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A GENERAL ITERATIVE ALGORITHM COMBINING VISCOSITY METHOD WITH PARALLEL METHOD FOR MIXED EQUILIBRIUM PROBLEMS FOR A FAMILY OF STRICT PSEUDO-CONTRACTIONS†

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Abstract. The purpose of this paper is to introduce a general iterative process by viscosity approximation method with parallel method to approximate a common element of the set of solutions of a mixed equilibrium problem and of the set of common fixed points of a finite family of k_i-strict pseudo-contractions in a Hilbert space. We obtain a strong convergence theorem of the proposed iterative method for a finite family of k_i-strict pseudo-contractions to the unique solution of variational inequality which is the optimality condition for a minimization problem under some mild conditions imposed on parameters. The results obtained in this paper improve and extend the corresponding results announced by Liu (2009), Plubtieng-Panpaeng (2007), Takahashi-Takahashi (2007), Peng et al. (2009) and some well-known results in the literature.

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1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively, and C is a nonempty closed convex subset of H. Recall that a mapping $T : C \to H$ is said to be k-strictly pseudo-contractive if there exists a constant $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C, \tag{1}$$

where I is an identity operator. We use $F(T)$ to denote the set of fixed points of T. Note that the class of k-strictly pseudo-contractive includes strictly the class

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of nonexpansive mappings which are mappings T on C such that $\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C$. This is, T is nonexpansive if and only if T is 0-strictly pseudo-contraction. The mapping T is also said to be pseudo-contraction if $k = 1$ and T is said to be strongly pseudo-contraction if there exists a positive constant $\lambda \in (0,1)$ such that $T - \lambda I$ is pseudo-contraction. Clearly, the class of k-strictly pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions. We also remark that the class of strongly pseudo-contractions is independent of the class of k-strictly pseudo-contractions (see [3]).

Let $\varphi : C \to \mathcal{R} \cup \{+\infty\}$ be a proper extended real-valued function and F be a bifunction of $C \times C$ into $\mathcal{R}$, where $\mathcal{R}$ is the set of real numbers. Flores-Bazán [1] considered the following mixed equilibrium problem for finding $x \in C$ such that

$$F(x, y) + \varphi(y) \geq \varphi(x) \text{ for all } y \in C. \tag{2}$$

The set of solutions of (2) is denoted by $MEP(F, \varphi)$. We see that x is a solution of problem (2) implies that $x \in \mathtt{dom}\varphi = \{x \in C \mid \varphi(x) < +\infty\}$. If $\varphi \equiv 0$, then the mixed equilibrium problem (2) becomes the following equilibrium problem: to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all} \quad y \in C. \tag{3}$$

The set of solutions of (3) is denoted by $EP(F)$. Given a mapping $B : C \to H$, let $F(x, y) = \langle Bx, y - x \rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Bz, y - z \rangle \geq 0$ for all $y \in C$. Some methods have been proposed to solve the equilibrium problem (see [2, 7, 8, 10, 11, 18, 19, 23]). The problem (3) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in non-cooperative games and others; see, for instance, [1, 2, 8]. In 2005, Combettes and Hirstoaga [8] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and they also proved a strong convergence theorem. Let $A : C \to H$ be a mapping. The classical variational inequality, denoted by $VI(C, A)$, is to find $x^* \in C$ such that $\langle Ax^*, v - x^* \rangle \geq 0$ for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [2, 9, 12, 25, 29] and the references therein. In 2008, Ceng and Yao [7] considered an iterative scheme for finding a common fixed point of a finite family of nonexpansive mappings and the set of solutions of a problem (2) in Hilbert spaces and obtained the strong convergence theorem. Let $K : C \to \mathcal{R}$ be a differentiable functional on a convex set C, which used the following condition (see [7]):

(H) $K : C \to \mathcal{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology.

Their results extend and improve the corresponding results in [8, 20]. We note that the condition (H) for the function $K : C \to \mathcal{R}$ is a very strong condition. We also note that the condition (H) does not cover the case $K(x) = \frac{\|x\|^2}{2}$ and $\eta(x, y) = x - y$ for each $(x, y) \in C \times C$. Motivated by Ceng and Yao [7], Peng and Yao [23] introduced a new iterative scheme based on only the extragradient method for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a family of finitely nonexpansive mappings and the set of the variational inequality for a monotone Lipschitz continuous mapping. They obtained a strong convergence theorem without the condition (H) for the sequences generated by these processes.

A mapping A of C into H is called α-inverse-strongly monotone [5] if there exists a positive real number α such that $\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$ for all $u, v \in C$. Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2 \quad \text{for all} \quad x \in H. \tag{4}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle, \tag{5}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H.

In 2007, S. Takahashi and W. Takahashi [18] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution (3) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Let $S : C \to C$ be a nonexpansive mapping. Starting with arbitrary initial $x_1 \in H$ and $u_n \in C$ define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{aligned} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad \forall n \in \mathbf{N}. \end{aligned} \tag{6}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)}f(z)$. Later, Plubtieng and Punpaeng [16] introduced an iterative scheme by the general iterative method for finding a common element of the set of solution (3) and the set of fixed points of a nonexpansive mapping in Hilbert space. Let $S : H \to H$ be a nonexpansive mapping. Starting with an arbitrary $x_1 \in H$ and $u_n \in C$ define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{aligned} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)Su_n, \quad \forall n \in \mathbf{N}, \end{aligned} \tag{7}$$

where A is strong positive bounded linear operators. They proved that if the sequences $\{\alpha_n\}$ and $\{r_n\}$ of parameters satisfies appropriate conditions, then $\{x_n\}$ generate by (7) converges strongly to the unique solution of variational inequality $\langle (A-\gamma f)z, x-z\rangle \geq 0, \quad \forall x \in F(S)\cap EP(F)$, which is the optimality condition for the minimization problem

$$\min_{x\in F(S)\cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{8}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

For finding a common element of the set of fixed points of a k-strictly pseudo-contraction and the set of solutions of an equilibrium problem in a real Hilbert space, very recently, by idea of Plubtieng-Punpaeng [16], Liu [14] introduced the following iterative scheme:

$$\begin{aligned} &F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in E,\\ &y_n = \beta_n u_n + (1-\beta_n)Su_n,\\ &x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A)y_n, \ \ \forall n \geq 1, \end{aligned} \tag{9}$$

where S is a k-strictly pseudo-contraction and $\{\epsilon_n\}$, $\{\beta_n\}$ are sequences in [0, 1]. They proved that under certain appropriate conditions over $\{\epsilon_n\}$, $\{\beta_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ both converge strongly to some $q \in F(S)\cap EP(F)$, which solves some variational inequality problems.

Very recently, Ceng et al. [6] and Peng et al. [22] established an iterative scheme for finding a common element of the set of solution of equilibrium problems (generalized and mixed equilibrium problems) and the set of fixed point of a k-strictly pseudo-contraction in the setting of a real Hilbert space. They also studied some weak and strong convergence theorem for k-strictly pseudo-contraction of the sequence generated by their algorithm under the control conditions. Many authors studied the problem to finding a common element of the set of fixed points and the set of solutions of an equilibrium problem for strictly pseudo-contractions in the frame work of Hilbert spaces; see, for instance, [6, 13, 14, 21, 22, 24, 25] and the references therein.

Motivated by Peng et al. [22], Plubtieng-Punpaeng [16] and Takahashi-Takahashi [19], we introduce an iterative scheme by combining the viscosity method with parallel method for finding a common element of the set of solution (2) and the set of fixed points of a finite family of strictly pseudo contractive mappings in a Hilbert space. Moreover, our results include Liu [14], Plubtieng-Panpaeng [16], Takahashi-Takahashi [18], Takahashi-Takahashi [19], Peng et al. [22] and some others as some special cases.

2. Preliminary

Let C be closed convex subset of a Hilbert space H, let P_C be the metric projection of H onto C i.e., for $x \in H$, $P_C x$ satisfies the property $\|x - P_C x\| =$

$\min_{y\in C}\|x-y\|$. It is well known that P_C is a nonexpansive mapping. Moreover, P_Cx is characterized by the following properties: $P_Cx \in C$ and

$$\langle x - P_Cx, y - P_Cx\rangle \le 0, \tag{10}$$

$$\|x-y\|^2 \ge \|x-P_Cx\|^2 + \|y-P_Cx\|^2 \quad \text{for all } x \in H, y \in C. \tag{11}$$

In the context of the variational inequality problem, this implies that

$$u \in VI(C,A) \Leftrightarrow u = P_C(u-\lambda Au), \text{ for all } \lambda > 0. \tag{12}$$

Let H be a real Hilbert space. Then for any $x, y \in H$, we have the following:
(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$
(ii) $\|x+y\|^2 \ge \|x\|^2 + 2\langle y, x\rangle$
(iii) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,\ \forall t \in [0,1]$.

It is also known that H satisfies the *Opial's condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty}\|x_n-x\| < \liminf_{n\to\infty}\|x_n-y\|$ holds for every $y \in H$ with $y \neq x$.

In order to prove our main results, we need the following lemmas.

Lemma 1. [30] *Let $T : C \to H$ be a k-strictly pseudo-contraction. Defined $S : C \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for each $x \in K$. Then, as $\lambda \in [k,1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.*

Lemma 2. [26] *Let $\{a_n\}$ be a sequence of nonnegative real numbers, satisfying the property, $a_{n+1} \le (1-\gamma_n)a_n + b_n,\ n \ge 0$, where $\{\gamma_n\} \subset (0,1)$, and $\{b_n\}$ is a sequence in $\mathcal{R}$ such that: (C1) $\Sigma_{n=1}^{\infty}\gamma_n = \infty$; (C2) $\limsup_{n\to\infty}\frac{b_n}{\gamma_n} \le 0$ or $\Sigma_{n=1}^{\infty}|b_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 3. [17] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 4. [15] *Let H be a Hilbert space. Let A be a strongly positive linear bounded selfadjoint operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $T : H \to H$ be a nonexpansive mapping with a fixed point x_t of the contraction $x \mapsto t\gamma f(x) + (1-tA)Tx$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point x^* os T, which solve the variational inequality $\langle (A-\gamma f)x^*, z - x^*\rangle \ge 0, \quad \forall z \in F(T)$.*

Lemma 5. [15] *Let H be a Hilbert space, C be a nonempty closed convex subset of H, and $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$, and A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Then, for $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$, $\langle x - y, (A-\gamma f)x - A(A-\gamma f)y\rangle \ge (\overline{\gamma} - \gamma\alpha)\|x-y\|^2, \quad x, y \in H$. That is, $A - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma\alpha$.*

Lemma 6. [15] *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.*

Lemma 7. [28] *Let H be a Hilbert space, C a nonempty closed convex subset of H. For any integer $N \geq 1$, assume that, for each $1 \leq i \leq N$, $T_i : C \to H$ be k_i-strictly pseudo-contractions for some $0 \leq k_i < 1$. Assume that $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\Sigma_{i=1}^N \eta_i = 1$. Then $\Sigma_{i=1}^N \eta_i T_i$ is a non-self-k-strictly pseudo-contraction with $k = \max\{k_i : 1 \leq i \leq N\}$.*

Lemma 8. [28] *Let $\{T_i\}_{i=1}^N$ and $\{\eta_i\}_{i=1}^N$ be given as in Lemma 7. Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point in C. Then $F(\Sigma_{i=1}^N \eta_i T_i) = \cap_{i=1}^{\infty} F(T_i)$.*

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, φ and the set C:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $y \in C, x \mapsto F(x, y)$ is weakly upper semicontinuous.
(A4) for each $x \in C, y \mapsto F(x, y)$ is convex;
(A5) for each $x \in C, y \mapsto F(x, y)$ is lower semicontinuous;

We need the following two conditions for the following lemma (see [22, 23] for more details):

(B1) for each $x \in H$ and $r > 0$, there exist abounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z);$$

(B2) C is a bounded set.

Lemma 9. ([21, 23]; see also [22, 24]) *Let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathcal{R}$ be a bifunction satisfies (A1)-(A4) and let $\varphi : C \to \mathcal{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C \quad : \quad F(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \quad \forall y \in C\}$$

for all $z \in H$. Assume that either (B1) or (B2) holds. Then, the following conclusions hold:

(1) *For each $x \in H, T_r(x) \neq \emptyset$;*
(2) *T_r is single-valued;*
(3) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
(4) *$F(T_r) = MEP(F, \varphi)$;*
(5) *$MEP(F, \varphi)$ is closed and convex.*

Remark We note that Lemma 9 is not a consequence of Lemma 3.1 in [1] because the condition of the sequential continuity from the weak topology to the strong topology for the derivative K' of the function $K : C \to \mathcal{R}$ does not cover the case $K(x) = \frac{\|x\|^2}{2}$.

3. Strong convergence theorem

In this section, we prove a strong convergence theorem of the iterative scheme (13) below to a common element of $MEP(F,\varphi)$ and $\bigcap_{i=1}^N F(T_i)$ for a finite family of k_i-strictly pseudo-contractions in the framework of Hilbert spaces.

Theorem 1. *Let H be a real Hilbert space, C a nonempty closed convex subset of H. Let $F : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)-(A5) and let $\varphi : C \to \mathcal{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap dom\varphi \neq \emptyset$. Let $T_i : C \to C$ be a k_i-strict pseudo-contraction for some $0 \leq k_i < 1$ such that $\Theta := \bigcap_{i=1}^N F(T_i) \cap MEP(F,\varphi) \neq \emptyset$ and let f be a contraction of H into itself with coefficient $\alpha \in (0,1)$. Assume that for each n, $\{\eta_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive number such that $\Sigma_{i=1}^N \eta_i^{(n)} = 1$ for all n and $\eta_i^{(n)} > 0$ for all $1 \leq i < N$. Let $k = \max\{k_i : 1 \leq i \leq N\}$. Assume that either (B1) or (B2) holds. Let A be a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Starting with an arbitrary $x_1 \in H, u_n \in C$ and define the sequences $\{x_n\}$ and $\{u_n\}$ by*

$$\begin{aligned} &F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ &y_n = \gamma_n u_n + (1-\gamma_n)\Sigma_{i=1}^N \eta_i^{(n)} T_i u_n \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) y_n, \end{aligned} \tag{13}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$. If the sequences $\{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^\infty \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(C3) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(C4) $\lim_{n\to\infty} |\eta_i^{(n+1)} - \eta_i^{(n)}| = 0$, *for all* $i = 1,2,3,...,N$,

(C5) $k \leq a < \gamma_n < b \leq 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$, *for some* $a, b \in \mathcal{R}$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to z, where $z = P_\Theta(I - A + \gamma f)z$, which solves the unique solution of the variational inequalities (14), i.e.,

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in \Theta, \tag{14}$$

which is the optimality condition for the minimization problem (8).

Proof. Note that by Lemma 9, u_n can be rewritten as $u_n = T_{r_n} x_n$ for each $n \in \mathbf{N}$. Let $p \in \Theta$, then $p = T_{r_n} p$. For any $n \in \mathbf{N}$, by nonexpansiveness of T_{r_n}, we have

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\|.$$

From the condition $\lim_{n\to\infty} \alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$. Since A is a strongly positive bounded linear operator on H, then $\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}$. Observe that

$$\langle ((1-\beta_n)I - \alpha_n A)x, x\rangle = 1 - \beta_n - \alpha_n \langle Ax, x\rangle \geq 1 - \beta_n - \alpha_n \|A\| \geq 0,$$

that is to say $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\alpha_nA\| &= \sup\{\langle((1-\beta_n)I-\alpha_nA)x,x\rangle : x\in H, \|x\|=1\}\\ &= \sup\{1-\beta_n-\alpha_n\langle Ax,x\rangle : x\in H, \|x\|=1\}\\ &\le 1-\beta_n-\alpha_n\overline{\gamma}.\end{aligned}$$

We now show that $\{x_n\}$ is bounded. Indeed pick any $p\in\Theta$, we define a mapping S_n by

$$S_nx=\Sigma_{i=1}^N\eta_i^{(n)}T_ix,\ \forall x\in C.$$

From Lemma 7, each S_n is a k-strict pseudo-contraction on C and by Lemma 8, $F(S_n)=\cap_{i=1}^N F(T_i)$. It follows that

$$\begin{aligned}\|y_n-p\|^2 &= \|\gamma_nu_n+(1-\gamma_n)S_nu_n-p\|^2\\ &= \|\gamma_n(u_n-p)+(1-\gamma_n)(S_nu_n-p)\|^2\\ &= \gamma_n\|u_n-p\|^2+(1-\gamma_n)\|S_nu_n-p\|^2-\gamma_n(1-\gamma_n)\|u_n-S_nu_n\|^2\\ &\le \gamma_n\|u_n-p\|^2+(1-\gamma_n)[\|u_n-p\|^2+k\|u_n-S_nu_n\|^2]\\ &\quad -\gamma_n(1-\gamma_n)\|u_n-S_nu_n\|^2\\ &= \|u_n-p\|^2+(1-\gamma_n)(k-\gamma_n)\|u_n-S_nu_n\|^2\le\|u_n-p\|^2,\end{aligned}$$

it follows that $\|y_n-p\|\le\|u_n-p\|$. We observe that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)y_n-p\|\\ &= \|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_nA)(y_n-p)\|\\ &\le \|\alpha_n(\gamma f(x_n)-Ap)\|+\beta_n\|x_n-p\|\\ &\quad +\|((1-\beta_n)I-\alpha_nA)\|\|y_n-p\|\\ &\le \|\alpha_n(\gamma f(x_n)-\gamma f(p)+\gamma f(p)-Ap)\|+\beta_n\|x_n-p\|\\ &\quad +(1-\beta_n-\alpha_n\overline{\gamma})\|\|u_n-p\|\\ &\le \alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|+\beta_n\|x_n-p\|\\ &\quad +(1-\beta_n-\alpha_n\overline{\gamma})\|\|x_n-p\|\\ &= (1-\alpha_n(\overline{\gamma}-\gamma\alpha))\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|\\ &= (1-\alpha_n(\overline{\gamma}-\gamma\alpha))\|x_n-p\|+\alpha_n(\overline{\gamma}-\gamma\alpha)\frac{\|\gamma f(p)-Ap\|}{(\overline{\gamma}-\gamma\alpha)}.\end{aligned}$$

By induction that $\|x_n-p\|\le\max\{\|x_1-p\|,\frac{\|\gamma f(p)-Ap\|}{(\overline{\gamma}-\gamma\alpha)}\},n\ge 0$, and hence $\{x_n\}$ is bounded. We also obtain that $\{u_n\}$, $\{f(x_n)\}$ and $\{y_n\}$ are also bounded. Define the mapping $V_n:C\to C$ by $V_n=\gamma_nI+(1-\gamma_n)S_n$, for any $x,y\in C$, we have

$$
\begin{aligned}
\|V_nx - V_ny\|^2 &= \|\gamma_n x + (1-\gamma_n)S_nx - (\gamma_n y + (1-\gamma_n)S_ny)\|^2 \\
&= \gamma_n\|x-y\|^2 + (1-\gamma_n)\|S_nx - S_ny\|^2 \\
&\quad -\gamma_n(1-\gamma_n)\|(I-S_n)x - (I-S_n)y\|^2 \\
&\le \gamma_n\|x-y\|^2 + (1-\gamma_n)[\|x-y\|^2 \\
&\quad +k\|(I-S_n)x - (I-S_n)y\|^2] \\
&\quad -\gamma_n(1-\gamma_n)\|(I-S_n)x - (I-S_n)y\|^2 \\
&= \|x-y\|^2 + (1-\gamma_n)(k-\gamma_n)\|(I-S_n)x - (I-S_n)y\|^2 \\
&\le \|x-y\|^2,
\end{aligned}
$$

which implies that V_n is nonexpansive. We compute

$$
\begin{aligned}
\|y_{n+1} - y_n\| &= \|V_{n+1}u_{n+1} - V_nu_n\| \\
&\le \|V_{n+1}u_{n+1} - V_{n+1}u_n\| + \|V_{n+1}u_n - V_nu_n\| \\
&\le \|u_{n+1} - u_n\| + \|\gamma_{n+1}u_n + (1-\gamma_{n+1})S_{n+1}u_n \\
&\quad -(\gamma_nu_n + (1-\gamma_n)S_nu_n)\| \\
&\le \|u_{n+1} - u_n\| + \|\gamma_{n+1}u_n + (1-\gamma_{n+1})S_{n+1}u_n \\
&\quad -(1-\gamma_{n+1})S_nu_n + (1-\gamma_{n+1})S_nu_n \\
&\quad -(\gamma_nu_n + (1-\gamma_n)S_nu_n)\| \\
&\le \|u_{n+1} - u_n\| + \|(\gamma_{n+1} - \gamma_n)u_n \\
&\quad +[(1-\gamma_{n+1}) - (1-\gamma_n)]S_nu_n\| \\
&\quad +\|(1-\gamma_{n+1})(S_{n+1}u_n - S_nu_n)\| \\
&\le \|u_{n+1} - u_n\| + |\gamma_{n+1} - \gamma_n|\|u_n - S_nu_n\| \\
&\quad +(1-\gamma_{n+1})\|S_{n+1}u_n - S_nu_n\| \\
&\le \|u_{n+1} - u_n\| + |\gamma_{n+1} - \gamma_n|M_1 \\
&\quad +(1-\gamma_{n+1})\Sigma_{i=1}^N|\eta_i^{(n+1)} - \eta_i^{(n)}|\|T_iu_n\| \qquad (15)
\end{aligned}
$$

where $M_1 = \sup\{\|u_n - S_nu_n\| : n \in \mathbf{N}\}$. Observing that $u_n = T_{r_n}x_n \in \text{dom}\,\varphi$ and $u_{n+1} = T_{r_{n+1}}x_{n+1} \in \text{dom}\,\varphi$, we get

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \ \forall y \in C, \qquad (16)$$

$$F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0, \ \forall y \in C. \qquad (17)$$

Take $y = u_{n+1}$ in (16) and $y = u_n$ in (17), by using condition (A2), we obtain

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \quad \ge \quad 0.$$

Thus $\langle u_{n+1} - u_n, u_n - u_{n+1} + x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \ge 0$. Without loss of generality, let us assume that there exists a real number c such that

$r_n > c, \forall n \geq 1$. Then, we have

$$\|u_{n+1} - u_n\|^2 \leq \|u_{n+1} - u_n\| \Big\{ \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}| \|u_{n+1} - x_{n+1}\| \Big\}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c} |r_{n+1} - r_n| M_2, \end{aligned} \tag{18}$$

where $M_2 = \sup\{\|u_n - x_n\| : n \in \mathbf{N}\}$. Substituting (18) into (15), we arrive at

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + d_n \tag{19}$$

where $d_n := \frac{1}{c}|r_{n+1} - r_n| M_2 + |\gamma_{n+1} - \gamma_n| M_1 + (1 - \gamma_{n+1}) \Sigma_{i=1}^N |\eta_i^{(n+1)} - \eta_i^{(n)}| \|T_i u_n\|$. Next, we show that $\|x_{n+1} - x_n\| \to 0$. Define the sequence $\{w_n\}$ such that

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) w_n, \;\; n \geq 0.$$

Observe that from the definition of w_n we obtain

$$\begin{aligned} w_{n+1} - w_n &= \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1} \gamma f(x_{n+1}) + ((1 - \beta_{n+1}) I - \alpha_{n+1} A) y_{n+1}}{1 - \beta_{n+1}} \\ &\quad - \frac{\alpha_n \gamma f(x_n) + ((1 - \beta_n) I - \alpha_n A) y_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\gamma f(x_{n+1}) - A y_{n+1}) \\ &\quad + \frac{\alpha_n}{1 - \beta_n} (A y_n - \gamma f(x_n)) + y_{n+1} - y_n. \end{aligned}$$

Thus,

$$\begin{aligned} \|w_{n+1} - w_n\| - \|x_n - x_{n+1}\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|\gamma f(x_{n+1}) - A y_{n+1}\| \\ &\quad + \frac{\alpha_n}{1 - \beta_n} \|A y_n - \gamma f(x_n)\| + \|y_{n+1} - y_n\| \\ &\quad - \|x_n - x_{n+1}\| \\ &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|\gamma f(x_{n+1}) - A v_{n+1}\| \\ &\quad + \frac{\alpha_n}{1 - \beta_n} \|A v_n - \gamma f(x_n)\| + d_n. \end{aligned}$$

By the conditions (C1)-(C5) and taking the limit superior that

$$\limsup_{n \to \infty} (\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{20}$$

From $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} < 1$, Lemma 3 and (20), we have

$$\lim_{n \to \infty} \|w_n - x_n\| = 0. \tag{21}$$

Note that $\|x_{n+1} - x_n\| = \|(1-\beta_n)w_n + \beta_n x_n - x_n\| = (1-\beta_n)\|w_n - x_n\|$, by (21), we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \tag{22}$$

applying (C2)-(C5) in (18) and (19), we obtain $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$ and $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$.
Next, we show that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. For any $p \in \Theta$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n} x_n - T_{r_n} p\|^2 \le \langle T_{r_n} x_n - T_{r_n} p, x_n - p\rangle = \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2). \end{aligned}$$

It follow that $\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2$. Therefore, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)y_n - p\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) \\ &\quad + ((1-\beta_n)I - \alpha_n A)(y_n - p)\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - Ap\|^2 + \beta_n\|x_n - p\|^2 \\ &\quad + (1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - p\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - Ap\|^2 + \beta_n\|x_n - p\|^2 \\ &\quad + (1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - p\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - Ap\|^2 + \beta_n\|x_n - p\|^2 \\ &\quad + (1 - \beta_n - \alpha_n\bar{\gamma})(\|x_n - p\|^2 - \|x_n - u_n\|^2) \\ &= \alpha_n\|\gamma f(x_n) - Ap\|^2 + (1 - \alpha_n\bar{\gamma})\|x_n - p\|^2 \\ &\quad - (1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - x_n\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} (1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - x_n\|^2 &\le \alpha_n\|\gamma f(x_n) - Ap\|^2 \\ &\quad + (1 - \alpha_n\bar{\gamma})\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - Ap\|^2 \\ &\quad + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|). \end{aligned}$$

By (C1), (C2) and (22), imply that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{23}$$

Since $\liminf_{n\to\infty} r_n > 0$, we have $\lim_{n\to\infty} \|\frac{x_n - u_n}{r_n}\| = \lim_{n\to\infty} \frac{1}{r_n}\|x_n - u_n\| = 0$.

Next, we prove that $\lim_{n\to\infty} \|S_n u_n - u_n\| = 0$. We consider

$$\begin{aligned} \|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)y_n - y_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - Ay_n\| + \beta_n\|x_n - y_n\|, \end{aligned}$$

it follows that $(1-\beta_n)\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - Ay_n\|$ from (C1), (C2) and (22), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{24}$$

We note that

$$\|y_n - u_n\| \le \|y_n - x_n\| + \|x_n - u_n\| \to 0 \text{ as } n \to \infty. \tag{25}$$

Then, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\gamma_n u_n + (1-\gamma_n)S_n u_n - p\|^2 \\
&= \|\gamma_n(u_n - p) + (1-\gamma_n)(S_n u_n - p)\|^2 \\
&= \gamma_n\|u_n - p\|^2 + (1-\gamma_n)\|S_n u_n - p\|^2 - \gamma_n(1-\gamma_n)\|u_n - S_n u_n\|^2 \\
&\le \gamma_n\|u_n - p\|^2 + (1-\gamma_n)[\|u_n - p\|^2 + k\|u_n - S_n u_n\|^2] \\
&\quad -\gamma_n(1-\gamma_n)\|u_n - S_n u_n\|^2 \\
&\le \|u_n - p\|^2 + (1-\gamma_n)(k-\gamma_n)\|u_n - S_n u_n\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
(1-\gamma_n)(\gamma_n - k)\|u_n - S_n u_n\|^2 &\le \|u_n - p\|^2 - \|y_n - p\|^2 \\
&\le \|u_n - y_n\|(\|u_n - p\| + \|y_n - p\|)
\end{aligned}$$

hence from (C5) and (25), we obtain that

$$\lim_{n\to\infty} \|S_n u_n - u_n\| = 0. \tag{26}$$

Next, we show that

$$\limsup_{n\to\infty} \langle (A-\gamma f)z, z - x_n\rangle \le 0, \tag{27}$$

where $z = P_\Theta(I - A + \gamma f)z$, is a unique solution of the variational inequality (14). We can choose a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\lim_{k\to\infty} \langle (A-\gamma f)z, z - u_{n_k}\rangle = \limsup_{n\to\infty} \langle (A-\gamma f)z, z - u_n\rangle. \tag{28}$$

Since $\{u_{n_k}\}$ is bounded, there exists a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ such that $u_{n_{k_j}} \rightharpoonup w$. Without loss of generality, we can assume that $u_{n_k} \rightharpoonup w$. Since C is closed and convex, $w \in C$. We first show that $w \in \cap_{i=1}^N F(T_i)$. To see that we observe that we may assume that $\eta_i^{(n_k)} \to \eta_i$ (as $k \to \infty$) for $i = 1, 2, 3, ..., N$. It is easy to see that $\eta_i > 0$ and $\Sigma_{i=1}^N \eta_i = 1$. We also have

$$S_{n_k} x \to Sx \;\; (\text{ as } k \to \infty) \;\forall x \in C, \tag{29}$$

where $S = \Sigma_{i=1}^N \eta_i T_i$. From Lemma 7, S is k-strictly pseudo-contraction and from Lemma 8, $F(S) = \cap_{i=1}^N F(T_i)$. Since

$$\begin{aligned}
\|u_{n_k} - Su_{n_k}\| &\le \|u_{n_k} - S_{n_k} u_{n_k}\| + \|S_{n_k} u_{n_k} - Su_{n_k}\| \\
&\le \|u_{n_k} - S_{n_k} u_{n_k}\| + \Sigma_{i=1}^N |\eta_i^{(n_k)} - \eta_i| \|T_i u_{n_k}\|,
\end{aligned}$$

it follows from (26) and $\eta_i^{(n_k)} \to \eta_i$ (as $k \to \infty$) that

$$\lim_{k\to\infty} \|u_{n_k} - Su_{n_k}\| = 0. \tag{30}$$

Thus, we get $Su_{n_k} \rightharpoonup w$. Now, we show that $w \in MEP(F, \varphi)$, Since $u_n = T_{r_n} x_n \in$ dom φ and (13) it follows from (A2), we also have $\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in C$, and hence

$$\varphi(y) - \varphi(u_n) + \langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\rangle \geq F(y, u_{n_k}), \quad \forall y \in C.$$

Since $\frac{u_{n_k} - x_{n_k}}{r_{n_k}} \to 0$ and $u_{n_k} \rightharpoonup w$, it follows by (A4), (A5) and the weakly lower semicontinuity of φ that

$$F(y, w) + \varphi(w) - \varphi(y) \leq 0, \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $F(y_t, w) + \varphi(w) - \varphi(y_t) \leq 0$. So, from (A1), (A4) and the convexity of φ, we have

$$\begin{aligned} 0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &\leq t(F(y_t, y) + \varphi(y) - \varphi(y_t)). \end{aligned}$$

Dividing by t, we get $F(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0$. From (A3) and the weakly lower semicontinuity of φ, we have $F(w, y) + \varphi(y) - \varphi(w) \geq 0$ for all $y \in C \cap dom\varphi$ and hence $w \in MEP(F, \varphi)$. Next, we show that $w \in F(S) = \bigcap_{i=1}^N F(T_i)$. We defined $H : C \to C$ by $Hx = kx + (1-k)Sx$ for all $x \in C$. It is clear that H is nonexpansive and from (30) we obtain

$$\|u_{n_k} - Hu_{n_k}\| = \|u_{n_k} - ku_{n_k} - (1-k)Su_{n_k}\| = (1-k)\|u_{n_k} - Su_{n_k}\| \to 0$$

as $k \to \infty$. From Lemma 1, we have $F(H) = F(S) = \bigcap_{i=1}^N F(T_i)$. We can show that $w \in F(H)$. Assume that $w \neq Hw$. From Opial's condition and $\|Hu_{n_k} - u_{n_k}\| \to 0$, we have

$$\begin{aligned} \liminf_{k\to\infty} \|u_{n_k} - w\| &< \liminf_{k\to\infty} \|u_{n_k} - Hw\| \\ &= \liminf_{k\to\infty} \|(u_{n_k} - Hu_{n_k}) + (Hu_{n_k} - Hw)\| \\ &= \liminf_{k\to\infty} \|Hu_{n_k} - Hw\| \leq \liminf_{k\to\infty} \|u_{n_k} - w\|. \end{aligned}$$

This is a contradiction. So, we have $w \in F(S)$. Therefore $w \in \Theta$. It follows that $\limsup_{n\to\infty}\langle (A-\gamma f)z, z-x_n\rangle = \limsup_{n\to\infty}\langle (A-\gamma f)z, z-u_n\rangle = \lim_{k\to\infty}\langle (A-\gamma f)z, z - u_{n_k}\rangle = \langle (A-\gamma f)z, z-w\rangle \leq 0$, as required. Finally, we prove that $x_n \to z$, where $z = P_\Theta(I - A + \gamma f)z$. From bounded of $\{x_n\}$ and $\{u_n\}$, we set

$M \geq \|\gamma f(x_n) - z\|^2 + \|T_n u_n - z\|\|\gamma f(x_n) - Az\|$. We note that

$$\begin{aligned}
& \|x_{n+1} - z\|^2 \\
= \;& \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)y_n - z\|^2 \\
= \;& \|\beta_n(x_n - z) + ((1-\beta_n)I - \alpha_n A)(y_n - z) + \alpha_n(\gamma f(x_n) - Az)\|^2 \\
\leq \;& \|\beta_n(x_n - z) + ((1-\beta_n)I - \alpha_n A)(y_n - z)\|^2 \\
& +2\alpha_n\langle \gamma f(x_n) - Az, x_{n+1} - z\rangle \\
\leq \;& [\beta_n\|x_n - z\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|y_n - z\|]^2 \\
& +2\alpha_n\langle \gamma f(x_n) - \gamma f(z), x_{n+1} - z\rangle + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle \\
\leq \;& [\beta_n\|x_n - z\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|u_n - z\|]^2 + 2\alpha_n\gamma\alpha\|x_n - z\|\|x_{n+1} - z\| \\
& +2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle \\
\leq \;& [\beta_n\|x_n - z\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - z\|]^2 \\
& +\alpha_n\gamma\alpha(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
& +2\alpha_n\langle \gamma f(x_n) - Az, x_{n+1} - z\rangle \\
= \;& (1 - \alpha_n\overline{\gamma})^2\|x_n - z\|^2 + \alpha_n\gamma\alpha(\|x_n - z\|^2 \\
& +\|x_{n+1} - z\|^2) + 2\alpha_n\langle \gamma f(x_n) - Az, x_{n+1} - z\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
(1 - \alpha_n\gamma\alpha)\|x_{n+1} - z\|^2 \;\leq\;& ((1 - \alpha_n\overline{\gamma})^2 + \alpha_n\gamma\alpha)\|x_n - z\|^2 \\
& +2\alpha_n\langle \gamma f(x_n) - Az, x_{n+1} - z\rangle,
\end{aligned}$$

and hence

$$\begin{aligned}
& \|x_{n+1} - z\|^2 \\
\leq \;& \frac{(1 - 2\alpha_n\overline{\gamma} + \alpha_n^2\overline{\gamma}^2 + \alpha_n\gamma\alpha)}{(1 - \alpha_n\gamma\alpha)}\|x_n - z\|^2 \\
& +\frac{2\alpha_n}{(1 - \alpha_n\gamma\alpha)}\langle \gamma f(x_n) - Az, x_{n+1} - z\rangle \\
\leq \;& \Big(1 - \frac{(2\alpha_n(\overline{\gamma} - \alpha\gamma)}{(1 - \alpha_n\gamma\alpha)} + \frac{\alpha_n^2\overline{\gamma}^2}{(1 - \alpha_n\gamma\alpha)}\Big)\|x_n - z\|^2 \\
& +\frac{2\alpha_n}{(1 - \alpha_n\gamma\alpha)}\langle \gamma f(x_n) - Az, x_{n+1} - z\rangle \\
\leq \;& \Big(1 - \frac{(2\alpha_n(\overline{\gamma} - \alpha\gamma))}{(1 - \alpha_n\gamma\alpha)}\Big)\|x_n - z\|^2 + \frac{\alpha_n^2\overline{\gamma}^2}{(1 - \alpha_n\gamma\alpha)}\|x_n - z\|^2 \\
& +\frac{2\alpha_n}{(1 - \alpha_n\gamma\alpha)}\langle \gamma f(x_n) - \gamma f(z), x_{n+1} - z\rangle \\
& +\frac{2\alpha_n}{(1 - \alpha_n\gamma\alpha)}\langle \gamma f(z) - Az, x_{n+1} - z\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big(1-\frac{(2\alpha_n(\overline{\gamma}-\alpha\gamma))}{(1-\alpha_n\gamma\alpha)}\Big)\|x_n-z\|^2+\frac{\alpha_n^2\overline{\gamma}^2}{(1-\alpha_n\gamma\alpha)}\|x_n-z\|^2\\
&\quad+\frac{2\alpha\gamma\alpha_n}{(1-\alpha_n\gamma\alpha)}\|x_n-z\|\|x_{n+1}-z\|+\frac{2\alpha_n}{(1-\alpha_n\gamma\alpha)}\langle\gamma f(z)-Az,x_{n+1}-z\rangle\\
&= (1-\gamma_n)\|x_n-z\|^2+\delta_n
\end{aligned}
$$

where $\gamma_n=\frac{(2\alpha_n(\overline{\gamma}-\alpha\gamma))}{(1-\alpha_n\gamma\alpha)}$ and $\delta_n=\frac{\alpha_n^2\overline{\gamma}^2}{(1-\alpha_n\gamma\alpha)}\|x_n-z\|^2+\frac{2\alpha\gamma\alpha_n}{(1-\alpha_n\gamma\alpha)}\|x_n-z\|\|x_{n+1}-z\|+\frac{2\alpha_n}{(1-\alpha_n\gamma\alpha)}\langle\gamma f(z)-Az,x_{n+1}-z\rangle$. From (C1), then $\Sigma_{n=1}^{\infty}\gamma_n=\infty$ and by (27), we obtain $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}\leq 0$. Hence, by Lemma 2, the sequence $\{x_n\}$ converges strongly to z. Moreover, since $\|x_n-u_n\|\to 0$, we also have $u_n\to z$. The proof is complete.

Corollary 1. [22, Theorem 3.1] *Let H be a real Hilbert space, C a nonempty closed convex subset of H. Let $F: C\times C\to\mathcal{R}$ be a bifunction satisfying (A1)-(A5) and let $\varphi: C\to\mathcal{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function such that $C\cap dom\varphi\neq\emptyset$. Let $T_i: C\to C$ be a k_i-strictly pseudo-contraction for some $0\leq k_i<1$ such that $\Theta:=\bigcap_{i=1}^N F(T_i)\cap MEP(F,\varphi)\neq\emptyset$ and let f be a contraction of H into itself with coefficient $\alpha\in(0,1)$. Assume that for each n, $\{\eta_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive number such that $\Sigma_{i=1}^N\eta_i^{(n)}=1$ for all n and $\eta_i^{(n)}>0$ for all $1\leq i<N$. Let $k=\max\{k_i:1\leq i\leq N\}$. Assume that either (B1) or (B2) holds. Starting with an arbitrary $x_1\in H, u_n\in C$ and define the sequences $\{x_n\}$ and $\{u_n\}$ by*

$$
\begin{aligned}
&F(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\ \forall y\in C\\
&y_n=\gamma_n u_n+(1-\gamma_n)\Sigma_{i=1}^N\eta_i^{(n)}T_iu_n\\
&x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+(1-\beta_n-\alpha_n)y_n, \qquad (31)
\end{aligned}
$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$. If the sequences $\{\alpha_n\},\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ satisfies the conditions (C1)-(C5) in Theorem 1. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to z, where $z=P_\Theta(f)z$.

Proof. Taking $A\equiv I$ and $\gamma\equiv 1$. By Theorem 1, the sequence $\{x_n\}$ converges strongly to $z=P_\Theta(f)z$.

Corollary 2. [22, Theorem 3.2] *Let H be a real Hilbert space, C a nonempty closed convex subset of H. Let $F: C\times C\to\mathcal{R}$ be a bifunction satisfying (A1)-(A5) and let $\varphi: C\to\mathcal{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function such that $C\cap dom\varphi\neq\emptyset$. Let $T_i: C\to C$ be a k_i-strictly pseudo-contraction for some $0\leq k_i<1$ such that $\Theta:=\bigcap_{i=1}^N F(T_i)\cap MEP(F,\varphi)\neq\emptyset$. Assume that for each n, $\{\eta_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive number such that $\Sigma_{i=1}^N\eta_i^{(n)}=1$ for all n and $\eta_i^{(n)}>0$ for all $1\leq i<N$. Let $k=\max\{k_i:1\leq i\leq N\}$. Assume that either (B1) or (B2) holds. Starting with an arbitrary*

$x_1 = u \in H, u_n \in C$ and define the sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{aligned} &F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ &y_n = \gamma_n u_n + (1-\gamma_n)\Sigma_{i=1}^N \eta_i^{(n)} T_i u_n \\ &x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \beta_n - \alpha_n) y_n, \end{aligned} \tag{32}$$

where $\{\alpha_n\}, \{\beta_n\}$,
$\{\gamma_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. If the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{r_n\}$ satisfies the conditions (C1)-C(5) in Theorem 1. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to z, where $z = P_\Theta u$.

Proof. If setting $f(x_n) \equiv u$ for all $x \in C$, by Theorem 1, we obtain that the desired result.

Theorem 2. *Let H be a real Hilbert space, C a nonempty closed convex subset of H. Let $F : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)-(A5) and let $\varphi : C \to \mathcal{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap dom\varphi \neq \emptyset$. Let $T : C \to C$ be a k-strictly pseudo-contraction for some $0 \leq k < 1$ such that $\Theta := F(T) \cap EP(F, \varphi) \neq \emptyset$ and let f be a contraction of H into itself with coefficient $\alpha \in (0, 1)$. Assume that either (B1) or (B2) holds. Let A be a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1 \in H, u_n \in C$ and*

$$\begin{aligned} &F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ &y_n = \gamma_n u_n + (1-\gamma_n) T u_n \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) y_n, \end{aligned} \tag{33}$$

for all $n \in \mathbf{N}$, where $u_n = T_{r_n}(x_n - r_n B x_n), \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. If the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{r_n\}$ satisfies the following conditions (C1)-(C3) and (C5) for some $a, b, c, d \in \mathcal{R}$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to z, where $z = P_\Theta(I - A + \gamma f)z$, which solves the unique solution of the variational inequalities (14), which is the optimality condition for the minimization problem (8).

Proof. For $i = 1, 2, 3, \ldots, N$, and set $T_1 = T_1 = \ldots = T_N = T$ by theorem 1, we obtain the desired result.

Put $\gamma_n \equiv 0$, for all $n \in \mathbf{N}$, we have the following corollary:

Corollary 3. *Let H be a real Hilbert space, C a nonempty closed convex subset of H. Let $F : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)-(A5) and let $\varphi : C \to \mathcal{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap dom\varphi \neq \emptyset$. Let $T : C \to C$ be a k-strictly pseudo-contractive mapping for some $0 \leq k < 1$ such that $\Theta := F(T) \cap EP(F, \varphi) \neq \emptyset$ and let f be a contraction of H into itself with coefficient $\alpha \in (0, 1)$. Assume that either (B1) or (B2) holds. Let A be a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$*

and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be the sequences generated by* $x_1 \in H$ *and* $u_n \in C$,

$$\begin{aligned} &F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)Tu_n, \end{aligned} \tag{34}$$

for all $n \in \mathbf{N}$, *where* $u_n = T_{r_n}(x_n - r_n Bx_n), \{\alpha_n\}, \{\beta_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$. *If the sequences* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{r_n\}$ *satisfies the following conditions (C1)-(C3) and (C5) in Theorem 2. Then* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* z, *where* $z = P_\Theta(I - A + \gamma f)z$, *which solves the unique solution of the variational inequalities (14), which is the optimality condition for the minimization problem (8).*

Remark

(1) If we take $\beta_n \equiv 0$ for all $n \in \mathbf{N}$ then the iterative scheme (34) reduces to the following scheme:

$$\begin{aligned} &F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ &x_{n+1} = \alpha_n \gamma f(x_n) + ((1 - \alpha_n A)Tu_n, \end{aligned} \tag{35}$$

which extend and improve Theorem 3.1 of Plubtieng and Panpaeng in [16] from $EP(F)$ to $MEP(F,\varphi)$

(2) If we take $\varphi \equiv 0$ in Corollary 3, the iterative scheme (34) reduces to the following scheme:

$$\begin{aligned} &F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)Tu_n, \end{aligned} \tag{36}$$

which is a modification of the iterative scheme in the previous results, and by Corollary 3, we obtain strong convergence of the sequence $\{x_n\}$ generated by (36) under some sufficient conditions.

(3) If we take $\beta_n \equiv 0$, for all $n \in \mathbf{N}$ then the iterative scheme (36) reduces to the iterative scheme in Theorem 3.1 of Plubtieng and Panpaeng in [16] from nonexpansive mappings to more general k-strictly pseudo-contractions in Hilbert spaces.

(4) If $\gamma = 1$ and $A \equiv I$ then the iterative scheme (36) reduces to Theorem 3.2 of S. Takahashi and W. Takahashi [18] from nonexpansive mappings to more general k-strictly pseudo-contractions in Hilbert spaces.

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