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SYMMETRY REDUCTIONS, VARIABLE TRANSFORMATIONS AND EXACT SOLUTIONS TO THE SECOND-ORDER PDES†

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ABSTRACT. In this paper, the Lie symmetry analysis is performed on the three mixed second-order PDEs, which arise in fluid dynamics, nonlinear wave theory and plasma physics, etc. The symmetries and similarity reductions of the equations are obtained, and the exact solutions to the equations are investigated by the dynamical system and power series methods. Then, the exact solutions to the general types of PDEs are considered through a variable transformation. At last, the symmetry and integration method is employed for reducing the nonlinear ODEs.

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1. Introduction

In [1], J. Li considered the following two equations by using the dynamical system method. One is the Vakhnenko equation (VE) is given by $\frac{\partial}{\partial x}\mathcal{D}u+u=0$, where $\mathcal{D}:=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}$, which describes high-frequency wave in a relaxing medium. The other is the reduced Ostrovsky equation (OE) $(u_t+c_0u_x+\alpha uu_x)_x=\gamma u$, where c_0 denotes the velocity of the dispersionless linear waves, α is a nonlinear coefficient, and γ is the dispersion coefficient. This equation was reported when finding of a so-called inverted loop-soliton solutions [2-4]. The exact explicit parametric representations of the solitary cusp wave solutions and the periodic cusp wave solutions are obtained. Essentially, these solutions are all traveling wave solutions.

We know that the Lie symmetry analysis is a powerful method for tackling exact solutions to partial differential equations (PDEs) (see e.g., [5-15] and references therein). Furthermore, the combination of Lie symmetry analysis and

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dynamical system method is a feasible approach for dealing with exact explicit solutions to nonlinear evolution equations (NLEEs). Recently, we have studied several nonlinear systems by means of the approach [9,13], the symmetries, similarity reductions and exact solutions (including traveling wave solutions) to these equations are obtained. In the present paper, we will consider the following mixed second-order PDEs:

1. The reduced Ostrovsky equation (OE):

$$u_{xt} + \alpha(uu_x)_x + \beta u_{xx} + \gamma u = 0, \tag{1}$$

where and in what follows, $u = u(x,t)$ denotes the unknown function, all the parameters are arbitrary real numbers.

2. The general Vakhnenko equation (VE):

$$u_{xt} + \lambda(uu_x)_x + \mu u = 0. \tag{2}$$

3. The special case, it is the following mixed second-order equation:

$$u_{xt} + \sigma u = 0. \tag{3}$$

Note that Eq. (1) is the general reduced Ostrovsky equation. When $\lambda = \mu = 1$, Eq. (2) is the Vakhnenko equation [1]. While Eq. (3) is the degenerate case for many important mixed second-order nonlinear NLEEs, such as the Vakhnenko equation (1), the reduced Ostrovsky equation (2) and the short pulse equation [9], etc. These mixed second-order nonlinear PDEs are of great importance in both nonlinear theory and physical applications. Furthermore, the three equations are of the general form as follows:

$$u_{xt} = P(u), \tag{4}$$

where $P(u) = P(u, u_x, u_{xx})$ is a given sufficiently smooth function of the variables. In particular, Eq. (3) is written as the following general form

$$u_{xt} = p(u), \tag{5}$$

where p is a given function with respect to the dependent variable u. In practice, many physical, mechanical and engineering models can be depicted by such equations (4) and (5). More importantly, we will show that the other types of equations

$$u_{tt} = au_{xx} + q(u) \tag{6}$$

can be transformed into Eq. (5) later, where $a \neq 0$ is an arbitrary parameter, q is a given function with respect to the dependent variable u.

The main purpose of this paper is to investigate the symmetries and exact solutions to the equations by Lie symmetry analysis and the dynamical system method. The remainder of this paper is organized as follows. In Section 2, we perform Lie symmetry analysis on Eqs. (1), (2) and (3), the symmetries and similarity reductions are presented. In Section 3, the exact solutions to the three equations are investigated by the dynamical system and power series methods. In Section 4, the general equation (6) is transformed into Eq. (5) through a variable transformation, then the exact solutions to such equations can be obtained successively. In Section 5, some further discussion on the ODE are made by the symmetry and integration method. Finally, the conclusion and some remarks will be given in Section 6.

2. Symmetry reductions for Eqs. (1), (2) and (3)

First of all, we note that Eqs. (1) and (2) can be written as the following forms:

$$u_{xt} + \alpha u_x^2 + \alpha u u_{xx} + \beta u_{xx} + \gamma u = 0, \tag{7}$$

and

$$u_{xt} + \lambda u_x^2 + \lambda u u_{xx} + \mu u = 0. \tag{8}$$

Recall that the vector field of a PDE is as follows:

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}, \tag{9}$$

where the coefficient functions $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$ of the vector field are to be determined later.

The symmetries of Eqs. (1), (2) and (3) will be generated by the vector field of the form (9), respectively. Applying the second prolongation $\mathrm{pr}^{(2)}V{=}\mathrm{pr}^{(1)}V + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}$ of V to the three equations, then the standard symmetry group calculation method leads to the following symmetries of the three PDEs:

For Eq. (1), we have

$$V_{11} = \frac{\partial}{\partial x}, \quad V_{12} = \frac{\partial}{\partial t}, \quad V_{13} = (x - 2\gamma t)\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}. \tag{10}$$

For Eq. (2), we have

$$V_{21} = \frac{\partial}{\partial x}, \quad V_{22} = \frac{\partial}{\partial t}, \quad V_{23} = x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}. \tag{11}$$

For Eq. (3), we have

$$V_{31} = \frac{\partial}{\partial x}, \quad V_{32} = \frac{\partial}{\partial t}, \quad V_{33} = x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t}, \quad V_{34} = u\frac{\partial}{\partial u}, \quad V_s = s\frac{\partial}{\partial u}, \tag{12}$$

where $s = s(x,t)$ satisfies Eq. (3).

Similar to Refs. [9,10,13], it is easy to check that the three vector fields (10), (11) and (12) are closed under the Lie bracket, respectively. Therefore the Lie algebra of symmetries of Eqs. (1) and (2) are generated by the three vector fields $\{V_{11}, V_{12}, V_{13}\}$ and $\{V_{21}, V_{22}, V_{23}\}$, respectively. For Eq. (3), it is spanned by the four vector fields $\{V_{31}, V_{32}, V_{33}, V_{34}\}$ and the infinite-dimensional subalgebra $V_s = s\frac{\partial}{\partial u}$, where $s = s(x,t)$ is an arbitrary solution to Eq. (3).

In what follows, we will reduce the three equations (1), (2) and (3) to ordinary differential equations (ODEs) by the similarity reduction method.

(i) For V_{13}, we have $u = t^{-2}f(\xi)$, where $\xi = xt - \gamma t^2$. Substituting into Eq. (1), we get

$$\alpha f'^2 + \alpha f f'' + \xi f'' - f' + \beta f = 0, \tag{13}$$

where $f' = \frac{df}{d\xi}$.

(ii) For $V = cV_{11} - V_{12}$, we have $u = f(\xi)$, where $\xi = x + ct$. Substituting into Eq. (1), we get

$$\alpha f'^2 + \alpha f f'' + (\beta + c) f'' + \gamma f = 0, \tag{14}$$

where $f' = \frac{df}{d\xi}$, $c \neq 0$ is an arbitrary constant.

(iii) For V_{23}, we have $u = t^{-2} f(\xi)$, where $\xi = xt$. Substituting into Eq. (2), we get

$$\lambda f'^2 + \lambda f f'' + \xi f'' - f' + \mu f = 0, \tag{15}$$

where $f' = \frac{df}{d\xi}$.

(iv) For $V = cV_{21} - V_{22}$, we have $u = f(\xi)$, where $\xi = x + ct$. Substituting into Eq. (2), we get

$$\lambda f'^2 + \lambda f f'' + c f'' + \mu f = 0, \tag{16}$$

where $f' = \frac{df}{d\xi}$, $c \neq 0$ is an arbitrary constant.

(v) For V_{33}, we have $u = f(\xi)$, where $\xi = xt$. Substituting into Eq. (3), we get

$$\xi f'' + f' + \sigma f = 0, \tag{17}$$

where $f' = \frac{df}{d\xi}$.

(vi) For $V = cV_{31} + V_{32}$, we have $u = f(\xi)$, where $\xi = x - ct$. Substituting into Eq. (3), we get

$$c f'' - \sigma f = 0, \tag{18}$$

where $f' = \frac{df}{d\xi}$, $v \neq 0$ is an arbitrary constant.

(vii) For $V = V_{31} + vV_{34}$, we have $u = e^{vx} f(\xi)$, where $\xi = t$. Substituting into Eq. (3), we get

$$v f' + \sigma f = 0, \tag{19}$$

where $f' = \frac{df}{d\xi}$, $v \neq 0$ is an arbitrary constant.

(viii) For $V = V_{32} + vV_{34}$, we have $u = e^{vt} f(\xi)$, where $\xi = x$. Substituting into Eq. (3), we get

$$v f' + \sigma f = 0, \tag{20}$$

where $f' = \frac{df}{d\xi}$, $v \neq 0$ is an arbitrary constant.

(ix) For $V = V_{33} + vV_{34}$, we have $u = t^{-v} f(\xi)$, where $\xi = xt$. Substituting into Eq. (3), we get

$$\xi f'' + (1 - v) f' + \sigma f = 0, \tag{21}$$

where $f' = \frac{df}{d\xi}$, $v \neq 0$ is an arbitrary constant.

3. Exact solutions to the equations

By exact solutions, we mean those that can be obtained from some ODEs or, in general, from PDEs of lower order than the original PDE [16]. In terms of this definition, the exact solutions to the three equations (1), (2) and (3) are obtained actually in Section 2. But we still want to investigate the explicit solutions to the equations. In this section, we deal with the exact explicit solutions to the three equations.

3.1. The traveling wave solutions. In view of Eqs. (14), (16) and (18), we can get the traveling wave solutions to Eqs. (1), (2) and (3), respectively. As is well known, the symmetries and first integrals are two fundamental structures for ordinary differential equations (ODEs). For example, based on the symmetries, the properties of a system can be explored. By means of symmetries or first integrals, the exact explicit solutions can be obtained immediately.

Firstly, it should be noted that Eq. (14) is equivalent to the planar system

$$\frac{df}{d\xi} = y, \quad \frac{dy}{d\xi} = -\frac{\alpha y^2 + \gamma f}{\alpha f + \beta + c}. \tag{22}$$

System (22) has the first integral

$$H(f, y) = y^2(\alpha f + \beta + c)^2 + \gamma f^2(\frac{2}{3}\alpha f + \beta + c) = h. \tag{23}$$

Similarly, Eq. (16) is equivalent to the planar system

$$\frac{df}{d\xi} = y, \quad \frac{dy}{d\xi} = -\frac{\lambda y^2 + \mu f}{\lambda f + c}. \tag{24}$$

System (24) has the first integral

$$H(f, y) = y^2(\lambda f + c)^2 + \mu f^2(\frac{2}{3}\lambda f + c) = h. \tag{25}$$

Then, based on the first integrals (23) and (25), by the dynamical system method, we can obtain the traveling wave solutions to Eqs. (1) and (2), respectively. The details are omitted here (see [1], pp. 911-913). Thus, we need to consider the traveling wave solutions to Eq. (3) only. In view of its reduced Eq. (18), we have the following results:

When $\sigma c > 0$, Eq. (18) has the solution $f(\xi) = c_1 e^{\sqrt{\frac{\sigma}{c}}\xi} + c_2 e^{-\sqrt{\frac{\sigma}{c}}\xi}$. The traveling wave solution to Eq. (3) is

$$u(x, t) = c_1 e^{\sqrt{\frac{\sigma}{c}}(x-ct)} + c_2 e^{-\sqrt{\frac{\sigma}{c}}(x-ct)}, \tag{26}$$

where c_1, c_2 are arbitrary constants.

When $\sigma c < 0$, Eq. (18) has the solution $f(\xi) = c_1 \cos\sqrt{-\frac{\sigma}{c}}\xi + c_2 \sin\sqrt{-\frac{\sigma}{c}}\xi$. The traveling wave solution to Eq. (3) is

$$u(x, t) = c_1 \cos\sqrt{-\frac{\sigma}{c}}(x - ct) + c_2 \sin\sqrt{-\frac{\sigma}{c}}(x - ct), \tag{27}$$

where c_1, c_2 are arbitrary constants.

3.2. The exact power series solutions. Note that Eqs. (13), (15), (17) and (21) are nonlinear nonautonomous ODEs. To our knowledge, there is no any general methods for dealing with such equations. Now, we tackle the solutions by using the power series method. As an example, we need to consider Eqs. (13) and (17) only, the other equations are handled similarly.

We will seek a solution to Eq. (13) in a power series of the form

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{28}$$

Substituting (28) into (13) and comparing coefficients, we obtain

$$c_2 = \frac{1}{2\alpha c_0}(c_1 - \alpha c_1^2 - \beta c_0). \tag{29}$$

Generally, for $n \geq 2$, we have

$$c_{n+1} = \frac{-1}{n(n+1)\alpha c_0}[(n-2)nc_n + \alpha \sum_{k=1}^{n} n(n+1-k)c_k c_{n+1-k} + \beta c_{n-1}]. \tag{30}$$

Thus, for arbitrary chosen c_0 and c_1, from (29), we can get c_2. On the other hand, in view of (30), we have $c_3 = \frac{-1}{6\alpha c_0}(6\alpha c_1 c_2 + \beta c_1)$, $c_4 = \frac{-1}{12\alpha c_0}(3c_3 + 12\alpha c_1 c_3 + 6\alpha c_2^2 + \beta c_2)$, and so on.

Therefore, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (30) in a unique manner. This implies that for Eq. (13), there exists a power series solution (28) with the coefficients given by (29) and (30). Furthermore, we can show that the convergence of the power series solution (28) to Eq. (13) (see [9,10,13]). Thus, the power series solution (28) is the exact analytic solution to this equation.

The power series solution to Eq. (13) can be written as following:

$$f(\xi) = c_0 + c_1\xi + c_2\xi^2 + \sum_{n=2}^{\infty} c_{n+1}\xi^{n+1} = c_0 + c_1\xi + \frac{1}{2\alpha c_0}(c_1 - \alpha c_1^2 - \beta c_0)\xi^2$$

$$+ \sum_{n=2}^{\infty} \frac{-1}{n(n+1)\alpha c_0}[(n-2)nc_n + \alpha \sum_{k=1}^{n} n(n+1-k)c_k c_{n+1-k} + \beta c_{n-1}]\xi^{n+1}. \tag{31}$$

Thus, the exact analytic solution to Eq. (1) is

$$u(x,t) = c_0 t^{-2} + c_1(x-\gamma t)t^{-1} + \frac{1}{2\alpha c_0}(c_1 - \alpha c_1^2 - \beta c_0)(x-\gamma t)^2$$

$$+ \sum_{n=2}^{\infty} \frac{-1}{n(n+1)\alpha c_0}[(n-2)nc_n + \alpha \sum_{k=1}^{n} n(n+1-k)c_k c_{n+1-k} + \beta c_{n-1}](x-\gamma t)^{n+1} t^{n-1}, \tag{32}$$

where c_i $(i = 0, 1)$ are arbitrary constants.

Suppose that Eq. (17) is a solution in a power series of the form (13). Similar to Eq. (13), we obtain

$$c_1 = -\sigma c_0. \tag{33}$$

Generally, for $n \geq 2$, we have

$$c_n = (-1)^n \frac{\sigma^n}{(n!)^2} c_0. \tag{34}$$

Thus, for arbitrary chosen c_0, from (33), we can get c_1. On the other hand, in view of (34), we have $c_2 = \frac{\sigma^2}{4} c_0$, $c_3 = -\frac{\sigma^3}{36} c_0$, $c_4 = \frac{\sigma^4}{576} c_0$, and so on.

The power series solution to Eq. (17) can be written as following:

$$f(\xi) = c_0 + c_1 \xi + \sum_{n=2}^{\infty} c_n \xi^n = c_0 - \sigma c_0 \xi + c_0 \sum_{n=2}^{\infty} (-1)^n \frac{\sigma^n}{(n!)^2} \xi^n. \tag{35}$$

Thus, the exact analytic solution to Eq. (3) is

$$u(x,t) = c_0 + c_1 xt + \sum_{n=2}^{\infty} c_n (xt)^n = c_0 - \sigma c_0 xt + c_0 \sum_{n=2}^{\infty} (-1)^n \frac{\sigma^n}{(n!)^2} (xt)^n, \tag{36}$$

where c_0 is an arbitrary constant. Moreover, it is easy to check that the convergence of the power series solution for all $x \in \mathbb{R}$ by using the elementary criterions.

3.3. The exact explicit solutions. In view of Eq. (19), we have the solution is $f(\xi) = ce^{-\frac{\sigma}{v}\xi}$. Thus, the solution to Eq. (3) is

$$u(x,t) = ce^{vx - \frac{\sigma}{v} t}, \tag{37}$$

where c and $v \neq 0$ are arbitrary constants.

On the other hand, for Eq. (20), we have the solution is $f(\xi) = ce^{-\frac{\sigma}{v}\xi}$. Thus, the other solution to Eq. (3) is

$$u(x,t) = ce^{vt - \frac{\sigma}{v} x}, \tag{38}$$

where c and $v \neq 0$ are arbitrary constants.

4. Transformation and exact solutions to Eq. (6)

In both of the preceding sections, the symmetry reductions and exact solutions to Eqs. (1), (2) and (3) are obtained. In this section, we consider the general equations (5) and (6). Setting

$$\xi = \frac{1}{2}\sqrt{-\frac{b}{a}}(x - \sqrt{a}t), \quad \eta = \frac{1}{2}\sqrt{-\frac{b}{a}}(x + \sqrt{a}t), \tag{39}$$

where $b \neq 0$ is a parameter related to the function q generally. Substituting (39) into Eq. (6), we have $bu_{\xi\eta} = q(u)$, that is $u_{\xi\eta} = \frac{1}{b} q(u)$. This is the form of Eq. (5) clearly. Thus, Eq. (6) is transformed into Eq. (5) through the transformation (39). Therefore, based on the solutions to Eq. (5), the exact solutions to Eq. (6) can be obtained successively.

For example, we consider the following one-dimensional wave equation

$$u_{tt} = au_{xx} + bu, \tag{40}$$

where a, b are arbitrary parameters. Plugging (39) into (40), we have

$$u_{\xi\eta} = u. \tag{41}$$

Thus, we transform Eq. (40) into Eq. (3) by the transformation (39). Furthermore, Eq. (3) has a solution (37), so we can get the exact solution to Eq. (40) is

$$u(x,t) = c\exp[\frac{v}{2}\sqrt{-\frac{b}{a}}(x-\sqrt{a}t) + \frac{1}{2v}\sqrt{-\frac{b}{a}}(x+\sqrt{a}t)], \tag{42}$$

where c and $v \neq 0$ are arbitrary constants.

On the other hand, Eq. (3) has a solution (38), so we can get the other exact solution to Eq. (40) is

$$u(x,t) = c\exp[\frac{v}{2}\sqrt{-\frac{b}{a}}(x+\sqrt{a}t) + \frac{1}{2v}\sqrt{-\frac{b}{a}}(x-\sqrt{a}t)], \tag{43}$$

where c and $v \neq 0$ are arbitrary constants.

While for the more general types of equations $u_{tt} = au_{xx} + Q(u)$, where $Q(u) = Q(u, u_x, u_{xx})$, it is difficult to transform this type of equation into Eq. (4) using the similar transformation generally. But for some specific equations, this approach is feasible. The details are omitted here.

5. Further discussion by the symmetry and integration method

In Subsection 3.2, we obtained the exact power series solutions to Eqs. (13) and (17). Now, we make some further discussion on Eq. (17). By the integration method based on the invariants of the group, we know that if we get a one-parameter symmetry group of an ODE, then we can reduce the order of the equation by one.

Clearly, Eq. (17) is invariant under the group of scale transformation $(\xi, f) \mapsto (\xi, e^{\epsilon} f)$, where ϵ is an arbitrary real number. This scaling group corresponds to the infinitesimal generator $V = f\frac{\partial}{\partial f}$. Furthermore, we have $V\xi = 0$, $Vf = f$. Letting $w = \log f$, we have $f_\xi = e^w w_\xi$, $f_{\xi\xi} = e^w w_\xi^2 + e^w w_{\xi\xi}$. Plugging into (17), we get

$$w_{\xi\xi} + w_\xi^2 + \frac{1}{\xi}w_\xi + \frac{\sigma}{\xi} = 0.$$

Setting $w_\xi = z$, we obtain

$$\frac{dz}{d\xi} = -z^2 - \frac{1}{\xi}z - \frac{\sigma}{\xi}. \tag{44}$$

Thus, we reduce Eq. (17) to a Riccati equation (44).

Furthermore, this equation can be simplified also. In fact, suppose that $z = u - \frac{1}{2\xi}$, substituting it into (44), we have

$$\frac{du}{d\xi} = -u^2 - \frac{\sigma}{\xi} - \frac{1}{4\xi^2}, \tag{45}$$

there is not any linear term in this equation.

6. Conclusion and remarks

In this paper, we have obtained the symmetries of the three mixed second-order PDEs. The exact solutions based on the similarity reductions are investigated. Then the first integrals and exact analytic solutions are given for the the first time in this paper. In addition, we have shown that Eq. (6) can be transformed into Eq. (5) by the transformation (39), and the exact solutions to Eq. (40) are given simultaneously. At last, we reduce the second-order nonlinear ODE to a first-order equation by using symmetry and integration method. This is a very powerful method for dealing with exact solutions of ODEs and is worthy of studying further.

On the other hand, from the geometric point of view, the symmetries (10) - (12) of the three equations are geometric symmetries (point transformations) since they act geometrically on the underlying space $X \times U$ (see e.g., [5,6]). So, are there generalized symmetries and any other forms of exact solutions to the equations? We hope to investigate in the future.

Remark 6.1. We would like to reiterate that the power series solutions which have been obtained in subsection 3.2 are exact analytic solutions. Moreover, we can see that such power series solutions converge quickly, so it is convenient for computations in both theory and applications.

Remark 6.2. Note that the Riccati equation (45) cannot be integrated by the elementary functions. For dealing with the exact solutions to such nonlinear ODEs, the power series method is a useful tool, especially in numerical analysis and physical applications.

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