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An Algorithm for Quartically Hyponormal Weighted Shifts

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Abstract. Examples of a quartically hyponormal weighted shift which is not 3-hyponormal are discussed in this note. In [7] Exner-Jung-Park proved that if $\alpha(x)$: $\sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$ with $0 < x \leq \frac{53252}{100000}$, then $W_{\alpha(x)}$ is quartically hyponormal but not 4-hyponormal. And, Curto-Lee([5]) improved their result such as that if $\alpha(x)$: $\sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$ with $0 < x \leq \frac{667}{990}$, then $W_{\alpha(x)}$ is quartically hyponormal but not 3-hyponormal. In this note, we improve slightly Curto-Lee's extremal value by using an algorithm and computer software tool.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denotes the set of all bounded linear operators on $\mathcal{H}$. We denote $\mathbb{N}$ for the set of all positive integers. An operator T is *weakly n-hyponormal* if $p(T)$ is hyponormal for all polynomial $p(x)$ with degree $\leq n$. For $A, B \in \mathcal{L}(\mathcal{H})$, let $[A, B] := AB - BA$. For $n \in \mathbb{N}$ and $T \in \mathcal{L}(\mathcal{H})$, T is (*strongly*) *n-hyponormal* if the operator matrix $([T^{*j}, T^i])_{i,j=1}^n$ is positive on the direct sum $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (n-copies), which is equivalent to $(T^{*j}T^i)_{i,j=0}^n$is positive([2]).

It is well-known that subnormal $\Rightarrow$ n-hyponormal $\Rightarrow$ weakly n-hyponormal, for every $n \geq 1$; the study for the gaps among those operator classes was discussed in [2], [3], [4] and [8]. In particular, unilateral weighted shifts were considered to study such gaps (cf. [2], [3], [4], [8] and [9]). In particular, an operator T is referred to be *quadratically* (*cubically*, or *quartically*, resp.) *hyponormal* if weakly 2- (weakly 3-, or weakly 4-, resp.) hyponormal.

In [6] Curto-Putinar proved that the polynomial hyponormality can not be always 2-hyponormality. But one does not know concrete examples of polynomially hyponormal but not 2-hyponormal yet. For a weight sequence $\alpha(x)$:

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$\sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$ with a positive real variable x, it is known that the corresponding weighted shift $W_{\alpha(x)}$ is quadratically hyponormal if and only if $0 < x \leq \sqrt{\frac{2}{3}}$ ([1]). And also in [1], one gave a question: "*Find the interval for cubic hyponormality of* $W_{\alpha(x)}$". This question is the one-step progress to find a concrete example for the gap between polynomially hyponormal and subnormal operators. In [7] Exner-Jung-Park found an example; if $\alpha : \sqrt{\frac{141}{250}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$, is a weight sequence, then W_α is cubically hyponormal but not 2-hyponormal. So it is worthwhile to solve the following problem.

Problem 1.1. *Find a quartically hyponormal weighted shift but not* 2*-hyponormal.*

In [7] Exner-Jung-Park proved that if $\alpha(x) : \sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$ with $0 < x \leq \frac{53252}{100000}$, then $W_{\alpha(x)}$ is quartically hyponormal but not 4-hyponormal, which also was improved by Curto-Lee in [5]; if $\alpha(x) : \sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$ with $0 < x \leq \frac{667}{990}$, then $W_{\alpha(x)}$ is quartically hyponormal but not 3-hyponormal. In this note we improve slightly some known results in the same context.

This note consists of four sections. In Section 2, we give some useful and crucial lemmas which will be used in the later sections. In Section 3, for a weighted shift $W_{\alpha(x)}$ with Bergmann tail beginning by positive real variable x, we construct the main algorithm for its quartic hyponormality. By using the main algorithm, we prove that if $\alpha(x) : \alpha_0 = \sqrt{x}, \alpha_n = \sqrt{\frac{n+2}{n+3}}$ $(n \geq 1)$ with $0 < x \leq \widehat{\delta}$ (see Theorem 3.1) for the number $\widehat{\delta} > \frac{667}{990}$, then $W_{\alpha(x)}$ is quartically hyponormal but not 3-hyponormal. In Section 4, we prove the main theorem and conclude our results.

Some of the calculations in this note were obtained through computer experiments using the software tool *Mathematica* [10].

2. Some lemmas

We begin with a characterization for a quartically hyponormal weighted shift W_α with a weight sequence $\alpha = \{\alpha_n\}_{n\in\mathbb{N}_0}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Lemma 2.1([7, Th. 2.3]). *Suppose W_α is a contractive hyponormal weighted shift with weight sequence $\alpha := \{\alpha_n\}_{n\in\mathbb{N}_0}$. Then W_a is quartically hyponormal if and only if the following holds:*

$$\Delta_4^\alpha(\phi, p, q) := \gamma_4 |\phi_4 p_0|^2 + \left\langle \begin{bmatrix} \gamma_3 & \gamma_4 \\ \gamma_4 & \gamma_5 \end{bmatrix} \begin{bmatrix} \phi_3 p_0 \\ \phi_4 p_1 \end{bmatrix}, \begin{bmatrix} \phi_3 p_0 \\ \phi_4 p_1 \end{bmatrix} \right\rangle \tag{2.1}$$

$$+ \left\langle \begin{bmatrix} \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_3 & \gamma_4 & \gamma_5 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{bmatrix} \begin{bmatrix} \phi_2 p_0 \\ \phi_3 p_1 \\ \phi_4 p_2 \end{bmatrix}, \begin{bmatrix} \phi_2 p_0 \\ \phi_3 p_1 \\ \phi_4 p_2 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \\ \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \end{bmatrix} \begin{bmatrix} p_0 \\ \phi_2 p_1 \\ \phi_3 p_2 \\ \phi_4 p_3 \end{bmatrix}, \begin{bmatrix} p_0 \\ \phi_2 p_1 \\ \phi_3 p_2 \\ \phi_4 p_3 \end{bmatrix} \right\rangle$$

$$+\sum_{k=0}^{\infty}\left\langle\begin{bmatrix}\gamma_k & \gamma_{k+1} & \gamma_{k+2} & \gamma_{k+3} & \gamma_{k+4}\\ \gamma_{k+1} & \gamma_{k+2} & \gamma_{k+3} & \gamma_{k+4} & \gamma_{k+5}\\ \gamma_{k+2} & \gamma_{k+3} & \gamma_{k+4} & \gamma_{k+5} & \gamma_{k+6}\\ \gamma_{k+3} & \gamma_{k+4} & \gamma_{k+5} & \gamma_{k+6} & \gamma_{k+7}\\ \gamma_{k+4} & \gamma_{k+5} & \gamma_{k+6} & \gamma_{k+7} & \gamma_{k+8}\end{bmatrix}\begin{bmatrix}q_k\\ p_{k+1}\\ \phi_2 p_{k+2}\\ \phi_3 p_{k+3}\\ \phi_4 p_{k+4}\end{bmatrix},\begin{bmatrix}q_k\\ p_{k+1}\\ \phi_2 p_{k+2}\\ \phi_3 p_{k+3}\\ \phi_4 p_{k+4}\end{bmatrix}\right\rangle \geq 0$$

for any $\phi :\equiv \{\phi_i\}_{i=2}^{3}$, $p :\equiv \{p_i\}_{i=0}^{\infty}$, *and* $q :\equiv \{q_i\}_{i=0}^{\infty}$ *in* $\mathbb{C}$.

For brevity we denote $\gamma_k := \alpha_0^2\alpha_1^2\cdots\alpha_{k-1}^2$, $k \in \mathbb{N}_0$, which sometimes are said to be *moments* (cf. [2]).

3. Results

The following theorem is contained in our main results of this note.

Theorem 3.1. *Let* $\alpha(x) : \alpha_0 = \sqrt{x}, \alpha_n = \sqrt{\frac{n+2}{n+3}}$ $(n \geq 1)$. *If* $0 < x \leq \widehat{\delta}$, *where* $\widehat{\delta} = \frac{844406080653836692089752}{1253126026358282939343663}$, *then* $W_{\alpha(x)}$ *is quartically hyponormal but not* 3*-hyponormal.*

Proof. The proof of this proposition will be appeared in the next section. □

To prove Theorem 3.1, we need several lemmas and algorithm. First we restate Lemma 2.1 with the weighted shift in Theorem 3.1.

Lemma 3.2. *Let* $\alpha(x) : \alpha_0 = \sqrt{x}, \alpha_n = \sqrt{\frac{n+2}{n+3}}$ $(n \geq 1)$. *Then* $W_{a(x)}$ *is quartically hyponormal if and only if the following holds:*

$$\frac{1}{3x}\Delta(a,b,c,p,q) := \frac{1}{6}\left|cp_0\right|^2 + \left\langle\begin{bmatrix}\frac{1}{5} & \frac{1}{6}\\ \frac{1}{6} & \frac{1}{7}\end{bmatrix}\begin{bmatrix}bp_0\\ cp_1\end{bmatrix},\begin{bmatrix}bp_0\\ cp_1\end{bmatrix}\right\rangle \tag{3.1}$$

$$+\left\langle\begin{bmatrix}\frac{1}{4} & \frac{1}{5} & \frac{1}{6}\\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7}\\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8}\end{bmatrix}\begin{bmatrix}ap_0\\ bp_1\\ cp_2\end{bmatrix},\begin{bmatrix}ap_0\\ bp_1\\ cp_2\end{bmatrix}\right\rangle + \left\langle\begin{bmatrix}\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6}\\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7}\\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8}\\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9}\end{bmatrix}\begin{bmatrix}p_0\\ ap_1\\ bp_2\\ cp_3\end{bmatrix},\begin{bmatrix}p_0\\ ap_1\\ bp_2\\ cp_3\end{bmatrix}\right\rangle$$

$$+\left\langle\begin{bmatrix}\frac{1}{3x} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6}\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7}\\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8}\\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9}\\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10}\end{bmatrix}\begin{bmatrix}q_0\\ p_1\\ ap_2\\ bp_3\\ cp_4\end{bmatrix},\begin{bmatrix}q_0\\ p_1\\ ap_2\\ bp_3\\ cp_4\end{bmatrix}\right\rangle$$

$$+\sum_{k=1}^{\infty}\left\langle\begin{bmatrix}\frac{1}{k+2} & \frac{1}{k+3} & \frac{1}{k+4} & \frac{1}{k+5} & \frac{1}{k+6}\\ \frac{1}{k+3} & \frac{1}{k+4} & \frac{1}{k+5} & \frac{1}{k+6} & \frac{1}{k+7}\\ \frac{1}{k+4} & \frac{1}{k+5} & \frac{1}{k+6} & \frac{1}{k+7} & \frac{1}{k+8}\\ \frac{1}{k+5} & \frac{1}{k+6} & \frac{1}{k+7} & \frac{1}{k+8} & \frac{1}{k+9}\\ \frac{1}{k+6} & \frac{1}{k+7} & \frac{1}{k+8} & \frac{1}{k+9} & \frac{1}{k+10}\end{bmatrix}\begin{bmatrix}q_k\\ p_{k+1}\\ ap_{k+2}\\ bp_{k+3}\\ cp_{k+4}\end{bmatrix},\begin{bmatrix}q_k\\ p_{k+1}\\ ap_{k+2}\\ bp_{k+3}\\ cp_{k+4}\end{bmatrix}\right\rangle \geq 0$$

for any a, b, c, $p :\equiv \{p_i\}_{i\in\mathbb{N}_0}$, *and* $q :\equiv \{q_i\}_{i\in\mathbb{N}_0}$ *in* $\mathbb{C}$.

We now discuss our main algorithm as following.

Algorithm 3.3.

Step I. 1° Fix $n \equiv 0 \pmod 3$. Say $n = 3k$.

2° Find $\varepsilon_{3k} > 0$ satisfying $\det \Phi_{3k}(\varepsilon_{3k}) = 0$, where

$$\Phi_j(t) := \begin{bmatrix} \frac{1}{j+2} & \frac{1}{j+3} & \frac{1}{j+4} & \frac{1}{j+5} & \frac{1}{j+6} \\ \frac{1}{j+3} & \frac{1}{j+4} - t & \frac{1}{j+5} & \frac{1}{j+6} & \frac{1}{j+7} \\ \frac{1}{j+4} & \frac{1}{j+5} & \frac{1}{j+6} & \frac{1}{j+7} & \frac{1}{j+8} \\ \frac{1}{j+5} & \frac{1}{j+6} & \frac{1}{j+7} & \frac{1}{j+8} & \frac{1}{j+9} \\ \frac{1}{j+6} & \frac{1}{j+7} & \frac{1}{j+8} & \frac{1}{j+9} & \frac{1}{j+10} \end{bmatrix}, \quad j \geq 1. \tag{3.2}$$

3° Find $\varepsilon_{3k-1} > 0$ satisfying $\det \Phi_{3k-1}(\varepsilon_{3k-1}) = 0$.

4° Find $\varepsilon_{3k-2} > 0$ satisfying $\det \Phi_{3k-2}(\varepsilon_{3k-2}) = 0$.

Step II. 1° Consider the matrix

$$\Psi_{3j-3}(s,t,u,v,a,b,c) =$$

$$\begin{bmatrix} \frac{1}{3j-1} & \frac{1}{3j} & \frac{a}{3j+1} & \frac{b}{3j+2} & \frac{c}{3j+3} \\ \frac{1}{3j} & \frac{1}{3j+1} - s & \frac{a}{3j+2} & \frac{b}{3j+3} & \frac{c}{3j+4} \\ \frac{\bar{a}}{3j+1} & \frac{\bar{a}}{3j+2} & \frac{|a|^2}{3j+3} + t & \frac{\bar{a}b}{3j+4} & \frac{\bar{a}c}{3j+5} \\ \frac{\bar{b}}{3j+2} & \frac{\bar{b}}{3j+3} & \frac{a\bar{b}}{3j+4} & \frac{|b|^2}{3j+5} + u & \frac{\bar{b}c}{3j+6} \\ \frac{\bar{c}}{3j+3} & \frac{\bar{c}}{3j+4} & \frac{a\bar{c}}{3j+5} & \frac{b\bar{c}}{3j+6} & \frac{|c|^2}{3j+7} + v \end{bmatrix}, \quad 2 \leq j \leq k,$$

and find $\varepsilon_{3k-3} := \varepsilon_{3k-3}(a,b,c) > 0$ such that

$$\det \Psi_{3k-3}(\varepsilon_{3k-3}, \varepsilon_{3k-2}, \varepsilon_{3k-1}, \varepsilon_{3k}, a, b, c) = 0 \tag{3.3}$$

for all $a, b, c \in \mathbb{C}$.

2° Find $\varepsilon_{3k-4} > 0$ satisfying $\det \Phi_{3k-4}(\varepsilon_{3k-4}) = 0$.

3° Find $\varepsilon_{3k-5} > 0$ satisfying $\det \Phi_{3k-5}(\varepsilon_{3k-5}) = 0$.

4° Find $\varepsilon_{3k-6} := \varepsilon_{3k-6}(a,b,c) > 0$ satisfying

$$\det \Psi_{3k-6}(\varepsilon_{3k-6}, \varepsilon_{3k-5}, \varepsilon_{3k-4}, \varepsilon_{3k-3}, a, b, c) = 0$$

for all $a, b, c \in \mathbb{C}$.

Step III. By repeating Step II, find $\varepsilon_{3k-7}, \varepsilon_{3k-8}, \varepsilon_{3k-9}, \cdots, \varepsilon_2, \varepsilon_1$.

Step IV. Consider the matrix

$$\Psi_0(x,a,b,c) := \begin{bmatrix} \frac{1}{3x} & \frac{1}{3} & \frac{1}{4}a & \frac{1}{5}b & \frac{1}{6}c \\ \frac{1}{3} & \phi(a,b,c) & \frac{1}{5}a & \frac{1}{6}b & \frac{1}{7}c \\ \frac{1}{4}\bar{a} & \frac{1}{5}\bar{a} & \frac{1}{6}|a|^2 + \varepsilon_1 & \frac{1}{7}b\bar{a} & \frac{1}{8}c\bar{a} \\ \frac{1}{5}\bar{b} & \frac{1}{6}\bar{b} & \frac{1}{7}a\bar{b} & \frac{1}{8}|b|^2 + \varepsilon_2 & \frac{1}{9}c\bar{b} \\ \frac{1}{6}\bar{c} & \frac{1}{7}\bar{c} & \frac{1}{8}a\bar{c} & \frac{1}{9}b\bar{c} & \frac{1}{10}|c|^2 + \varepsilon_3 \end{bmatrix}, \tag{3.4}$$

where

$$\phi(a,b,c) = \frac{1}{4} + \frac{1}{252}|c|^2 + \frac{1}{7350}|b|^2 + \frac{1+200|c|^2}{17640(2+25|c|^2)}|a|^2,$$

and find a range $\mathcal{R}$ of x such that $\Psi_0(x,a,b,c) \geq 0$, for all $a,b,c \in \mathbb{C}$ by using Determinant Nested Test (cf. [2]).

Proposition 3.4. *For given $k \in \mathbb{N}$, the values ε_j, $1 \leq j \leq 3k$, can be obtained recursively.*

Proof. The proof of this proposition also will be appeared in the next section. $\square$

The following lemma shows the reason why we give Algorithm 3.3 and its legitimacy.

Lemma 3.5. *Under the same notation of Algorithm 3.3, it holds that if*

$$\begin{aligned}
\frac{1}{3x}\Delta(a,b,c,p,q) &= \left\langle \begin{bmatrix} \frac{1}{5} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{7} - \frac{1}{252} \end{bmatrix} \begin{bmatrix} bp_0 \\ cp_1 \end{bmatrix}, \begin{bmatrix} bp_0 \\ cp_1 \end{bmatrix} \right\rangle \\
&+ \left\langle \begin{bmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} - \frac{1}{7350} & \frac{1}{7} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} ap_0 \\ bp_1 \\ cp_2 \end{bmatrix}, \begin{bmatrix} ap_0 \\ bp_1 \\ cp_2 \end{bmatrix} \right\rangle \\
&+ \left\langle \begin{bmatrix} \frac{1}{3} + \frac{|c|^2}{6} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} - \frac{200|c|^2+1}{17640(25|c|^2+2)} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} p_0 \\ ap_1 \\ bp_2 \\ cp_3 \end{bmatrix}, \begin{bmatrix} p_0 \\ ap_1 \\ bp_2 \\ cp_3 \end{bmatrix} \right\rangle \\
&+ \left\langle \Psi_0(x,a,b,c) \begin{bmatrix} q_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}, \begin{bmatrix} q_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \right\rangle + \sum_{\substack{j \not\equiv 0 \ (\mathrm{mod} 3) \\ 1 \leq j \leq n}} \left\langle \Phi_j(\varepsilon_j) \begin{bmatrix} q_j \\ p_{j+1} \\ ap_{j+2} \\ bp_{j+3} \\ cp_{j+4} \end{bmatrix}, \begin{bmatrix} q_j \\ p_{j+1} \\ ap_{j+2} \\ bp_{j+3} \\ cp_{j+4} \end{bmatrix} \right\rangle \\
&+ \sum_{\substack{j \equiv 0 \ (\mathrm{mod} 3) \\ 1 \leq j \leq n}} \left\langle \Psi_j(\varepsilon_j, \varepsilon_{j+1}, \varepsilon_{j+2}, \varepsilon_{j+3}, a, b, c) \begin{bmatrix} q_j \\ p_{j+1} \\ p_{j+2} \\ p_{j+3} \\ p_{j+4} \end{bmatrix}, \begin{bmatrix} q_j \\ p_{j+1} \\ p_{j+2} \\ p_{j+3} \\ p_{j+4} \end{bmatrix} \right\rangle \\
&+ \sum_{k=n+1}^{\infty} \left\langle \begin{bmatrix} \frac{1}{k+2} & \frac{1}{k+3} & \frac{1}{k+4} & \frac{1}{k+5} & \frac{1}{k+6} \\ \frac{1}{k+3} & \frac{1}{k+4} & \frac{1}{k+5} & \frac{1}{k+6} & \frac{1}{k+7} \\ \frac{1}{k+4} & \frac{1}{k+5} & \frac{1}{k+6} & \frac{1}{k+7} & \frac{1}{k+8} \\ \frac{1}{k+5} & \frac{1}{k+6} & \frac{1}{k+7} & \frac{1}{k+8} & \frac{1}{k+9} \\ \frac{1}{k+6} & \frac{1}{k+7} & \frac{1}{k+8} & \frac{1}{k+9} & \frac{1}{k+10} \end{bmatrix} \begin{bmatrix} q_k \\ p_{k+1} \\ ap_{k+2} \\ bp_{k+3} \\ cp_{k+4} \end{bmatrix}, \begin{bmatrix} q_k \\ p_{k+1} \\ ap_{k+2} \\ bp_{k+3} \\ cp_{k+4} \end{bmatrix} \right\rangle \geq 0,
\end{aligned}$$

for any a,b,c, $p :\equiv \{p_i\}_{i=0}^{\infty}$, and $q :\equiv \{q_i\}_{i=0}^{\infty}$ in $\mathbb{C}$, then $\Delta_4^{\alpha}(a,b,c,p,q) \geq 0$ for any a,b,c, $p :\equiv \{p_i\}_{i\in\mathbb{N}_0}$, and $q :\equiv \{q_i\}_{i\in\mathbb{N}_0}$ in $\mathbb{C}$. Hence if $x \in \mathcal{R}$, then $W_{\alpha(x)}$ is *quartically hyponormal.*

4. Proof of Theorem 3.1

First we find the regions of 2- and 3-hyponormality as following.

Lemma 4.1. *Let* $\alpha(x,y) : \alpha_0 = \sqrt{x}, \alpha_1 = \sqrt{y}, \alpha_n = \sqrt{\frac{n+2}{n+3}}$ $(n \geq 2)$. *Then we have the following assertions*:

1° $W_{\alpha(x,y)}$ *is* 2-*hyponormal if and only if* $0 < y \leq \frac{25}{33}$, $0 < x \leq \frac{2y}{5(15y^2-24y+10)}$.

2° $W_{\alpha(x,y)}$ *is* 3-*hyponormal if and only if* $0 < y \leq \frac{100}{133}$, $0 < x \leq \frac{50y}{5(6433y^2-9800y+3750)}$.

Proof of Proposition 3.4. For $k \geq 1$ in the Algorithm 3.3 Step I, we may compute $\varepsilon_{3k}, \varepsilon_{3k-1}, \varepsilon_{3k-2}$ as following:

$$\begin{aligned}
\varepsilon_{3k} &= \frac{4}{9(k+1)^2(k+2)^2(3k+4)(3k+5)^2(3k+7)^2}, \\
\varepsilon_{3k-1} &= \frac{4}{3(k+1)(k+2)^2(3k+2)^2(3k+4)^2(3k+5)^2}, \\
\varepsilon_{3k-2} &= \frac{4}{(k+1)^2(3k+1)^2(3k+2)(3k+4)^2(3k+5)^2}.
\end{aligned}$$

By using these values, we obtain an expression ε_{3k-3} in the step II and we compute the value $\varepsilon_{27}(a,b,c)$ with $k = 10$; in fact, $\varepsilon_{27}(a,b,c) = \frac{\varphi(a,b,c)}{\phi(a,b,c)}$ with

$$\begin{aligned}
\varphi(a,b,c) &= a_0 + a_1|a|^2 + a_2|b|^2 + a_3|c|^2 + a_4|a|^2|b|^2 + a_5|a|^2|c|^2 \\
&\quad + a_6|b|^2|c|^2 + a_7|a|^2|b|^2|c|^2; \\
\phi(a,b,c) &= 41738400(b_0 + +b_1|a|^2 + b_2|b|^2 + b_3|c|^2 + b_4|a|^2|b|^2 \\
&\quad + b_5|a|^2|c|^2 + b_6|b|^2|c|^2 + b_7|a|^2|b|^2|c|^2),
\end{aligned}$$

where

$$\begin{aligned}
a_0 &= 4488, a_1 = 728229425, a_2 = 7131317760, a_3 = 31015353600, \\
a_4 &= 1000984446000, a_5 = 16432968182400, a_6 = 38026703232000, \\
a_7 &= 17428905648000; \\
b_0 &= 3, b_1 = 498467200, b_2 = 1297794960, b_3 = 2662934400, \\
b_4 &= 186536395584000, b_5 = 1444773283430400, b_6 = 888874188048000, \\
b_7 &= 417178285590528000.
\end{aligned}$$

And we repeat the method of Step I to obtain $\varepsilon_{3k-4}, \varepsilon_{3k-5}$ as following:

$$\begin{aligned}
\varepsilon_{3k-4} &= \frac{4}{3k(3k-1)^2(k+1)^2(3k+1)^2(3k+2)^2}, \\
\varepsilon_{3k-5} &= \frac{4}{k^2(3k-2)^2(3k-1)(3k+1)^2(3k+2)^2}.
\end{aligned}$$

Because the next step follows similar process, we obtain the values ε_j for $0 < j \leq 3k-6$ recursively. $\square$

Before proving Theorem 3.1, we obtain the following diagram for our convenience. We take $k = 10$, and obtain the following table.

Table 4.1

j	30	29	28	26	25	23
ε_j	$\frac{1}{2235366995400}$	$\frac{1}{1722702643200}$	$\frac{1}{1317324192800}$	$\frac{1}{751045025280}$	$\frac{1}{559341977600}$	$\frac{1}{300858667200}$
j	22	20	19	17	16	14
ε_j	$\frac{1}{216965385000}$	$\frac{1}{108622215000}$	$\frac{1}{75252320000}$	$\frac{1}{34411238400}$	$\frac{1}{22645104020}$	$\frac{1}{921817800}$
j	13	11	10	8	7	5
ε_j	$\frac{1}{5655859200}$	$\frac{1}{1957616640}$	$\frac{1}{1094038400}$	$\frac{1}{300600300}$	$\frac{1}{145745600}$	$\frac{1}{27878400}$
j	4	2	1	*	*	*
ε_j	$\frac{1}{10672200}$	$\frac{1}{1058400}$	$\frac{1}{250880}$	*	*	*

We now prove Theorem 3.1 by using Proposition 3.4.

Proof of Theorem 3.1. To apply Step II 1° of Algorithm 3.3, we first consider $\det \Psi_{27}(\varepsilon_{27}, \varepsilon_{28}, \varepsilon_{29}, \varepsilon_{30}, a, b, c)$ with

$$\varepsilon_{28} = \frac{1}{1317324192800}, \ \varepsilon_{29} = \frac{1}{1722702643200}, \ \varepsilon_{30} = \frac{1}{2235366995400},$$

and also obtain an expression $\varepsilon_{27} := \varepsilon_{27}(a, b, c)$ in Step II 1° of Algorithm 3.3 such that

$$\det \Psi_{27}(\varepsilon_{27}, \varepsilon_{28}, \varepsilon_{29}, \varepsilon_{30}, a, b, c) = 0.$$

With $\varepsilon_{25} = \frac{1}{559341977600}$, $\varepsilon_{26} = \frac{1}{751045025280}$, and $\varepsilon_{27} = \varepsilon_{27}(a, b, c)$, we find $\varepsilon_{24} := \varepsilon_{24}(a, b, c) > 0$ satisfying

$$\det \Psi_{24}(\varepsilon_{24}, \varepsilon_{25}, \varepsilon_{26}, \varepsilon_{27}, a, b, c) = 0.$$

Continuing this process 9-times, we obtain $\varepsilon_3 := \varepsilon_3(a, b, c)$ such that

$$\det \Psi_3(\varepsilon_3, \frac{1}{10672200}, \frac{1}{27878400}, \varepsilon_6, a, b, c) = 0,$$

where ε_6 is in the value of 8-times process. Finally, we substitute the values $\varepsilon_1 = \frac{1}{250880}$, $\varepsilon_2 = \frac{1}{1058400}$, and ε_3 in the matrix $\Psi_0(x, a, b, c)$ of (3.4) in Step IV. Then we obtain

$$\det \Psi_0(x, a, b, c) = F(x, a, b, c) \cdot \sum_{(i,j,k) \in \Gamma} (A_{ijk} - B_{ijk} x) \left|a\right|^{2i} \left|b\right|^{2j} \left|c\right|^{2k}$$

for some positive real function $F(x, a, b, c)$, positive real numbers A_{ijk} and B_{ijk} (note that the cardinality of Γ is large, in fact, more than at least 10^3). By long and boring computations, we obtain that

$$\min\{\frac{A_{ijk}}{B_{ijk}} : (i, j, k) \in \Gamma\} = \frac{8444060806538366920892752}{12531260263582829393436663} =: \widehat{\delta}.$$

Hence obviously, if $0 < x \leq \widehat{\delta}$, then $W_{\alpha(x)}$ is quartically hyponormal but not 3-hyponormal. $\square$

Remark 4.2. If we give repeat the processes in proof of Theorem 3.1 with $n > 10$, we are able to obtain the bigger value $\widehat{\delta}_n > \widehat{\delta}$. We do not know whether or not there exists a sufficient large $N \in \mathbb{N}$ such that $W_{\alpha(\widehat{\delta}_N)}$ is quartically hyponormal but not 2-hyponormal. In fact, the existence of such a large number N solves Problem 1.1.

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