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On Opial Type Inequalities with Nonlocal Conditions and Applications

Lazhar Bougoffa*
Department of Mathematics, Faculty of Science, Al-imam University, P. O. Box 90950, Riyadh 11623, Saudi Arabia
e-mail : bougoffa@hotmail.com

Jamal Ibrahim Daoud
Department of Science in Engineering, International Islamic University, P. O. Box 10, Kuala Lumpur 50728, Malaysia
e-mail : jamal58@iium.edu.my

Abstract. The purpose of this note is to give Opial type inequalities with nonlocal conditions. Also, a reverse of the original inequality with $y(a) = y(b) = 0$ is derived. We apply these inequalities to second-order differential equations with nonlocal conditions to derive several necessary conditions for the existence of solutions.

1. Introduction

In 1960 Z. Opial [10] proved the following integral inequality

Theorem 1. *Let $y(x)$ be of class $C^{(1)}[0, b]$ and $y(0) = y(b) = 0$. Then*

$$\int_0^b \mid y(x)y'(x) \mid dx \le \frac{b}{4}\int_0^b \mid y'(x) \mid^2 dx. \tag{1.1}$$

Olech [9] also showed that

$$\int_a^b \mid y(x)y'(x) \mid dx \le \frac{b-a}{4}\int_a^b \mid y'(x) \mid^2 dx \tag{1.2}$$

is valid for any function which is absolutely continuous on $[a, b]$ and satisfies $y(a) = y(b) = 0$. In 1962 Beesack [1] used this result to obtain a simplification of proofs given earlier by Olech [9] and Opial [10] of the following result.

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* Corresponding Author.

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Theorem 2. *Let $y(x)$ be real, continuously differentiable on $[a,b]$ and $y(a) = y(b) = 0$. Then, inequality* (1.2) *holds.*

For other generalizations of Opial's original inequality in different directions, see [3] and [11].
Also, a generalization of (1.2) obtained by Das [5] is the following:

Theorem 3. *If $y \in C^{(n-1)}[a,b]$ with $y^{(i)}(a) = y^{(i)}(b) = 0$ for $i = 0, 1, ..., n-1$. Let $y^{(n-1)}$ be absolutely continuous and $y^{(n)} \in L_2$. Then*

$$\int_a^b \mid y(x)y^{(n)}(x) \mid dx \leq K(\frac{b-a}{2})^n \int_a^b \mid y^{(n)}(x) \mid^2 dx, \tag{1.3}$$

where $K = \frac{1}{2n!}\left(\frac{n}{2n-1}\right)^{\frac{1}{2}}$.

Recently, Brown [2] considered the inequality

$$\int_a^b \mid y(x)y'(x) \mid dx \leq K(b-a) \int_a^b y'^2(x)dx, \tag{1.4}$$

where $y : [a,b] \to \Re$ is absolutely continuous function such that $y' \in L_2$ and $\int_a^b y(x)dx = 0$, and proved the following result which was conjectured by one of the authors in 2001 and presented as an open problem in the meeting " General Inequalities 8" at Noszvaj, Hungary, in September 2002.

Theorem 4. *The best value of K in* (1.4) *is* $\frac{1}{4}$.

Also, Kwong obtained an answer to this problem in [7].
The purpose of this note is to give new Opial type inequalities with nonlocal conditions

$$\int_a^{\frac{a+b}{2}} y(x)dx = 0 \text{ and } \int_{\frac{a+b}{2}}^b y(x)dx = 0.$$

Also, a reverse of the original inequality with $y(a) = y(b) = 0$ is derived. We apply these inequalities to second-order differential equations with nonlocal conditions to derive several necessary conditions for the existence of solutions.

2. Opial type inequalities with nonlocal conditions

Here, we shall give simpler proof of Opial type inequalities with nonlocal conditions. Our main tool is the following Lemma.

Lemma 1. *Let $y : [\alpha, \beta] \to \Re$ be an absolutely continuous function such that $y' \in L_2$.*
If $\int_\alpha^\beta y(x)dx = 0$. Then,

$$\int_\alpha^\beta \mid y(x) \mid^2 dx \leq \frac{(\beta-\alpha)^2}{\pi^2} \int_\alpha^\beta \mid y'(x) \mid^2 dx. \tag{2.1}$$

This inequality is called Wirtinger's inequality [8].

Theorem 5. *Let $y : [a, b] \to \Re$ be an absolutely continuous function such that $y' \in L_2$. If*

$$\int_a^{\frac{a+b}{2}} y(x)dx = 0 \text{ and } \int_{\frac{a+b}{2}}^b y(x)dx = 0,$$

then

$$\int_a^b | y(x)y'(x) | \, dx \leq K(b-a) \int_a^b | y'(x) |^2 \, dx, \tag{2.2}$$

where $K = \frac{1}{2\pi}$.

Proof. Apply Schwarz's inequality to $[a, \frac{a+b}{2}]$, we get

$$\int_a^{\frac{a+b}{2}} | y(x)y'(x) | \, dx \leq \left(\int_a^{\frac{a+b}{2}} | y(x) |^2 \, dx \right)^{\frac{1}{2}} \left(\int_a^{\frac{a+b}{2}} | y'(x) |^2 \, dx \right)^{\frac{1}{2}} \tag{2.3}$$

and use Lemma 1 with $\alpha = a$ and $\beta = \frac{a+b}{2}$, to obtain that

$$\int_a^{\frac{a+b}{2}} | y(x)y'(x) | \, dx \leq \frac{b-a}{2\pi} \int_a^{\frac{a+b}{2}} | y'(x) |^2 \, dx. \tag{2.4}$$

Similarly, and again on $[\frac{a+b}{2}, b]$, we get

$$\int_{\frac{a+b}{2}}^b | y(x)y'(x) | \, dx \leq \frac{b-a}{2\pi} \int_{\frac{a+b}{2}}^b | y'(x) |^2 \, dx, \tag{2.5}$$

and then add (2.4) and (2.5), we get (2.2). This completes the proof. □

The following is a generalization of Theorem 5.

Theorem 6. *Let $y : [a, b] \to \Re$ and $z : [a, b] \to \Re$ be absolutely continuous functions such that $y' \in L_2$ and $z' \in L_2$. If*

$$\int_a^{\frac{a+b}{2}} y(x)dx = \int_{\frac{a+b}{2}}^b y(x)dx = 0$$

and

$$\int_a^{\frac{a+b}{2}} z(x)dx = \int_{\frac{a+b}{2}}^b z(x)dx = 0.$$

Then

$$\int_a^b [| y(x)z'(x) | + | y'(x)z(x) |] \, dx \leq \frac{b-a}{2\pi} \int_a^b \left[| y'(x) |^2 + | z'(x) |^2\right] dx. \tag{2.6}$$

Proof. On $[a, \frac{a+b}{2}]$, we apply Schwarz's inequality to obtain

$$\int_a^{\frac{a+b}{2}} | y(x)z'(x) | \, dx \leq \left(\int_a^{\frac{a+b}{2}} | y(x) |^2 \, dx \right)^{\frac{1}{2}} \left(\int_a^{\frac{a+b}{2}} | z'(x) |^2 \, dx \right)^{\frac{1}{2}}. \tag{2.7}$$

Using Lemma 1 with $\alpha = a$ and $\beta = \frac{a+b}{2}$, we obtain

$$\int_a^{\frac{a+b}{2}} | y(x)z'(x) | \, dx \leq \frac{b-a}{2\pi} \left(\int_a^{\frac{a+b}{2}} | y'(x) |^2 \, dx \right)^{\frac{1}{2}} \left(\int_a^{\frac{a+b}{2}} | z'(x) |^2 \, dx \right)^{\frac{1}{2}}. \tag{2.8}$$

Similarly, and again on $[\frac{a+b}{2}, b]$, we get

$$\int_{\frac{a+b}{2}}^b | y(x)z'(x) | \, dx \leq \frac{b-a}{2\pi} \left(\int_{\frac{a+b}{2}}^b | y'(x) |^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\frac{a+b}{2}}^b | z'(x) |^2 \, dx \right)^{\frac{1}{2}}. \tag{2.9}$$

Thus, summing up (2.8) and (2.9), we get

$$\begin{aligned} &\int_a^b | y(x)z'(x) | \, dx \\ &\leq \frac{b-a}{2\pi} \left[\left(\int_a^{\frac{a+b}{2}} | y' |^2 \, dx \int_a^{\frac{a+b}{2}} | z' |^2 \, dx \right)^{\frac{1}{2}} + \left(\int_{\frac{a+b}{2}}^b | y' |^2 \, dx \int_{\frac{a+b}{2}}^b | z' |^2 \, dx \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Applying the arithmetic-geometric mean inequality to the right-hand side of this inequality to get

$$\begin{aligned} &\int_a^b | y(x)z'(x) | \, dx \\ &\leq \frac{b-a}{2\pi} \left[\frac{\int_a^{\frac{a+b}{2}} | y' |^2 \, dx + \int_a^{\frac{a+b}{2}} | z' |^2 \, dx}{2} + \frac{\int_{\frac{a+b}{2}}^b | y' |^2 \, dx + \int_{\frac{a+b}{2}}^b | z' |^2 \, dx}{2} \right]. \end{aligned}$$

Therefore,

$$\int_a^b | y(x)z'(x) | \, dx \leq \frac{b-a}{4\pi} \int_a^b \left[| y'(x) |^2 + | z'(x) |^2 \right] dx. \tag{2.10}$$

Similarly, we also obtain,

$$\int_a^b | y'(x)z(x) | \, dx \leq \frac{b-a}{4\pi} \int_a^b \left[| y'(x) |^2 + | z'(x) |^2 \right] dx. \tag{2.11}$$

Adding side to side (2.10) and (2.11), we obtain (2.6). □

Remark 1. *Theorem 6 with $z = y$ reduces to Theorem 5.*

Theorem 7. *Let $y : [a,b] \to \Re$ be an absolutely continuous function such that $y' \in L_2$. If*

$$\int_a^{\frac{a+b}{2}} y(x)dx = k_1 \text{ and } \int_{\frac{a+b}{2}}^b y(x)dx = k_2,$$

where $k_i \in \Re,\ i = 1,2$. Then

(2.12)
$$\int_a^b |\, y(x)y'(x) \,|\, dx \le \frac{b-a}{2\pi} \int_a^b |\, y'(x) \,|^2\, dx + \sup(|\, \gamma_1 \,|,\ |\, \gamma_2 \,|) \int_a^b |\, y'(x) \,|\, dx,$$

where $\gamma_i = \frac{2k_i}{b-a},\ i = 1,2$.

Proof. Since $\int_a^{\frac{a+b}{2}} y(x)dx = k_1$ and $\int_a^{\frac{a+b}{2}} y(x)dx = k_2$. It follows from the first value theorem of integral that there exist $c_1 \in (a, \frac{a+b}{2})$ and $c_2 \in (\frac{a+b}{2}, b)$ such that $y(c_1) = \frac{2k_1}{b-a}$ and $y(c_2) = \frac{2k_2}{b-a}$.

Let $z_i(x) = y(x) - \gamma_i,\ i = 1,2$. This gives $\int_a^{\frac{a+b}{2}} z_1(x)dx = 0$ and $\int_{\frac{a+b}{2}}^b z_2(x)dx = 0$ and then we can apply Lemma 1 and Schwarz's inequality to obtain

$$\int_a^{\frac{a+b}{2}} |\, z_1(x)z_1'(x) \,|\, dx \le \frac{b-a}{2\pi} \int_a^{\frac{a+b}{2}} |\, z_1'(x) \,|^2\, dx \tag{2.13}$$

and

$$\int_{\frac{a+b}{2}}^b |\, z_2(x)z_2'(x) \,|\, dx \le \frac{b-a}{2\pi} \int_{\frac{a+b}{2}}^b |\, z_2'(x) \,|^2\, dx. \tag{2.14}$$

Adding side to side (2.13) and (2.14), we obtain

$$\begin{aligned}\int_a^{\frac{a+b}{2}} |\, z_1(x)z_1'(x) \,|\, dx + \int_{\frac{a+b}{2}}^b |\, z_2(x)z_2'(x) \,|\, dx \\ \le \frac{b-a}{2\pi} \left(\int_a^{\frac{a+b}{2}} |\, z_1'(x) \,|^2\, dx + \int_{\frac{a+b}{2}}^b |\, z_2'(x) \,|^2\, dx \right).\end{aligned} \tag{2.15}$$

So that

$$\begin{aligned}\int_a^{\frac{a+b}{2}} |\, (y(x)-\gamma_1)y'(x) \,|\, dx + \int_{\frac{a+b}{2}}^b |\, (y(x)-\gamma_2)y'(x) \,|\, dx \\ \le \frac{b-a}{2\pi} \int_a^b |\, y'(x) \,|^2\, dx.\end{aligned} \tag{2.16}$$

Now, applying the inequality $\mid a-b\mid\geq\mid a\mid-\mid b\mid$ to the left-hand side of (2.16), we get

$$\begin{aligned}(2.17)\quad &\int_a^b \mid y(x)y'(x)\mid dx\\ &\leq \frac{b-a}{2\pi}\int_a^b \mid y'(x)\mid^2 dx+\mid\gamma_1\mid\int_a^{\frac{a+b}{2}}\mid y'(x)\mid dx+\mid\gamma_2\mid\int_{\frac{a+b}{2}}^b \mid y'(x)\mid dx,\end{aligned}$$

which is indeed (2.12). □

3. Reverse inequalities of opial

In the following the reverse of Opial's inequalities are derived.

Theorem 8. *Let $y:[a,b]\to\Re$ be an absolutely continuous function such that $y'\in L_2$. If $y(a)=0$, then*

$$\frac{1}{2}\int_a^b \mid y(x)\mid^2 dx\leq\int_a^b (b-x)\mid y(x)y'(x)\mid dx. \tag{3.1}$$

Equality holds only for $y=c(x-a)$, where c is a constant.
Also, if $y(b)=0$, then

$$\frac{1}{2}\int_a^b \mid y(x)\mid^2 dx\leq\int_a^b (x-a)\mid y(x)y'(x)\mid dx \tag{3.2}$$

and equality holds only for $y=c(b-x)$.

Proof. For $y(a)=0$, we have

$$\frac{y^2(x)}{2}=\int_a^x y(t)y'(t)dt.$$

Therefore

$$\frac{1}{2}\int_a^b \mid y(x)\mid^2 dx\leq\int_a^b\left(\int_a^x \mid y(t)y'(t)\mid dt\right)dx=\int_a^b (b-x)\mid y(x)y'(x)\mid dx. \tag{3.3}$$

Since the equality holds in (3.1) only if $y'=c$, substitution of $y=cx+d$ into (3.1) and standard argument leads to $d=-ca$.
The proof of (3.2) is similar using

$$-\frac{y^2(x)}{2}=\int_x^b y(t)y'(t)dt.$$ □

We have therefore the following corollary of Theorem 8.

Corollary 1. *Let* $y : [a, b] \to \Re$ *be an absolutely continuous function such that* $y' \in L_2$. *If* $y(a) = y(b) = 0$, *then*

$$\frac{1}{b-a}\int_a^b \mid y(x) \mid^2 dx \leq \int_a^b \mid y(x)y'(x) \mid dx \tag{3.4}$$

with equality if and only if $y = 0$.

Remark 2. *The Opial's original inequality can be stated as*

$$\frac{1}{b-a}\int_a^b \mid y(x) \mid^2 dx \leq \int_a^b \mid y(x)y'(x) \mid dx \leq \frac{b-a}{4}\int_a^b \mid y'(x) \mid^2 dx. \tag{3.5}$$

4. Applications

Opial inequalities with boundary conditions have various applications in the theory of differential equations. In [6], Harris and Kong obtained the following results:
If y is solution of $y'' + q(x)y = 0,\ a \leq x \leq b$ with no zeros in (a, b) and such that $y'(a) = y(b) = 0$, then

$$(b-a)\max_{a\leq x\leq b} \mid \int_a^x q(t)dt \mid > 1$$

and if $y(a) = y'(b) = 0$, then

$$(b-a)\max_{a\leq x\leq b} \mid \int_x^b q(t)dt \mid > 1.$$

Brown [3] also obtained several results which related to this problem. Two of his results state that

Theorem 9. *If* y *is a nontrivial solution of* $y'' + q(x)y = 0$ *with* $y(a) = y'(b) = 0$, *then*

$$1 < 2\int_a^b Q^2(x)(x-a)dx,$$

where $Q(x) = \int_x^b q(t)dt$. *If* $y'(a) = y(b) = 0$, *then*

$$1 < 2\int_a^b Q^2(x)(b-x)dx,$$

where $Q(x) = \int_a^x q(t)dt$.

Remark 3. *These results can be extended to a class of second order differential equations with integral conditions.*

We have

Theorem 10. *Let y be a nontrivial solution of*

$$y'' + p(x)y' + q(x)y = 0, \ \ a \le x \le b, \tag{4.1}$$

where $p, q \in \mathcal{C}[a,b]$. If $\int_a^{\frac{a+b}{2}} y(x)dx = 0$ and $\int_{\frac{a+b}{2}}^b y(x)dx = 0$, then there exists $[\alpha, \beta] \subset [a,b]$ such that

$$\frac{\beta-\alpha}{4}\left(\max_{\alpha\le x\le\beta} \mid p(x) \mid +2 \max_{\alpha\le x\le\beta} \mid \int_\alpha^x q(t)dt \mid\right) \ge 1. \tag{4.2}$$

Proof. By the first value theorem of integral there exist $\alpha \in (a, \frac{a+b}{2})$ and $\beta \in (\frac{a+b}{2}, b)$ such that $y(\alpha) = 0$ and $y(\beta) = 0$.
Multiplying (4.1) by y and integrating by parts over $[\alpha, \beta]$ gives

$$-\int_\alpha^\beta y'^2(x)dx + \int_\alpha^\beta p(x)y(x)y'(x)dx + \int_\alpha^\beta q(x)y^2(x)dx = 0.$$

Since $\int_\alpha^\beta q(x)y^2(x)dx = -2\int_\alpha^\beta (\int_\alpha^x q(t)dt)y(x)y'(x)dx$.
Thus

$$\int_\alpha^\beta y'^2(x)dx \le \int_\alpha^\beta \mid p(x) \mid\mid y(x)y'(x) \mid dx + 2\int_\alpha^\beta \mid \int_\alpha^x q(t)dt \mid\mid y(x)y'(x) \mid dx.$$

Consequently,

$$\int_\alpha^\beta y'^2(x)dx \le \left(\max_{\alpha\le x\le\beta} \mid p(x) \mid +2 \max_{\alpha\le x\le\beta} \mid \int_\alpha^x q(t)dt \mid\right)\int_\alpha^\beta \mid y(x)y'(x) \mid dx. \tag{4.3}$$

Applying Opial inequality (1.2) to (4.3), we obtain

$$\int_\alpha^\beta y'^2(x)dx \le \frac{\beta-\alpha}{4}\left(\max_{\alpha\le x\le\beta} \mid p(x) \mid +2 \max_{\alpha\le x\le\beta} \mid \int_\alpha^x q(t)dt \mid\right)\int_\alpha^\beta y'^2(x)dx. \tag{4.4}$$

By canceling $\int_\alpha^\beta y'^2(x)dx$, we get (4.2). □

In order to illustrate a possible practical use of inequality (4.2), we present a simple example.

Example 1. Consider the following nonlocal boundary value problem

$$\begin{cases} y'' + p(x)y' + q(x)y &= 0, \ 0 \le x \le 2, \\ \int_0^1 y(x)dx = \int_1^2 y(x)dx &= 0, \end{cases} \tag{4.5}$$

where

$$p(x) = \begin{cases} 2x-1, \ 0 \le x \le 1, \\ 2x-3, \ 1 \le x \le 2 \end{cases} \tag{4.6}$$

and $q(x) = -2$. The exact solution of (4.5) is

$$y(x) = \begin{cases} x - \frac{1}{2}, & 0 \leq x \leq 1, \\ x - \frac{3}{2}, & 1 \leq x \leq 2. \end{cases} \tag{4.7}$$

A direct calculation produces there exists $[\alpha, \beta] = [\frac{1}{2}, \frac{3}{2}]$ such that $y(\alpha) = y(\beta) = 0$,

$$\max_{\frac{1}{2} \leq x \leq \frac{3}{2}} \mid p(x) \mid = 1$$

and

$$\max_{\frac{1}{2} \leq x \leq \frac{3}{2}} \mid \int_{\frac{1}{2}}^{x} q(t)dt \mid = \max_{\frac{1}{2} \leq x \leq \frac{3}{2}} 2\left(x - \frac{1}{2}\right) = 2.$$

Thus,

$$\frac{\beta - \alpha}{4}\left(\max_{\alpha \leq x \leq \beta} \mid p(x) \mid + 2\max_{\alpha \leq x \leq \beta} \mid \int_{\alpha}^{x} q(t)dt \mid\right) = \frac{5}{4} > 1.$$

Now, we shall use our main result to derive a new estimate for the following nonlocal boundary value problem

$$\begin{cases} (p(x)y')' + \lambda y &= 0, \ a \leq x \leq b, \\ \int_a^{\frac{a+b}{2}} y(x)dx = \int_{\frac{a+b}{2}}^{b} y(x)dx &= 0, \end{cases} \tag{8}$$

where $p \in \mathcal{C}^1[a, b]$ and $\lambda \neq 0$.
For that, we reduce problem (8) to an equivalent problem.

Lemma 2. *Problem* (8) *is equivalent to the following problem*

$$\begin{cases} (p(x)y')' + \lambda y &= 0, \ a \leq x \leq b, \\ \alpha y'(a) - \beta y'(b) &= 0, \end{cases} \tag{9}$$

where $\alpha = p(a)$ *and* $\beta = p(b)$.

Proof. Let y be a solution of (8). Integrating $(p(x)y')' + \lambda y = 0$ over $[a, \frac{a+b}{2}]$, and again on $[\frac{a+b}{2}, b]$ and taking into account $\int_a^{\frac{a+b}{2}} y(x)dx = \int_{\frac{a+b}{2}}^{b} y(x)dx = 0$, we obtain

$$p(\frac{a+b}{2})y'(\frac{a+b}{2}) - p(a)y'(a) = 0 \tag{10}$$

and

$$p(b)y'(b) - p(\frac{a+b}{2})y'(\frac{a+b}{2}) = 0. \tag{11}$$

From (10) and (11), it follows that

$$\alpha y'(a) - \beta y'(b) = 0. \tag{12}$$

Let now y be a solution of (9). For this end we integrate the same equation over $[a, \frac{a+b}{2}]$ and using (10), we get $\int_a^{\frac{a+b}{2}} y(x)dx = 0$. The proof of $\int_{\frac{a+b}{2}}^b y(x)dx = 0$ is similar using integration on $[\frac{a+b}{2}, b]$ and inequality (11). □

Now, by using Opial's inequality (2.2) and Lemma 2, we prove the following result

Theorem 11. *Let y be a nontrivial solution of* (8)*. If $0 < m \leq p'(x) \leq M$ for $a \leq x \leq b$ and $\alpha = \beta$, then,*

$$m \leq 2 \mid \lambda \mid \frac{b-a}{\pi}. \tag{13}$$

Proof. Multiplying $(p(x)y')' + \lambda y = 0$ by y', integrating by parts over $[a, b]$ and taking into account $\alpha y'(a) - \beta y'(b) = 0$ with $\alpha = \beta$ gives

$$-\frac{1}{2}\int_a^b p'(x)y'^2(x)dx + \lambda \int_a^b yy'dx = 0.$$

Thus,

$$\frac{1}{2}\int_a^b p'(x)y'^2(x)dx \leq \mid \lambda \mid \int_a^b \mid y(x)y'(x) \mid dx. \tag{14}$$

Hence,

$$\frac{m}{2}\int_a^b y'^2(x)dx \leq \mid \lambda \mid \int_a^b \mid y(x)y'(x) \mid dx. \tag{15}$$

Applying Opial's inequality with nonlocal conditions (2.2) to the RHS of (15), we obtain

$$\frac{m}{2}\int_a^b y'^2(x)dx \leq \mid \lambda \mid \frac{b-a}{\pi}\int_a^b y'^2(x)dx. \tag{16}$$

By canceling $\int_a^b y'^2(x)dx$, we obtain (13). □

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