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Principally Small Injective Rings

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Abstract. A right ideal I of a ring R is small in case for every proper right ideal K of R, $K + I \neq R$. A right R-module M is called PS-injective if every R-homomorphism $f : aR \to M$ for every principally small right ideal aR can be extended to $R \to M$. A ring R is called right PS-injective if R is PS-injective as a right R-module. We develop, in this article, PS-injectivity as a generalization of P-injectivity and small injectivity. Many characterizations of right PS-injective rings are studied. In light of these facts, we get several new properties of a right GPF ring and a semiprimitive ring in terms of right PS-injectivity. Related examples are given as well.

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary. Let R be a ring. The Jacobson radical and nil radical of R are denoted by $J(R)$ and $Nil(R)$, respectively. The right singular ideal is denoted by $Z(R_R)$, the socles are denoted by $soc(R_R)$ and $soc({}_RR)$. If X is a subset of R, the right (resp. left) annihilator of X in R is denoted by $r_R(X)$ (resp. $l_R(X)$). Let M and N be right R-modules. $\mathrm{Ext}^n(M,N)$ (resp. $\mathrm{Tor}_n(R/aR,M)$) means $\mathrm{Ext}^n_R(M,N)$ (resp. $\mathrm{Tor}^R_n(R/aR,M)$). If N is a submodule of M, we write $N \leq^{ess} M$ and $N \ll M$ to indicate that N is an essential submodule and a small submodule of M, respectively. The character module M^+ is defined by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. We will use the usual notations from [1, 5, 6, 9].

The concept of injectivity was firstly introduced by Baer in [2]. In recent decades, the generalizations of injective rings are extensively studied by many authors (see [3-4, 7-12]). Let R be a ring. A right ideal I of R is called *small* if for every proper right ideal K of R, $K + I \neq R$. A ring R is called right *small injective* [10] if every R-homomorphism $f : I \to R$ for every small right ideal I can be extended to $R \to R$. A ring R is said to be right *P-injective* [7] (resp. *mininjective* [4 or 8]) if every R-homomorphism $f : aR \to R$ for every principally (resp. minimal) right ideal aR can be extended to $R \to R$. In this paper, we say

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that R is *principally small injective* (abbr. *PS-injective*) if every R-homomorphism $f : aR \to R$ for every principally small right ideal aR can be extended to $R \to R$. The concept of PS-injective rings is introduced as a generalization of P-injective rings and small injective rings. Some examples of PS-injective rings are given. We show that if R is right PS-injective and satisfies the ACC on right annihilators of elements, then $J(R) = Z(R_R)$. In [7], Nicholson and Yousif proved that, if R is a right P-injective ring and $R/soc(R_R)$ satisfies the ACC on right annihilators, then $J(R)$ is nilpotent. We extend their results from a right P-injective ring to a right PS-injective ring. If R is semiregular, we prove that R is right P-injective if and only if R is right PS-injective. It is shown that being a right PS-injective ring is not a Morita invariant. A ring R is a right *GPF ring* [7] if it is right P-injective, semiperfect and $soc(R_R) \leq^{ess} R_R$. Here we give a new characterization of a right GPF ring in terms of right PS-injectivity. Finally, we also give a characterization of a semiprimitive ring.

2. Main results

Definition 2.1. Let R be a ring. A right R-module M is called *principally small injective* (abbr. *PS-injective*) if every R-homomorphism $f : aR \to M$ for every principally small right ideal aR can be extended to $R \to M$, equivalently, if $f = m\cdot$ is left multiplication by some element $m \in M$. A ring R is called *right PS-injective* if R is *PS-injective* as a right R-module. Similarly, we have the concept of left PS-injective rings.

Remark 2.2. It is easy to see that a right R-module M is PS-injective if and only if every R-homomorphism $f : aR \to M$ for every principally right ideal aR in $J(R)$ can be extended to $R \to M$.

The following lemma is frequently used in the sequel.

Lemma 2.3. *The following are equivalent for a ring R.*

(1) *R is right PS-injective.*

(2) *For all $a \in J(R)$, $l_R r_R(a) = Ra$.*

(3) *$r_R(a) \subseteq r_R(b)$, where $a \in J(R), b \in R$, implies that $Rb \subseteq Ra$.*

(4) *For all $a \in J(R), b \in R$, $l_R[bR \cap r_R(a)] = l_R(b) + Ra$.*

(5) *If $f : aR \to R,\ a \in J(R)$, is R-linear, then $f(a) \in Ra$.*

Proof. The proof is modeled on that of [9, Lemma 5.1].

(1)⇒(2). If $m \in l_R r_R(a)$, then $r_R(a) \subseteq r_R(m)$, so $f : aR \to R$ by $f(ar) = mr$ is well defined. By assumption, $f = c\cdot$ for some $c \in R$, whence $m = f(a) = ca \in Ra$. The other inclusion is clear.

(2)⇒(3). If $r_R(a) \subseteq r_R(b)$, for $a \in J(R), b \in R$, then $b \in l_R r_R(a)$, so $b \in Ra$ by (2). Thus $Rb \subseteq Ra$.

(3)⇒(4). For any $a \in J(R), b \in R$, it is clear that $l_R(b) + Ra \subseteq l_R[bR \cap r_R(a)]$. If $x \in l_R[bR \cap r_R(a)]$, then $bR \cap r_R(a) \subseteq r_R(x)$. If $y \in r_R(ab)$, then $aby = 0$, so $by \in r_R(a)$, and hence $by \in bR \cap r_R(a) \subseteq r_R(x)$, implies that $y \in r_R(xb)$. Thus $r_R(ab) \subseteq r_R(xb)$. Note that $ab \in J(R)$, so $xb = rab$ for some $r \in R$ by (3). Then $x - ra \in l_R(b)$, proving that $l_R[bR \cap r_R(a)] \subseteq l_R(b) + Ra$.

(4)⇒(2). Let $b = 1$ in (4).

(2)⇒(5). Let $f : aR \to R,\ a \in J(R)$, be R-linear, and write $f(a) = d$. Then $r_R(a) \subseteq r_R(d)$, so $d \in l_R r_R(a) = Ra$.

(5)⇒(1). Let $f : aR \to R$. By (5) write $f(a) = ca, c \in R$. Then $f = c\cdot$. □

Corollary 2.4. *A direct product of rings $R = \prod_{i\in I} R_i$ is right PS-injective if and only if R_i is right PS-injective for all $i \in I$.*

Proof. By [6, Exercises 4.12], $J(R) = \prod J(R_i)$. Then the result follows by lemma 2.3. □

Remark 2.5. Here give some examples of PS-injective rings.

(1) Obviously, every right P-injective ring is right PS-injective.

(2) Every right small injective ring is right PS-injective. Moreover, a semiprimitive ring (that is, a ring such that $J(R) = 0$) is right and left PS-injective.

Example 1. Let $R = \mathbb{Z}$, the ring of integers. Then R is semiprimitive, and hence is PS-injective. But R is not a P-injective ring.

Example 2(Björk Example). Let F be a field and assume that $a \mapsto \bar{a}$ is an isomorphism $F \to \bar{F} \subseteq F$, where the subfield $\bar{F} \neq F$. Let R denote the left vector space on basis $\{1, t\}$, and make R into an F-algebra by defining $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$. Then R is right P-injective. But R is not right small injective by [10, Example 3.7].

(3) Every right PS-injective ring is right mininjective. Moreover, every right PS-injective ring is right minsymmetric (if kR simple, $k \in R$, implies that Rk is simple). In fact, in view of [6, Lemma 10.22], every minimal right ideal of R is either nilpotent or a direct summand of R. But the converse is not true as the next example.

Example 3. Let $R = \{ \begin{bmatrix} a & v \\ 0 & a \end{bmatrix} \mid a \in F, v \in V\}$ be the trivial extension of a field F by a two-dimensional vector space V over F. By [9, Example 5.12], R is a commutative, local, artinian ring. Then $R[x]$, the polynomial ring over R, is a commutative mininjective ring by [9, Example 2.3]. But $R[x]$ is not a PS-injective ring. In fact, let $V = uF \oplus wF$, and write $\bar{u} = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \in J(R)$. Then $\bar{u}R = \begin{bmatrix} 0 & uF \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & ua \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & wa \\ 0 & 0 \end{bmatrix}$ is an R-linear map from $\bar{u}R \to R$ that can not be extended to $R \to R$ because $w \notin uF$. Hence R is not a PS-injective ring. By Lemma 2.3, there exists $0 \neq a \in J(R)$ such that $l_R r_R(a) \neq Ra$.

By [6, Proposition 10.27], $J(R) = Nil(R)$ because R is commutative artinian, so $a \in Nil(R)$. Then $a \in Nil(R)[x] = J(R[x])$ by [6, Theorem 5.1], and hence $l_{R[x]}r_{R[x]}(a) = (l_R r_R(a))[x] \neq (Ra)[x] = (R[x])a$. So $R[x]$ is not PS-injective by Lemma 2.3 again.

(4) If R is the direct product of R_1 and R_2 where R_1 is a right P-injective ring that is not right small injective and R_2 is a right small injective ring that is not right P-injective, observe that R is a right PS-injective ring that is neither right P-injective nor right small injective. Following Example 1, Example 2 and Example 3, we have the following relations in which every inclusion is proper:

$$\left.\begin{array}{c}\text{right } P\text{-injective rings}\\ \text{right small injective rings}\end{array}\right\} \subsetneqq \text{ right } PS\text{-injective rings} \subsetneqq \text{ right mininjective rings.}$$

Theorem 2.6. *If R is a right PS-injective ring, then $J(R) \subseteq Z(R_R)$. Moreover, if R is right PS-injective and satisfies the* ACC *on right annihilators of elements, then $J(R) = Z(R_R)$.*

Proof. Let $a \in J(R)$, and $bR \cap r_R(a) = 0$ for any $b \in R$. By Lemma 2.3, $l_R(b) + Ra = l_R[bR \cap r_R(a)] = l_R(0) = R$, so $l_R(b) = R$ because $a \in J(R)$, implies that $b = 0$. Thus $a \in Z(R_R)$. The second assertion follows from [5, Theorem 7.15 (1)]. □

Corollary 2.7. *Let R be a right PS-injective and reduced ring. Then R is semiprimitive.*

Proof. By [5, Lemma 7.8], $Z(R_R) = 0$ since R is reduced. Then $J(R) = 0$ by Theorem 2.6. □

Example 4. Let $R = \mathbb{Z}_{(p)}$, the localization ring of $\mathbb{Z}$ at the prime p. Then R is a commutative local mininjective ring because it has no minimal ideals. Since R is a domain, $Z(R_R) = 0$ and $J(R) = p\mathbb{Z}_p \neq 0$. Therefore, R is not a PS-injective ring by Theorem 2.6. However, we claim that the polynomial ring $R[x]$ over R is PS-injective. Because R is a domain, it is a reduced ring. By [6, Corollary 5.2], $R[x]$ is semiprimitive. Therefore, $R[x]$ is PS-injective in terms of Remark 2.5 (2).

Example 5. Let R be a non-semiprimitive reduced ring. Then R is a right and left mininjective ring but not a right PS-injective ring. In fact, R has not nonzero nilpotent ideal. Thus R is a right and left mininjective ring. Suppose that R is a right PS-injective ring. Then R is semiprimitive by Corollary 2.7, a contradiction.

Example 6. Let R be the ring of all $\mathbb{N}$-square upper triangular matrices over a field F that are constant on the diagonal and have only finitely many nonzero entries off the diagonal ([12, Example 1.7]). So R is right mininjective, $Z(R_R) = 0$ and $J(R) \neq 0$. By Theorem 2.6, R is not right PS-injective.

The next result is a generalization of [7, Theorem 2.2].

Theorem 2.8. *If R is a right PS-injective ring and $R/soc(R_R)$ satisfies the* ACC *on right annihilators, then $J(R)$ is nilpotent.*

Proof. Write $S = soc(R_R)$ and $\overline{R} = R/S$. For any sequence $a_1, a_2, a_3, \cdots \in J(R)$, there is an ascending chain

$$r_{\overline{R}}(\overline{a_1}) \subseteq r_{\overline{R}}(\overline{a_2}\ \overline{a_1}) \subseteq r_{\overline{R}}(\overline{a_3}\ \overline{a_2}\ \overline{a_1}) \subseteq \cdots .$$

By hypothesis, there exists a positive integer m such that

$$r_{\overline{R}}(\overline{a_m} \cdots \overline{a_2}\ \overline{a_1}) = r_{\overline{R}}(\overline{a_{m+k}} \cdots \overline{a_m} \cdots \overline{a_2}\ \overline{a_1}),\ k = 1, 2, \cdots .$$

Since $a_{n+1}a_n \cdots a_1 \in J(R) \subseteq Z(R_R)$ by Theorem 2.6, $r_R(a_{n+1}a_n \cdots a_1)$ is the essential right ideal of R. Then $S \subseteq r_R(a_{n+1}a_n \cdots a_1)$.

Now we prove that

$$r_{\overline{R}}(\overline{a_n} \cdots \overline{a_2}\ \overline{a_1}) \subseteq r_R(a_{n+1}a_n \cdots a_1)/S \subseteq r_{\overline{R}}(\overline{a_{n+1}}\ \overline{a_n} \cdots \overline{a_1}) \quad (1)$$

In fact, for any $b + S \in r_{\overline{R}}(\overline{a_n} \cdots \overline{a_2}\ \overline{a_1})$, $a_n \cdots a_1 b \in S$. Then $a_{n+1}a_n \cdots a_1 b = 0$ because $S \subseteq r_R(a_{n+1})$. So $b \in r_R(a_{n+1}a_n \cdots a_1)$, and hence $b + S \in r_R(a_{n+1}a_n \cdots a_1)/S$. But the second inclusion is clear.

Since $r_{\overline{R}}(\overline{a_m} \cdots \overline{a_2}\ \overline{a_1}) = r_{\overline{R}}(\overline{a_{m+2}}\ \ \overline{a_{m+1}} \cdots \overline{a_2}\ \overline{a_1})$, by (1), $r_R(a_{m+1}a_m \cdots a_1)/S = r_R(a_{m+2}a_{m+1} \cdots a_1)/S$. Then $r_R(a_{m+1}a_m \cdots a_1) = r_R(a_{m+2}a_{m+1} \cdots a_1)$, and so $(a_{m+1}a_m \cdots a_1)R \cap r_R(a_{m+2}) = 0$. Since $r_R(a_{m+2})$ is also an essential right ideal of R, $a_{m+1}a_m \cdots a_1 = 0$. So $J(R)$ is a right T-nilpotent ideal and the ideal $(J(R)+S)/S$ of $\overline{R}$ is also a right T-nilpotent. By [1, Proposition 29.1], $(J(R)+S)/S$ is nilpotent. Then there exists a positive integer t such that $(J(R))^t \subseteq S$, so $(J(R))^{t+1} \subseteq J(R)S = 0$, as desired. □

Proposition 2.9. *If R is right PS-injective, so is eRe for all $e^2 = e \in R$ satisfying $ReR = R$.*

Proof. Let $S = eRe$ and $r_S(a) \subseteq r_S(b)$, where $a \in J(S)$, $b \in S$. Since $J(S) = J(eRe) = eJe$, aR is a principally small ideal of R. Since $ReR = R$, we write $1 = \sum_{i=1}^{n} a_i e b_i$, where $a_i, b_i \in R$. Let $ax = 0$, $x \in R$. Then $a(exa_ie) = axa_ie = 0$ for each i, so $b(exa_ie) = 0$ because $r_S(a) \subseteq r_S(b)$. Thus $bx = \sum_{i=1}^{n} bxa_ieb_i = 0$ because $b = be$. Then $r_R(a) \subseteq r_R(b)$. By Lemma 2.3, $b = eb \in eRa = Sa$. Therefore, S is right PS-injective by Lemma 2.3 again. □

Corollary 2.10. *If the matrix ring $M_n(R)$ over a ring R is right PS-injective, so is R.*

Proof. If $S = M_n(R)$ is right PS-injective, so is $R \cong e_{11}Se_{11}$ by Proposition 2.9 because $Se_{11}S = S$ (here e_{11} denotes the $n \times n$ matrix whose $(1,1)$-entry is 1 and others are zero). □

Let R be a ring and I, K be two right ideals. Recall that R is called *right (I,K)-m-injective* (see [12, Definition 1.1]) if, for any m-generated right ideal $U \subseteq I$

and any R-homomorphism $f : U \to K$, $f = c\cdot$ for some $c \in R$. R is *right* (I,K)-FP-*injective* if, for any $n \geq 1$ and any finitely generated R-submodule N of I_n, every R-homomorphism $f : N \to K$ can be extended to an R-homomorphism $g : R_n \to R$, where I_n (resp. R_n) denotes the set of all $1 \times n$ matrices over I (resp. R). It is clear that a right (J,R)-FP-injective ring is right J-injective in the sense of [3].

It has been shown that R is a right FP-injective ring if and only if $M_n(R)$ is a right P-injective ring for each $n \geq 1$ (see [9, Theorem 5.41]). Similarly, we have the following result.

Theorem 2.11. *The following are equivalent for a ring R.*

(1) *R is right (J,R)-FP-injective.*

(2) *$M_n(R)$ is right PS-injective for all integers $n \geq 1$.*

Proof. By [12, Lemma 1.3], R is a right (J,R)-FP-injective ring if and only if $M_n(R)$ is a right $(M_n(J), M_n(R))$-1-injective ring for every $n \geq 1$; equivalently $M_n(R)$ is a right PS-injective ring because $M_n(J) = J(M_n(R))$. □

A ring R is called *semiregular* if $R/J(R)$ is (Von Neumann) regular and idempotents lift modulo $J(R)$, equivalently if, for any $a \in R$, there exists $e^2 = e \in Ra$ such that $a(1-e) \in J(R)$ (cf. [9, Lemma B.40]).

Proposition 2.12. *If R is a semiregular ring. Then R is right P-injective if and only if R is right PS-injective.*

Proof. ($\Rightarrow$) follows by Remark 2.5 (1).

($\Leftarrow$). Let $f : aR \to R$, $a \in R$, be an R-homomorphism. Since R is semiregular, $Ra = Re \oplus Rb$ where $e^2 = e$ and $b \in J(R)$. Thus $r_R(a) = r_R(Re \oplus Rb) = r_R(Re) \cap r_R(b) = (1-e)R \cap r_R(b)$, and hence $l_R r_R(a) = l_R[(1-e)R \cap r_R(b)]$. Let $x = f(a)$. Then $x \in l_R r_R(a) = l_R[(1-e)R \cap r_R(b)]$. Thus $r_R(b(1-e)) \subseteq r_R(x(1-e))$. So $g : b(1-e)R \to R$ given by $b(1-e)y \mapsto x(1-e)y$ is a well defined R-homomorphism. Since $b(1-e) \in J(R)$, $g = c\cdot$ for some $c \in R$ because R is right PS-injective. Thus $x(1-e) = g(b(1-e)) = cb(1-e)$, and hence $f(a) = x = xe + x(1-e) = xe + cb(1-e) = (x-cb)e + cb \in Re + Rb = Ra$. So R is a right P-injective. □

A ring R is called *semiperfect* if $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$. So a semiperfect ring is semiregular.

Corollary 2.13. *If R is a semiperfect and right PS-injective ring, then $R \cong R_1 \times R_2$, where R_1 is semisimple and every simple right ideal of R_2 is nilpotent.*

Proof. It follows from Proposition 2.12 and [7, Theorem 1.4]. □

Remark 2.14. (1) By Theorem 2.11 and Corollary 2.10, every right (J,R)-FP-injective ring is right PS-injective. But the converse is not true in general. For example, the Björk example (see Example 2) is a local, left artinian right GPF ring R. By Example 2, R is right PS-injective. Now we prove that R is not right

(J, R)-FP-injective.

Proof. It is mentioned in [9, Example 5.34] that R is not a left GPF ring. Suppose that R is right (J, R)-FP-injective. Then $M_n(R)$ is right PS-injective by Theorem 2.11. But R is semiperfect, and hence $M_n(R)$ is also semiperfect. So $M_n(R)$ is right P-injective by Proposition 2.12. Then R is right FP-injective by [9, Theorem 5.41], so R is right 2-injective. In view of [9, Corollary 5.32], R is a left GPF ring, a contradiction. □

By the above conclusion, being a right PS-injective ring is not Morita invariant property.

(3) The Björk example shows that R is right PS-injective but is not left PS-injective.

A ring R is said to be *right Kasch* if every simple right R-module embeds in R, equivalently $l_R(T) \neq 0$ for every maximal right ideal T of R. A ring is called *left min-CS* if every minimal left ideal is essential in a direct summand of ${}_RR$. Now we give a new characterization of a right GPF ring by using right PS-injectivity.

Theorem 2.15. *The following are equivalent for a ring R.*

(1) *R is a right GPF ring.*

(2) *R is a semiperfect, right PS-injective ring with $soc({}_RR) = soc(R_R) \leq^{ess} R_R$.*

(3) *R is a right PS-injective, right and left Kasch and left min-CS ring.*

Proof. (1)⇒(2) is trivial.

(2)⇒(3). By [9, Theorem 5.31], R is a right and left Kasch ring. For every minimal left ideal K of R, in view of [9, Theorem 2.32 and Theorem 5.31], $l_Rr_R(K) = K$. Since R is semiperfect, write $r_R(K) = (1-e)R + bR$, where $e^2 = e$ and $b \in J(R)$. Then $l_Rr_R(K) = Re \cap l_R(b)$. Note that $b \in J(R)$, so $l_R(b) \supseteq l_R(J(R)) = soc(R_R)$. By (2), $soc(R_R) \leq^{ess} R_R$, so $l_R(b)$ is essential in R_R. Then, $l_Rr_R(K) \leq^{ess} Re$ by [9, Lemma 1.1 (2)], and hence $K \leq^{ess} Re$. Thus R is left min-CS.

(3)⇒(2). Let T be a maximal right ideal of R. Then $l_R(T) \neq 0$ because R is right Kasch. Then there is $0 \neq a \in l_R(T)$, and hence $T \subseteq r_R(a) \neq R$. So $T = r_R(a)$. Then $aR \cong R/r_R(a) = R/T$ is a simple right ideal. Note that Ra is also a simple left ideal by [9, Theorem 2.21] because R is right mininjective. If $(Ra)^2 \neq 0$, then Ra is a direct summand of R, and so $l_Rr_R(a) = Ra$. Otherwise, $a \in J(R)$, so $l_Rr_R(a) = Ra$ by Lemma 2.3. By hypothesis, $l_R(T) = l_Rr_R(a) = Ra \leq^{ess} Re$ for some $e^2 = e \in R$. Thus, by [9, Lemma 4.1], R is semiperfect. By [9, Lemma 4.5], $soc({}_RR) = soc(R_R) \leq^{ess} R_R$.

(2)⇒(1). By Proposition 2.12, R is right P-injective since R is semiperfect, as desired. □

Corollary 2.16. *If R is right PS-injective with $soc(R_R) \leq^{ess} R_R$ and the ascending chain $r_R(a_1) \subseteq r_R(a_1a_2) \subseteq \cdots \subseteq r_R(a_1a_2\cdots a_n) \subseteq \cdots$ terminates for every infinite sequence $a_1, a_2, \cdots$ in R, then R is a right GPF ring.*

Proof. Note that R is right minsymmetric. So, in view of [11, Lemma 2.2], R is right perfect. Then R is a right GPF ring by Theorem 2.15. □

Remark 2.17. The condition $soc(R_R) \leq^{ess} R_R$ can not be omitted. If $R = \mathbb{Z}$ is the ring of integers, then R is a PS-injective and noetherian ring but R is not a GPF ring because R is not P-injective.

It is easy to see that a right R-module M is PS-injective if and only if $\mathrm{Ext}^1(R/aR, M) = 0$ for any right principally small ideal aR. At the end of this paper, we give a characterization of a semiprimitive ring.

Proposition 2.18. *The following are equivalent for a ring R.*

(1) *R is semiprimitive.*

(2) *Every right (or left) R-module is PS-injective.*

(3) *Every right (or left) simple R-module is PS-injective.*

(4) *Every right (or left) principally small ideal is PS-injective.*

(5) *Every right (or left) principally small ideal is pure in R.*

Proof. (1)⇒(2), (2)⇒(3) and (2)⇒(4) are trivial.

(3)⇒(1). Let $a \in J(R)$. If $J(R) + r_R(a) < R$, then we take a maximal right ideal K of R such that $J(R) + r_R(a) \leq K$. Then R/K is PS-injective by (3). Note that the homomorphism $f : aR \to R/K$ given by $f(ax) = x + K$, $x \in R$ is a well defined homomorphism. So there exists $c \in R$ such that $f = (c + K)\cdot$. Then $1 + K = f(a) = (c + K)a = ca + K$, implies that $1 - ca \in K$. But $ca \in K$, which yields $1 \in K$, a contradiction. Therefore $J(R) + r_R(a) = R$ and so $r_R(a) = R$ because $J(R) \ll R$. So $a = 0$. Hence $J(R) = 0$.

(4)⇒(1). Let $a \in J(R)$. By (4), aR is PS-injective. Thus the inclusion map $aR \to R$ splits, so aR is a direct summand of R. Since $aR \ll R$, $aR = 0$. Therefore, $J(R) = 0$.

(2)⇒(5). For any right principally small ideal aR and any left R-module M, there exists the standard isomorphism $\mathrm{Ext}^1(R/aR, M^+) \cong \mathrm{Tor}_1(R/aR, M)^+$. By (2), $\mathrm{Ext}^1(R/aR, M^+) = 0$, so $\mathrm{Tor}_1(R/aR, M) = 0$. Then R/aR is flat, and hence aR is pure in R.

(5)⇒(2). Let aR be a right principally small ideal. Then R/aR is flat, and hence is projective. Thus aR is a direct summand of R. Hence every right R-module is PS-injective. □

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