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APPLICATION OF EXP-FUNCTION METHOD FOR A CLASS OF NONLINEAR PDE'S ARISING IN MATHEMATICAL PHYSICS

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Abstract. In this paper we apply the Exp-function method to obtain traveling wave solutions of three nonlinear partial differential equations, namely, generalized sinh-Gordon equation, generalized form of the famous sinh-Gordon equation, and double combined sinh-cosh-Gordon equation. These equations play a very important role in mathematical physics and engineering sciences. The Exp-Function method changes the problem from solving nonlinear partial differential equations to solving a ordinary differential equation. Mainly we try to present an application of Exp-function method taking to consideration rectifying a commonly occurring errors during some of recent works.

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1. Introduction

The study of exact traveling wave solutions of nonlinear partial differential equations (NPDE) plays an important role in mathematical physics, engineering and the other sciences. The wave phenomena are observed in plasma, kink dynamics, fluid dynamics, elastic media, etc. In the past several decades, various methods for obtaining solutions of NPDEs and ODEs have been presented, such as , tanh-function method [44, 46, 45], Adomian decomposition method [18, 40], Homotopy perturbation method [32, 7, 5], variational iteration method [34, 37, 14], spectral method [28, 29, 27], sine-cosine method [39, 42], radial basis method [41, 10] and so on. Recently, He and Wu [20] proposed a novel method, so called Exp-function method, which is easy, succinct and powerful

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to implement to nonlinear partial differential equations arising in mathematical physics. The Exp-function method has been successfully applied to many kinds of NPDEs, such as, KdV equation with variable coefficients [50], Maccari's system [51], Kawahara equation [4], Boussinesq equations [1], Burger's equations [12, 6, 13], Double Sine-Gordon equation [11, 19], Fisher equation [26], Jaulent-Miodek equations [22] and the other important nonlinear partial differential equations [9, 35, 52]. In this paper we apply the Exp-function method [20] to obtain exact solutions of three nonlinear partial differential equations, namely, generalized sinh-Gordon equation, generalized form of the famous sinh-Gordon equation, and double combined sinh-cosh-Gordon equation given by

$$u_{tt} - au_{xx} + b\sinh(nu) = 0\ ,\ n \geq 1$$
$$u_{xt} + b\sinh(nu) = 0\ ,\ n \geq 1$$
$$u_{tt} - mu_{xx} + \alpha\sinh(u) + \alpha\cosh(u) + \beta\sinh(2u) + \beta\cosh(2u) = 0\ .$$

respectively.

It is well known that the sinh-Gordon type equations admits geometric interpretation as the differential equation which determines time-like surfaces of constant positive curvature in the same spaces [16, 33]. This type equations are known to be completely integrable because it possesses similarity reductions to the third Painlevé equation [43, 17, 8]. The sinh-Gordon type equations appear in wide range of physical applications including fluid flow, relativistic field theory, string dynamics, hydrodynamics, quantum field theory, kink dynamics, fluid dynamics, thermodynamics, differential geometry, solid-state physics, dislocations in metals and nonlinear optics [38, 3, 36, 15, 47, 24, 31, 2, 43]. Also, sinh-Gordon equation appeared in the propagation of fluxons in Josephson junctions [30] between two superconductors. Sinh-Gordon type equations have been investigated with analytical and numerical methods by some authors [44, 45, 33, 8, 36, 48, 49]. Recently, Wazwaz [44, 46, 45, 43] studied the generalized sinh-Gordon type equations by using the tanh method and a variable separated ODE method. He derived families of exact traveling wave solutions of them. Also, Sirendaoreji [36] applied a direct method for solving sinh-Gordon equation. Hu et. al.[23] solved sinh-Gordon by the complex multi-symplectic scheme.

The rest of the paper is organized as follows: Section 2 describes Exp-function method for finding exact traveling wave solutions to the NPDEs. The applications of the proposed analytical scheme presented in Section 3. The conclusions are discussed in the section 4.

2. Summary of Exp-function method

We consider a general nonlinear PDE in the following form

$$N(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...) = 0\ , \tag{1}$$

where N is a polynomial function with respect to the indicated variables or some functions which can be reduced to a polynomial function by using some transformation. We introduce a complex variation as

$$u(x,t) = U(\eta)\ ,\ \eta = k(x - ct) + \varphi_0\ . \tag{2}$$

where k and c are constants and φ_0 is an arbitrary constant. We can rewrite Eq.(1) in the following nonlinear ordinary differential equations

$$N(U, kU', -kcU', k^2U'', ...) = 0\ ,$$

where the prime denotes the derivative with respect to η. According to the Exp-function method [20], we assume that the solution can be expressed in the form

$$U(\eta) = \frac{\sum_{i=-d}^{f} a_i \exp(i\eta)}{\sum_{j=-q}^{p} b_j \exp(j\eta)}\ , \tag{3}$$

where f, d, p and q are positive integers which can be freely chosen, a_i and b_j are unknown constants to be determined. To determine the values of f and p, we balance the highest order linear term with the highest order nonlinear term in Eq.(3). Similarly, to determine the values of d and q. So by means of the Exp-function method, we obtain the traveling wave solution for nonlinear evolution equations arising in mathematical physics.

3. Applications of Exp-function method

3.1.The generalized sinh-Gordon equation. Let us consider the generalized sinh-Gordon equation [44, 45] in the form

$$u_{tt} - au_{xx} + b\sinh(nu) = 0\ . \tag{4}$$

By using the complex variation

$$u(x,t) = U(\eta)\ ,\ \eta = k(x - ct) + \varphi_0\ , \tag{5}$$

where k and c are constants to be determined later and φ_0 is an arbitrary constant, Eq.(4) can be converted to the ODE

$$k^2(c^2 - a)U'' + b\sinh(nU) = 0\ ,$$

where the prime denotes the derivative with respect to η. Now, using the Painlevé transformation,

$$v(\eta) = e^{nU(\eta)}\ , \tag{6}$$

we have

$$\begin{aligned} U' &= \frac{1}{nv}v'\ , \\ U'' &= \frac{1}{nv}v'' - \frac{1}{nv^2}(v')^2\ . \end{aligned} \tag{7}$$

By using Eq.(6) we have

$$\sinh(nU) = \frac{v - v^{-1}}{2} \ , \ \cosh(nU) = \frac{v + v^{-1}}{2} \ , \tag{8}$$

and

$$U = \frac{1}{n} arccosh\left(\frac{v + v^{-1}}{2}\right) \ . \tag{9}$$

Substituting the transformations (7) into the sinh-Gordon equation gives the ODE,

$$2k^2(c^2 - a)vv'' - 2k^2(c^2 - a)(v')^2 + nbv^3 - nbv = 0 \ , \tag{10}$$

According to the Exp-function method [20, 19, 21], we assume that the solution of Eq.(10) can be expressed in the form

$$v(\eta) = \frac{a_f \exp(f\eta) + ... + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + ... + b_{-q} \exp(-q\eta)} \ ,$$

where f, d, p and q are positive integers which are unknown to be determined later. In order to determine values of f and p, we balance the linear term of the highest order with the highest order nonlinear terms in Eq.(10), i.e. vv'' and v^3. By simplify calculation, we have

$$vv'' = \frac{c_1 \exp[(2f + 3p)\eta] + ...}{c_2 \exp[5p\eta] + ...} \ , \tag{11}$$

and

$$v^3 = \frac{c_3 \exp[(3f + p)\eta] + ...}{c_4 \exp[4p\eta] + ...} = \frac{c_3 \exp[(3f + 2p)\eta] + ...}{c_4 \exp[5p\eta] + ...} \ , \tag{12}$$

where c_i are coefficients only for simplicity. By balancing highest order of Exp-function in Eqs.(11) and (12), we have

$$3f + 2p = 2f + 3p \ ,$$

which leads to the result

$$p = f \ .$$

Similarly, to determine values of d and q, we balance the linear term of lowest order in Eq.(10)

$$vv'' = \frac{... + d_1 \exp[-(3q + 2d)\eta]}{... + d_2 \exp[-5q\eta]} \ , \tag{13}$$

and

$$v^3 = \frac{... + d_3 \exp[-(q + 3d)\eta]}{... + d_4 \exp[-4q\eta]} = \frac{... + d_3 \exp[-(2q + 3d)\eta]}{... + d_4 \exp[-5q\eta]} \ , \tag{14}$$

where d_i are determined coefficients only for simplicity, we have

$$-(2q + 3d) = -(3q + 2d) \ ,$$

which leads to results

$$q = d \ .$$

For simplicity, we set $p = f = 1$ and $q = d = 1$, so Eq.(3) reduces to

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)} \ . \tag{15}$$

Substituting Eq.(15) into Eq.(10) and equating to zero the coefficients of all powers of $\exp(n\eta)$, yields a set of algebraic equations in terms of a_0, b_0, a_{-1}, a_1, b_1, k and c. To determine the unknowns we can solve the obtained system of algebraic equations by a symbolic professional mathematical software

case 1.

$$\begin{cases} a_1 = 1, \ b_{-1} = \frac{1}{4} b_0^2, \\ a_0 = -b_0, \ b_0 = b_0, \\ a_{-1} = \frac{1}{4} b_0^2, \ k = \sqrt{\frac{nb}{a-c^2}}. \end{cases}$$

Substituting these result into Eq.(15), we obtain

$$v(x,t) = \frac{\exp[\sqrt{\frac{nb}{a-c^2}}(x-ct) + \varphi_0] - b_0 + \frac{1}{4} b_0^2 \exp[-\sqrt{\frac{nb}{a-c^2}}(x-ct) + \varphi_0]}{\exp[\sqrt{\frac{nb}{a-c^2}}(x-ct) + \varphi_0] + b_0 + \frac{1}{4} b_0^2 \exp[-\sqrt{\frac{nb}{a-c^2}}(x-ct) + \varphi_0]} \ . \tag{16}$$

where b_0, c and φ_0 are free parameters which can be determined by initial or boundary conditions. These results cover some of special solutions of Eq.(4) regarding to initial value conditions. By considering $u(x, 0)$ as a initial value condition, we have

$$\begin{cases} u_{tt} - a u_{xx} + b \sinh(nu) = 0 \ , \\ u(x,0) = \frac{1}{n} arccosh \left\{ \frac{1}{2} \left(\tanh^2 \left[\frac{1}{2} \sqrt{\frac{nb}{a-c^2}} x \right] + \coth^2 \left[\frac{1}{2} \sqrt{\frac{nb}{a-c^2}} x \right] \right) \right\} \ . \end{cases} \tag{17}$$

From Eq.(17) and Eq.(16), we obtain

$$b_0 = 2 \ , \ \varphi_0 = 0 \ , \tag{18}$$

Thus, from substituting Eq.(18) into Eq.(16), we obtain

$$v(x,t) = \frac{\exp[\sqrt{\frac{nb}{a-c^2}}(x-ct)] - 2 + \exp[-\sqrt{\frac{nb}{a-c^2}}(x-ct)]}{\exp[\sqrt{\frac{nb}{a-c^2}}(x-ct)] + 2 + \exp[-\sqrt{\frac{nb}{a-c^2}}(x-ct)]} \ .$$

or equivalently,

$$v(x,t) = \tanh^2 \left[\frac{1}{2} \sqrt{\frac{nb}{a-c^2}} (x-ct) \right] \ .$$

By Eq.(9), we can obtain the solution

$$u(x,t) = \frac{1}{n} arccosh\left\{ \frac{1}{2}\left(\tanh^2\left[\frac{1}{2}\sqrt{\frac{nb}{a-c^2}}(x-ct)\right] + \coth^2\left[\frac{1}{2}\sqrt{\frac{nb}{a-c^2}}(x-ct)\right]\right)\right\} .$$

which is the traveling wave solution obtained by tanh method in [44, 45].
If Eq.(4) be in the following form

$$\begin{cases} u_{tt} - a u_{xx} + b\sinh(nu) = 0 \ , \\ u(x,0) = \frac{1}{n} arccosh\left\{ -\frac{1}{2}\left(\tan^2\left[\frac{1}{2}\sqrt{\frac{nb}{c^2-a}}x\right] + \cot^2\left[\frac{1}{2}\sqrt{\frac{nb}{c^2-a}}x\right]\right)\right\} . \end{cases} \quad (19)$$

then taking to account Eq.(16), we can obtain

$$b_0 = -2 \ , \ \varphi_0 = 0 \ , \quad (20)$$

Thus, substituting Eq.(20) into Eq.(16), we have

$$v(x,t) = \frac{\exp[-i\sqrt{\frac{nb}{c^2-a}}(x-ct)] - 2 + \exp[i\sqrt{\frac{nb}{c^2-a}}(x-ct)]}{\exp[-i\sqrt{\frac{nb}{c^2-a}}(x-ct)] + 2 + \exp[i\sqrt{\frac{nb}{c^2-a}}(x-ct)]} .$$

or equivalently,

$$v(x,t) = -\tan^2\left[\frac{1}{2}\sqrt{\frac{nb}{c^2-a}}(x-ct)\right]$$

By Eq.(9), we can obtain the solution

$$u(x,t) = \frac{1}{n} arccosh\left\{ -\frac{1}{2}\left(\tan^2\left[\frac{1}{2}\sqrt{\frac{nb}{c^2-a}}(x-ct)\right] + \cot^2\left[\frac{1}{2}\sqrt{\frac{nb}{c^2-a}}(x-ct)\right]\right)\right\} .$$

which is the traveling wave solution obtained by tanh method in [44, 45] and complex multi-symplectic method in [23].

case 2.

$$\begin{cases} a_1 = -1, \ b_{-1} = \frac{1}{4}b_0^2, \\ a_0 = b_0, \ b_0 = b_0, \\ a_{-1} = -\frac{1}{4}b_0^2, \ k = \sqrt{\frac{nb}{c^2-a}}. \end{cases}$$

Substituting these result into Eq.(15), we obtain

$$v(x,t) = -\frac{\exp[\sqrt{\frac{nb}{c^2-a}}(x-ct)+\varphi_0] - b_0 + \frac{1}{4}b_0^2\exp[-\sqrt{\frac{nb}{c^2-a}}(x-ct)+\varphi_0]}{\exp[\sqrt{\frac{nb}{c^2-a}}(x-ct)+\varphi_0] + b_0 + \frac{1}{4}b_0^2\exp[-\sqrt{\frac{nb}{c^2-a}}(x-ct)+\varphi_0]} . \quad (21)$$

where b_0, c and φ_0 are free parameters which can be determined by initial or boundary conditions. If Eq.(4) be in the following form

$$\begin{cases} u_{tt} - au_{xx} + b\sinh(nu) = 0 \ , \\ u(x,0) = \frac{1}{n}arccosh\Bigg\{ -\frac{1}{2}\Bigg(\tanh^2\left[\frac{1}{2}\sqrt{\frac{nb}{a-c^2}}x\right] + \coth^2\left[\frac{1}{2}\sqrt{\frac{nb}{a-c^2}}x\right]\Bigg)\Bigg\} \ . \end{cases} \tag{22}$$

then, taking to account Eq.(21), we can obtain

$$b_0 = 2 \ , \ \varphi_0 = 0 \ , \tag{23}$$

Thus, substituting Eq.(23) into Eq.(21), we have:

$$v(x,t) = -\frac{\exp[\sqrt{\frac{nb}{c^2-a}}(x-ct)] - 2 + \exp[-\sqrt{\frac{nb}{c^2-a}}(x-ct)]}{\exp[\sqrt{\frac{nb}{c^2-a}}(x-ct)] + 2 + \exp[-\sqrt{\frac{nb}{c^2-a}}(x-ct)]} \ .$$

or equivalently,

$$v(x,t) = -\tanh^2\left[\sqrt{\frac{nb}{c^2-a}}(x-ct)\right] \ ,$$

By Eq.(9), we can obtain the solution

$$u(x,t) = \frac{1}{n}arccosh\Bigg\{ -\frac{1}{2}\Bigg(\tanh^2\left[\frac{1}{2}\sqrt{\frac{nb}{a-c^2}}(x-ct)\right] + \coth^2\left[\frac{1}{2}\sqrt{\frac{nb}{a-c^2}}(x-ct)\right]\Bigg)\Bigg\} \ .$$

which is the traveling wave solution obtained by tanh method in [44, 45] and complex multi-symplectic method in [23].

If Eq.(4) be in the following form

$$\begin{cases} u_{tt} - au_{xx} + b\sinh(nu) = 0 \ , \\ u(x,0) = \frac{1}{n}arccosh\Bigg\{ -\frac{1}{2}\Bigg(\tanh^2\left[\frac{1}{2}\sqrt{\frac{nb}{a-c^2}}x\right] + \coth^2\left[\frac{1}{2}\sqrt{\frac{nb}{a-c^2}}x\right]\Bigg)\Bigg\} \ . \end{cases} \tag{24}$$

then, taking to account Eq.(21), we can obtain

$$b_0 = 2 \ , \ \varphi_0 = 0 \ , \tag{25}$$

Thus, substituting Eq.(25) into Eq.(21), we have:

$$v(x,t) = -\frac{\exp[-i\sqrt{\frac{nb}{c^2-a}}(x-ct)] - 2 + \exp[i\sqrt{\frac{nb}{c^2-a}}(x-ct)]}{\exp[-i\sqrt{\frac{nb}{c^2-a}}(x-ct)] + 2 + \exp[i\sqrt{\frac{nb}{c^2}}(x-ct)]} \ .$$

or equivalently,

$$v(x,t) = \tan^2\left[\frac{1}{2}\sqrt{\frac{nb}{c^2-a}}(x-ct)\right]$$

By Eq.(9), we can obtain the solution

$$u(x,t) = \frac{1}{n} arccosh \left\{ \frac{1}{2} \left(\tan^2 \left[\frac{1}{2} \sqrt{\frac{nb}{c^2 - a}} (x - ct) \right] + \cot^2 \left[\frac{1}{2} \sqrt{\frac{nb}{c^2 - a}} (x - ct) \right] \right) \right\} .$$

which is the traveling wave solution obtained by tanh method in [44, 45].

3.2.The generalized famous sinh-Gordon equation. Now we consider the generalized famous sinh-Gordon equation [44, 45, 8, 36]

$$u_{xt} + b \sinh(nu) = 0 . \tag{26}$$

Using the transformation (2), Eq.(26) becomes

$$-k^2 cU'' + b \sinh(nU) = 0 .$$

where prime denotes the differential with respect to η. Proceeding as before subsection we use the transformation (6), Eqs.(7, 8 and 9) gives

$$-2k^2 cvv'' + 2k^2 c(v')^2 + nbv^3 - nbv = 0 \tag{27}$$

The highest nonlinear term vv'' is now given by

$$vv'' = \frac{c_1 \exp[(2f + 3p)\eta] + ...}{c_2 \exp[5p\eta] + ...} , \tag{28}$$

and the highest linear term v^3 is given by

$$v^3 = \frac{c_3 \exp[(3f + p)\eta] + ...}{c_4 \exp[4p\eta] + ...} = \frac{c_3 \exp[(3f + 2p)\eta] + ...}{c_4 \exp[5p\eta] + ...} . \tag{29}$$

Balancing the highest order of Exp-function in Eqs.(28) and (29), we have $2f + 3p = 3f + 2p$, so $p = f$. As mentioned in the previous subsection, we can obtain $q = d$. Here, we only consider the simplest case $p = f = 1$ and $q = d = 1$, so Eq.(3) reduces to

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)} . \tag{30}$$

Substituting Eq.(30) into Eq.(27) and equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations in term of a_0, b_0, a_{-1}, a_1, b_1, k and c. To determine the unknowns we can solve the obtained system of algebraic equations by a symbolic professional mathematical software

case 1.

$$\begin{cases} a_1 = 1, \; b_{-1} = \frac{1}{4} b_0^2, \\ a_0 = -b_0, \; b_0 = b_0, \\ a_{-1} = \frac{1}{4} b_0^2, \; k = \sqrt{\frac{nb}{c}}. \end{cases}$$

Substituting these result into Eq.(30), we obtain

$$v(x,t) = \frac{\exp[\sqrt{\frac{nb}{c}}(x-ct)+\varphi_0] - b_0 + \frac{1}{4}b_0^2 \exp[-\sqrt{\frac{nb}{c}}(x-ct)+\varphi_0]}{\exp[\sqrt{\frac{nb}{c}}(x-ct)+\varphi_0] + b_0 + \frac{1}{4}b_0^2 \exp[-\sqrt{\frac{nb}{c}}(x-ct)+\varphi_0]} \ . \qquad (31)$$

where b_0, c and φ_0 are free parameters. If Eq.(26) be in the following form

$$\begin{cases} u_{xt} + b\sinh(nu) = 0 \ , \\ u(x,0) = \frac{1}{n} arccosh\left\{ \frac{1}{2}\left(\tanh^2\left[\frac{1}{2}\sqrt{\frac{nb}{c}}x\right] + \coth^2\left[\frac{1}{2}\sqrt{\frac{nb}{c}}x\right]\right)\right\} \ . \end{cases} \qquad (32)$$

then, taking to account Eq.(31), we can obtain

$$b_0 = 2 \ , \ \varphi_0 = 0 \ , \qquad (33)$$

Thus, substituting Eq.(33) into Eq.(31), we have:

$$v(x,t) = \frac{\exp[\sqrt{\frac{nb}{c}}(x-ct)] - 2 + \exp[-\sqrt{\frac{nb}{c}}(x-ct)]}{\exp[\sqrt{\frac{nb}{c}}(x-ct)] + 2 + \exp[-\sqrt{\frac{nb}{c}}(x-ct)]} \ .$$

or equivalently,

$$v(x,t) = \tanh^2\left[\frac{1}{2}\sqrt{\frac{nb}{c}}(x-ct)\right] \ ,$$

By Eq.(9), we can obtain the solution

$$u(x,t) = \frac{1}{n} arccosh\left\{ \frac{1}{2}\left(\tanh^2\left[\frac{1}{2}\sqrt{\frac{nb}{c}}(x-ct)\right] + \coth^2\left[\frac{1}{2}\sqrt{\frac{nb}{c}}(x-ct)\right]\right)\right\} \ .$$

which is the traveling wave solution obtained by tanh method in [44, 45]. Also, if Eq.(26) be in the following form

$$\begin{cases} u_{xt} + b\sinh(nu) = 0 \ , \\ u(x,0) = \frac{1}{n} arccosh\left\{ -\frac{1}{2}\left(\tan^2\left[\frac{1}{2}\sqrt{-\frac{nb}{c}}x\right] + \cot^2\left[\frac{1}{2}\sqrt{-\frac{nb}{c}}x\right]\right)\right\} \ . \end{cases} \qquad (34)$$

then, taking to account Eq.(31), we can obtain

$$b_0 = 2 \ , \ \varphi_0 = 0 \ , \qquad (35)$$

Thus, substituting Eq.(35) into Eq.(31), we have:

$$v(x,t) = \frac{\exp[-i\sqrt{-\frac{nb}{c}}(x-ct)] - 2 + \exp[i\sqrt{-\frac{nb}{c}}(x-ct)]}{\exp[-i\sqrt{-\frac{nb}{c}}(x-ct)] + 2 + \exp[i\sqrt{-\frac{nb}{c}}(x-ct)]} \ .$$

or equivalently,

$$v(x,t) = -\tan^2\left[\frac{1}{2}\sqrt{-\frac{nb}{c}}(x-ct)\right] ,$$

By Eq.(9), we can obtain the solution

$$u(x,t) = \frac{1}{n}arccosh\Bigg\{ -\frac{1}{2}\Bigg(\tan^2\left[\frac{1}{2}\sqrt{-\frac{nb}{c}}(x-ct)\right] + \cot^2\left[\frac{1}{2}\sqrt{-\frac{nb}{c}}(x-ct)\right]\Bigg)\Bigg\} .$$

which is the traveling wave solution obtained by tanh method in [44, 45].

case 2.

$$\begin{cases} a_1 = -1,\ b_{-1} = \frac{1}{4}b_0^2, \\ a_0 = b_0,\ b_0 = b_0, \\ a_{-1} = -\frac{1}{4}b_0^2,\ k = \sqrt{-\frac{nb}{c}}. \end{cases}$$

Substituting these result into Eq.(30), we obtain

$$v(x,t) = -\frac{\exp[\sqrt{-\frac{nb}{c}}(x-ct)+\varphi_0] - b_0 + \frac{1}{4}b_0^2\exp[-\sqrt{-\frac{nb}{c}}(x-ct)+\varphi_0]}{\exp[\sqrt{-\frac{nb}{c}}(x-ct)+\varphi_0] + b_0 + \frac{1}{4}b_0^2\exp[-\sqrt{-\frac{nb}{c}}(x-ct)+\varphi_0]} . \quad (36)$$

where b_0, c and φ_0 are free parameters. If Eq.(26) be in the following form

$$\begin{cases} u_{xt} + b\sinh(nu) = 0\ , \\ u(x,0) = \frac{1}{n}arccosh\Bigg\{ -\frac{1}{2}\Bigg(\tanh^2\left[\frac{1}{2}\sqrt{-\frac{nb}{c}}x\right] + \coth^2\left[\frac{1}{2}\sqrt{-\frac{nb}{c}}x\right]\Bigg)\Bigg\} , \end{cases} \quad (37)$$

then, taking to account Eq.(36), we can obtain

$$b_0 = 2\ ,\ \varphi_0 = 0\ , \quad (38)$$

Thus, substituting Eq.(38) into Eq.(36), we have:

$$v(x,t) = -\frac{\exp[\sqrt{-\frac{nb}{c}}(x-ct)] - 2 + \exp[-\sqrt{-\frac{nb}{c}}(x-ct)]}{\exp[\sqrt{-\frac{nb}{c}}(x-ct)] + 2 + \exp[-\sqrt{-\frac{nb}{c}}(x-ct)]} .$$

or equivalently,

$$v(x,t) = -\tanh^2\left[\frac{1}{2}\sqrt{-\frac{nb}{c}}(x-ct)\right] ,$$

By Eq.(9), we can obtain the solution

$$u(x,t) = \frac{1}{n}arccosh\Bigg\{ -\frac{1}{2}\Bigg(\tanh^2\left[\frac{1}{2}\sqrt{-\frac{nb}{c}}(x-ct)\right] + \coth^2\left[\frac{1}{2}\sqrt{-\frac{nb}{c}}(x-ct)\right]\Bigg)\Bigg\}$$

which is the traveling wave solution obtained by tanh method in [44, 45].
Also, if Eq.(26) be in the following form

$$\begin{cases} u_{xt} + b\sinh(nu) = 0 \ , \\ u(x,0) = \frac{1}{n} arccosh\left\{ \frac{1}{2}\left(\tan^2\left[\frac{1}{2}\sqrt{\frac{nb}{c}}x\right] + \cot^2\left[\frac{1}{2}\sqrt{\frac{nb}{c}}x\right]\right)\right\} \ . \end{cases} \tag{39}$$

then, taking to account Eq.(36), we can obtain

$$b_0 = 2 \ , \ \varphi_0 = 0 \ , \tag{40}$$

Thus, substituting Eq.(40) into Eq.(36), we have:

$$v(x,t) = -\frac{\exp[i\sqrt{\frac{nb}{c}}(x-ct)] - 2 + \exp[-i\sqrt{\frac{nb}{c}}(x-ct)]}{\exp[i\sqrt{\frac{nb}{c}}(x-ct)] + 2 + \exp[-i\sqrt{\frac{nb}{c}}(x-ct)]} \ .$$

or equivalently,

$$v(x,t) = \tan^2\left[\frac{1}{2}\sqrt{\frac{nb}{c}}(x-ct)\right] \ , \tag{41}$$

By Eq.(9), we can obtain the solution

$$u(x,t) = \frac{1}{n} arccosh\left\{ \frac{1}{2}\left(\tan^2\left[\frac{1}{2}\sqrt{\frac{nb}{c}}(x-ct)\right] + \cot^2\left[\frac{1}{2}\sqrt{\frac{nb}{c}}(x-ct)\right]\right)\right\} \ .$$

which is the traveling wave solution obtained by tanh method in [44, 45].
Also, if we set $b = -1$, $n = 1$ and $c = -\frac{4}{a^2}$ in the Eq.(41), where a is free parameter, then gives

$$u(x,t) = 2\ln\left[\tan\left(\frac{a}{4}x + \frac{t}{a} + \varphi_0\right)\right] \ ,$$

which is the solution obtained in [8, 36].

3.3.The double combined sinh-cosh-Gordon equation. In this section, we examine the double combined sinh-cosh-Gordon equation given by [46]

$$u_{tt} - mu_{xx} + \alpha\sinh(u) + \alpha\cosh(u) + \beta\sinh(2u) + \beta\cosh(2u) = 0 \ . \tag{42}$$

The transformation (5) converts Eq.(42) into the nonlinear ODE:

$$k^2(c^2-m)U'' + \alpha\sinh(U) + \alpha\cosh(U) + \beta\sinh(2U) + \beta\cosh(2U) = 0 \ .$$

We can use the transformations

$$v = e^U \ ,$$

or

$$U = \ln(v) \ .$$

so we have

$$\sinh(U) = \frac{v - v^{-1}}{2}\ ,\ \cosh(U) = \frac{v + v^{-1}}{2}\ ,$$
$$\sinh(2U) = \frac{v^2 - v^{-2}}{2}\ ,\ \cosh(2U) = \frac{v^2 + v^{-2}}{2}\ , \tag{43}$$

that also give

$$U = arccosh\left(\frac{v + v^{-1}}{2}\right)\ .$$

Substituting the transformations (43) into the sinh-cosh-Gordon equation gives the ODE,

$$2k^2(c^2 - m)vv'' - 2k^2(c^2 - m)(v')^2 + 2\alpha v^3 + 2\beta v^4 = 0\ .$$

We assume that the solutions can be expressed in the following forms [20]

$$v(\eta) = \frac{a_f \exp(f\eta) + ... + a_{-d}\exp(-d\eta)}{b_p \exp(p\eta) + ... + b_{-q}\exp(-q\eta)}\ ,$$

where f, d, p and q are positive integers which are unknown to be determined later. By balancing linear term of highest order (v^4) and balancing the highest order nonlinear term (vv'') we have

$$vv'' = \frac{c_1 \exp[(2f + 3p)\eta] + ...}{c_2 \exp[5p\eta] + ...}\ , \tag{44}$$

and

$$v^4 = \frac{c_3 \exp[(4f + p)\eta] + ...}{c_4 \exp[5p\eta] + ...}\ . \tag{45}$$

where c_i are coefficients only for simplicity. Balancing the highest order of the Exp-function in Eqs.(44) and (45), we have

$$4f + p = 2f + 3p\ ,$$

which leads to the result

$$f = p\ .$$

Similarly, to determine valuee of d and q, we balance the linear term of the lowest order in Eqs.(44) and (45)

$$vv'' = \frac{... + d_1 \exp[-(3q + 2d)\eta]}{... + d_2 \exp[-5q\eta]}\ , \tag{46}$$

and

$$v^4 = \frac{... + d_3 \exp[-(2q + 3d)\eta]}{... + d_4 \exp[-5q\eta]}\ . \tag{47}$$

where d_i are determined coefficients only for simplicity, we have

$$-(3q + 2d) = -(2q + 3d)\ ,$$

which leads to results

$$q = d\ .$$

For simplicity, we set $p = f = 1$ and $q = d = 1$, so Eq.(3) reduces to

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}\ . \tag{48}$$

Substituting Eq.(48) into Eq.(42) and equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations in terms of a_0, b_0, a_{-1}, a_1, b_1, k and c. To determine the unknowns we can solve the obtained system of algebraic equations by a symbolic professional mathematical software

$$\begin{aligned} &a_1 = 0,\ a_0 = 0,\ a_{-1} = a_{-1}, \\ &b_0 = 0,\ b_{-1} = -\frac{\beta}{\alpha} a_{-1},\ k = \pm \frac{\alpha}{2\sqrt{\beta(m - c^2)}}\ . \end{aligned}$$

Substituting these result into Eq.(30), we obtain

$$v(x,t) = \frac{a_{-1} \exp[\pm \frac{\alpha}{2\sqrt{\beta(m-c^2)}}(x - ct)]}{\exp[\pm \frac{\alpha}{2\sqrt{\beta(m-c^2)}}(x - ct)] - \frac{\beta}{\alpha} a_{-1} \exp[\pm \frac{\alpha}{2\sqrt{\beta(m-c^2)}}(x - ct)]}\ . \tag{49}$$

To compare our results with those obtained in [46], we present the following discussion

I. At $k = \pm \frac{\alpha}{2\sqrt{\beta(m-c^2)}}$, $a_{-1} = -\frac{\alpha}{\beta}$
According to Eq.(49) we can obtain

$$v(x,t) = -\frac{\alpha}{2\beta}\left(1 \mp \tanh\left[\frac{\alpha}{2\sqrt{\beta(m - c^2)}}(x - ct)\right]\right),\ \beta(m - c^2) > 0$$

II. At $k = \pm \frac{\alpha}{2\sqrt{\beta(m-c^2)}}$, $a_{-1} = \frac{\alpha}{\beta}$
According to Eq.(49) we can obtain

$$v(x,t) = -\frac{\alpha}{2\beta}\left(1 \mp \coth\left[\frac{\alpha}{2\sqrt{\beta(m - c^2)}}(x - ct)\right]\right),\ \beta(m - c^2) > 0$$

Recall that

$$U = arccosh\left(\frac{v + v^{-1}}{2}\right)\ .$$

therefore, we can obtain the solutions

$$u(x,t) = arccosh\left[\frac{1}{2}\left(-\frac{\alpha}{2\beta}\left(1 \mp \tanh\left[\frac{\alpha}{2\sqrt{\beta(m-c^2)}}(x-ct)\right]\right)\right.\right.$$
$$\left.\left. -\frac{2\beta}{\alpha\left(1 \mp \tanh\left[\frac{\alpha}{2\sqrt{\beta(m-c^2)}}(x-ct)\right]\right)}\right)\right],$$

and

$$u(x,t) = arccosh\left[\frac{1}{2}\left(-\frac{\alpha}{2\beta}\left(1 \mp \coth\left[\frac{\alpha}{2\sqrt{\beta(m-c^2)}}(x-ct)\right]\right)\right.\right.$$
$$\left.\left. -\frac{2\beta}{\alpha\left(1 \mp \coth\left[\frac{\alpha}{2\sqrt{\beta(m-c^2)}}(x-ct)\right]\right)}\right)\right].$$

4. Conclusions

The sinh-Gordon type equations appear in wide range of physical applications in branches of nonlinear science. In this work we used the Exp-function method to obtain traveling wave solutions of the sinh-Gordon type equations. In applications of Exp-function method in past decade common errors in finding exact solutions of nonlinear problems have been omitted [25]. In this paper we present an application of this method with tackling these common errors. Also, we do not claim that we have obtained new solutions. This method changes the problem from solving nonlinear partial differential equations to solving a ordinary differential equations by chosen free parameters. The obtained result clarify that the Exp-function method is a very effective and powerful mathematical tool for finding traveling wave solutions of the nonlinear partial differential equations.

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