beta (\langle a \rangle_{(0,n)})^n \\
& = a^m \alpha e \beta a^n.
\end{aligned}$$

(2) $\Rightarrow$ (4). It is clear.

(4) $\Rightarrow$ (2). If $a \in I_{(m,n)}$, then $\langle a \rangle_{(m,n)} = a$ and by 4), we have $a^m \alpha e \beta a^n = a$.

(3) $\Rightarrow$ (5). The proof is similar with the previous case of (m, n)-ideal elements. $\square$

Remark 3.10. From all above, it is clear that the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold in *poe*-Γ-semigroups, and the equivalence (2) $\Leftrightarrow$ (4) in $\vee e$-Γ-semigroups in general.

Remark 3.11. In regular *poe*-Γ-semigroups, we have $I_{(m,n)} = Q_{(m,n)}$. For every element a of a *poe*-Γ-semigroup M and $\alpha, \beta, \delta, \rho \in \Gamma$, we have

$$\begin{aligned} a^m\alpha e \wedge e\beta a^n &\leq (a^m\alpha e \wedge e\beta a^n)\delta e\rho(a^m\alpha e \wedge e\beta a^n) \leq (a^m\alpha e)\delta e\rho(e\beta a^n) \\ &\leq a^m\alpha e\beta a^n \leq a^m\alpha e \wedge e\beta a^n. \end{aligned}$$

So, in regular *poe*-Γ-semigroups, we have $a^m\alpha e\beta a^n = a^m\alpha e \wedge e\beta a^n$.

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