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All Regular Elements in $Hyp_G(2)$

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Abstract. In this paper we consider mappings σ which map the binary operation symbol f to the term $\sigma(f)$ which do not necessarily preserve the arities. We call these mappings generalized hypersubstitutions. Any generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma}$ on the set of all terms of type $\tau = (2)$. We define a binary operation on the set $Hyp_G(2)$ of all generalized hypersubstitutions of type $\tau = (2)$ by using this extension. The set $Hyp_G(2)$ together with the identity generalized hypersubstitution σ_{id} which maps f to the term $f(x_1, x_2)$ forms a monoid. We determine all regular elements of this monoid.

1. Introduction

The main tool for studying hyperidentities is the concept of a hypersubstitution which was introduced by K. Denecke, D. Lau, R.Pöschel and D. Schweigert [1] (see also in [3]). In [5], S. Leeratanavalee and K. Denecke generalized the concept of a hypersubstitution to a generalized hypersubstitution and that of hyperidentities to strong hyperidentities. Let $\{f_i | i \in I\}$ be an indexed set of operation symbols of type τ where f_i is n_i-ary, $n_i \in \mathbb{N}$. Let $W_\tau(X)$ be the set of all terms of type τ built up by operation symbols from $\{f_i \mid i \in I\}$ and variables from $X := \{x_1, x_2, x_3, \ldots\}$. A generalized hypersubstitution is a mapping σ which maps each n_i-ary operation symbol of type τ to a term of this type which does not necessarily preserve the arity. To define the extension $\hat{\sigma}$ of σ, we define inductively the concept of superposition of terms $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ as follows:

for $t \in W_\tau(X)$,

(i) if $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$,

(ii) if $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$,

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

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Then we extend the generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ as follows:

(i) $\hat{\sigma}[x_j] := x_j \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for an n_i-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \le j \le n_i$ are already known.

Let $Hyp_G(\tau)$ be the set of all generalized hypersubstitutions of type τ. We define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ for every $\sigma_1, \sigma_2 \in Hyp_G(\tau)$, where $\circ$ denotes the usual composition of mappings. Let σ_{id} be the generalized hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. Then we have the following two propositions.

Proposition 1.1([5]). *For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

Proposition 1.2([5]). $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ *is a monoid where σ_{id} is the identity element and the monoid* $\underline{Hyp(\tau)} = (Hyp(\tau); \circ_h, \sigma_{id})$ *of all arity preserving hypersubstitutions of type τ forms a submonoid of* $\underline{Hyp_G(\tau)}$.

For more details on generalized hypersubstitutions see [4] and [5]. Next, we will determine all regular elements of the monoid $Hyp_G(2)$.

2. Regular elements in $Hyp_G(2)$

From now on, we assume that the type is $\tau = (2)$, i.e. we have only one binary operation symbol, say f. By σ_t we denote the generalized hypersubstitution which maps f to the term t and by $var(t)$ we denote the set of all variables occurring in the term t. We will determine all regular elements of $Hyp_G(2)$. Firstly, we recall the definition of a regular element.

Definition 2.1([2]). An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$. The semigroup S is called *regular* if all its elements are regular.

Theorem 2.2. *Let $t \in W_{(2)}(X)$. Then σ_t is regular iff t has one of the following forms:*

(a) $t = f(x_2, s)$ *for* $s \in W_{(2)}(X)$ *with* $x_1 \notin var(s)$,

(b) $t = f(s, x_2)$ *for* $s \in W_{(2)}(X)$ *with* $x_1 \notin var(s)$,

(c) $t = f(x_1, s)$ *for* $s \in W_{(2)}(X)$ *with* $x_2 \notin var(s)$,

(d) $t = f(s, x_1)$ *for* $s \in W_{(2)}(X)$ *with* $x_2 \notin var(s)$,

(e) $t \in \{x_1, x_2, f(x_1, x_2), f(x_2, x_1)\}$,

(f) $var(t) \cap \{x_1, x_2\} = \emptyset$.

Proof. In the right-to-left direction, it is easy to verify that

(a) $\sigma_t \circ_G \sigma_{f(x_1,x_1)} \circ_G \sigma_t = \sigma_t \circ_G \sigma_{f(x_2,x_2)} = \sigma_t$.

(b) $\sigma_t \circ_G \sigma_{f(x_2,x_2)} \circ_G \sigma_t = \sigma_t \circ_G \sigma_{f(x_2,x_2)} = \sigma_t$.

(c) and (d) are similarly as (a) and (b), respectively.

(e) is trivial since $\sigma_t^3 = \sigma_t$.

(f) $\sigma_t \circ_G \sigma_{id} \circ_G \sigma_t = \sigma_t \circ_G \sigma_t = \sigma_t$.

For the converse direction, we have the following situation. Let σ_t be regular. Then there is $\sigma_s \in Hyp_G(2)$ such that $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$. If $t \notin X \cup \{f(x_1, x_2), f(x_2, x_1)\}$ and $var(t) \cap \{x_1, x_2\} \neq \emptyset$, then we consider four cases for t:

Case 1: $t = f(t_1, x_1)$ where $t_1 \in W_{(2)}(X) \setminus X$ and $x_2 \in var(t_1)$ or $t = f(x_1, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X$ and $x_2 \in var(t_2)$.

Case 2: $t = f(t_1, x_2)$ where $t_1 \in W_{(2)}(X) \setminus X$ and $x_1 \in var(t_1)$ or $t = f(x_2, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X$ and $x_1 \in var(t_2)$.

Case 3: $t = f(t_1, x_i)$ where $t_1 \in W_{(2)}(X) \setminus X$ and $x_1 \in var(t_1)$ or $t = f(x_i, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X$ and $x_2 \in var(t_2)$.

Case 4: $t = f(t_1, t_2)$ where $t_1, t_2 \in W_{(2)}(X) \setminus X$ and $x_1 \in var(t_1) \cup var(t_2)$ or $x_2 \in var(t_1) \cup var(t_2)$.

Let $t = f(t_1, t_2)$ where $t_1, t_2 \in W_{(2)}(X)$. Consider $(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) = (\sigma_{f(t_1,t_2)} \circ_G \sigma_s \circ_G \sigma_{f(t_1,t_2)})(f) = \hat{\sigma}_{f(t_1,t_2)}[\hat{\sigma}_s[f(t_1, t_2)]] = f(t_1, t_2)$. We put $u = \hat{\sigma}_s[f(t_1, t_2)]$. We have $u \notin X$ and thus $u = f(u_1, u_2)$ for some $u_1, u_2 \in W_{(2)}(X)$, i.e.

$$(1) \qquad \hat{\sigma}_{f(t_1,t_2)}[f(u_1, u_2)] = S^2(f(t_1, t_2), \hat{\sigma}_{f(t_1,t_2)}[u_1], \hat{\sigma}_{f(t_1,t_2)}[u_2]) = f(t_1, t_2)$$

Case 1: $t = f(t_1, x_1)$ where $t_1 \in W_{(2)}(X) \setminus X$ and $x_2 \in var(t_1)$ or $t = f(x_1, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X$ and $x_2 \in var(t_2)$.

Case 1.1: $t = f(t_1, x_1)$ where $t_1 \in W_{(2)}(X) \setminus X$ and $x_2 \in var(t_1)$. Then by (1), we have

$$S^2(f(t_1, x_1), \hat{\sigma}_{f(t_1,x_1)}[u_1], \hat{\sigma}_{f(t_1,x_1)}[u_2]) = f(t_1, x_1).$$

Thus $\hat{\sigma}_{f(t_1,x_1)}[u_1] = x_1$ and since $x_2 \in var(t_1)$, we have $\hat{\sigma}_{f(t_1,x_1)}[u_2] = x_2$. We have $u_1 = x_1, u_2 = x_2$. Thus $u = f(x_1, x_2)$ and $\hat{\sigma}_s[f(t_1, x_1)] = f(x_1, x_2)$. It is clear that $s \notin X$. So $s = f(s_1, s_2)$ for some $s_1, s_2 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(s_1,s_2)}[f(t_1, x_1)] = S^2(f(s_1, s_2), \hat{\sigma}_{f(s_1,s_2)}[t_1], x_1) = f(x_1, x_2)$. Since $t_1 \notin X$, we have $\hat{\sigma}_{f(s_1,s_2)}[t_1] \notin X$.

Therefore $s_2 = x_1$ and $\hat{\sigma}_{f(s_1,s_2)}[t_1] = x_2$ which contradicts to $\hat{\sigma}_{f(s_1,s_2)}[t_1] \notin X$. Hence $\sigma_{f(t_1,x_1)}$ is not regular.

Case 1.2: $t = f(x_1, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X$ and $x_2 \in var(t_2)$. We can prove that $\sigma_{f(x_1,t_2)}$ is not regular by the similar way as in Case 1.1.

Case 2: $t = f(t_1, x_2)$ where $t_1 \in W_{(2)}(X) \setminus X$ and $x_1 \in var(t_1)$ or $t = f(x_2, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X$ and $x_1 \in var(t_2)$. We can prove that $\sigma_{f(t_1,x_2)}$ and $\sigma_{f(x_2,t_2)}$ are not regular by the similar way as in Case 1.

Case 3: $t = f(t_1, x_i)$ where $t_1 \in W_{(2)}(X) \setminus X$ and $x_1 \in var(t_1)$ or $t = f(x_i, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X$ and $x_2 \in var(t_2)$.

Case 3.1: $t = f(t_1, x_i)$ where $t_1 \in W_{(2)}(X) \setminus X$ and $x_1 \in var(t_1)$. Then by (1), we have

$$S^2(f(t_1, x_i), \hat{\sigma}_{f(t_1,x_i)}[u_1], \hat{\sigma}_{f(t_1,x_i)}[u_2]) = f(t_1, x_i).$$

Since $x_1 \in var(t_1)$, we have $\hat{\sigma}_{f(t_1,x_i)}[u_1] = x_1$. We have $u_1 = x_1$. Thus $u = f(x_1, u_2)$ and $\hat{\sigma}_s[f(t_1, x_i)] = f(x_1, u_2)$. It is clear that $s \notin X$. So $s = f(s_1, s_2)$ for some $s_1, s_2 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(s_1,s_2)}[f(t_1, x_i)] = S^2(f(s_1, s_2), \hat{\sigma}_{f(s_1,s_2)}[t_1], x_i) = f(x_1, u_2)$. Therefore $s_1 = x_1 = \hat{\sigma}_{f(s_1,s_2)}[t_1]$. Since $t_1 \notin X$, we have $\hat{\sigma}_{f(s_1,s_2)}[t_1] \notin X$ which contradicts to $\hat{\sigma}_{f(s_1,s_2)}[t_1] = x_1$. Hence $\sigma_{f(t_1,x_i)}$ is not regular.

Case 3.2: $t = f(x_i, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X$ and $x_2 \in var(t_2)$. We can prove that $\sigma_{f(x_i,t_2)}$ is not regular by the similar way as in Case 3.1.

Case 4: $t = f(t_1, t_2)$ where $t_1, t_2 \in W_{(2)}(X) \setminus X$ and $x_1 \in var(t_1) \cup var(t_2)$ or $x_2 \in var(t_1) \cup var(t_2)$.

Case 4.1: $t = f(t_1, t_2)$ where $x_1 \in var(t_1) \cup var(t_2)$. Then by (1), we have

$$S^2(f(t_1, t_2), \hat{\sigma}_{f(t_1,t_2)}[u_1], \hat{\sigma}_{f(t_1,t_2)}[u_2]) = f(t_1, t_2).$$

Since $x_1 \in var(t_1) \cup var(t_2)$ then $\hat{\sigma}_{f(t_1,t_2)}[u_1] = x_1$. We have $u_1 = x_1$. Thus $u = f(x_1, u_2)$ and $\hat{\sigma}_s[f(t_1, t_2)] = f(x_1, u_2)$. It is clear that $s \notin X$. So $s = f(s_1, s_2)$ for some $s_1, s_2 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(s_1,s_2)}[f(t_1, t_2)] = S^2(f(s_1, s_2), \hat{\sigma}_{f(s_1,s_2)}[t_1], \hat{\sigma}_{f(s_1,s_2)}[t_2]) = f(x_1, u_2)$. Therefore $t_1 = x_1$ which contradicts to $t_1 \notin X$. Hence $\sigma_{f(t_1,t_2)}$ is not regular.

Case 4.2: $t = f(t_1, t_2)$ where $x_2 \in var(t_1) \cup var(t_2)$. We can prove that $\sigma_{f(t_1,t_2)}$ is not regular by the similar way as in Case 4.1. □

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