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Some Properties of Choquet Integrals with Respect to a Fuzzy Complex Valued Fuzzy Measure

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Abstract

In this paper, we consider fuzzy complex valued fuzzy measures and Choquet integrals with respect to a fuzzy measure of real-valued measurable functions. In doing so, we investigate some basic properties and convergence theorems.

Key Words: Choquet integrals, fuzzy measures, fuzzy complex valued fuzzy measures.

1. Introduction

Burkley [1-3] have been studied the concept of fuzzy complex numbers, the differentiability and integrability of fuzzy complex valued functions on the complex plane $\mathbb{C}$. Recently, Wang and Li [14] studied generalized Lebesgue integrals with respect to a complex valued functions. We first defined interval-valued Choquet integral with respect to a fuzzy measure and have been studied some properties and various convergence theorems of them.

In this paper, we define the Choquet integral with respect to a fuzzy complex valued fuzzy measure of real-valued measurable functions. In section 2, we list the definitions and some basic properties of fuzzy measures, Choquet integrals, interval-valued sets, and fuzzy numbers. In section 3, we introduce fuzzy complex numbers , interval-valued fuzzy measures. In section 4, we consider fuzzy complex valued fuzzy measures and define the Choquet integral with respect to a fuzzy complex valued fuzzy measure of a real-valued measurable function. In particular, we discuss the existence of them and investigate some basic properties.

2. Definitions and Preliminaries

In this section, we assume that (X, Ω) is a measurable space and $\mathbb{R}^+ = [0, \infty)$ and $\bar{\mathbb{R}}^+ = [0, \infty]$. We first list the definitions of fuzzy measures and Choquet integrals(see[4-14]). A fuzzy measure μ on a measurable space (X, Ω) is a set function $\mu : \Omega \longrightarrow \bar{\mathbb{R}}^+$ satisfying

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(A) \le \mu(B)$, whenever $A, B \in \Omega$ and $A \subset B$.

(iii) for every increasing sequence $\{A_n\}$ of measurable sets, $\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$,

(iv) for every decreasing sequence $\{A_n\}$ of measurable sets and $\mu(A_1) < \infty$, $\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

We note that a fuzzy measure satisfies the conditions (i) and (ii) in many papers. But, in this paper, we assume that a fuzzy measure satisfies the four conditions (i)-(iv). Now, we introduce the Choquet integral with respect to a fuzzy measure which was defined by M. Sugeno (see[9-13]) as follows.

Definition 2.1. (1) The Choquet integral of a measurable function $f : X \to \mathbb{R}^+$ with respect to a fuzzy measure μ on $A \in \Omega$ is defined by

$$(C)\int_A f d\mu = \int_0^{\infty} \mu(\{x|f(x) > r\} \cap A)dr$$

where the integral on the right-hand side is an ordinary one.

(2) A measurable function f is said to be C-integrable if the Choquet integral of f on X can be defined and its value is finite.

Instead of $(C)\int_X f d\mu$, we will write $(C)\int f d\mu$. We note that if we define the distribution function $G_f(r) = \mu(\{x|f(x) > r\})$ of a measurable function f for any $r \in \mathbb{R}^+ = [0, \infty)$, then $G_f(r)$ is a decreasing function for $r \in \mathbb{R}^+$.

Definition 2.2. Let μ be a fuzzy measure on Ω and f a measurable function. We say that f and g are comonotonic, in symbol, $f \sim g$ if $f(x) < f(y) \longrightarrow g(x) \le g(y)$ for all $x, y \in X$.

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From Definition 2.1 and Definition 2.2, we introduce the following comonotonicity and basic properties of Choquet integrals

Theorem 2.3. ([6,7,8]) Let f, g, and h be measurable functions. Then we have
(1) $f \sim f$,
(2) $f \sim g \Longrightarrow g \sim f$,
(3) $f \sim a$ for all $a \in \mathbb{R}^+$,
(4) $f \sim g$ and $g \sim h \Longrightarrow f \sim g + h$.

Theorem 2.4. ([6,7,8]) Let f and g be measurable functions. Then we have
(1) if $f \leq g$, then $(C)\int f d\mu \leq (C)\int g d\mu$,
(2) $E_1 \subset E_2$ and $E_1, E_2 \in \Omega$, then $(C)\int_{E_1} f d\mu \leq (C)\int_{E_2} f d\mu$,
(3) if $f \sim g$ and $a, b \in \mathbb{R}^+$, then

$$(C)\int (af + bg) d\mu = a(C)\int f d\mu + b(C)\int g d\mu,$$

(4) if we define $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in X$, then

$$(C)\int f \vee g d\mu \geq (C)\int f d\mu \vee (C)\int g d\mu$$

and

$$(C)\int f \wedge g d\mu \leq (C)\int f d\mu \wedge (C)\int g d\mu.$$

Throughout this paper, $I(\mathbb{R}^+)$ is the class of all closed intervals in $\mathbb{R}^+$, that is,

$$I(\mathbb{R}^+) = \{[a^-, a^+] | a^-, a^+ \in \mathbb{R}^+ \text{ and } a^- \leq a^+\}.$$

For any $a \in \mathbb{R}^+$, we define $a = [a, a]$. Obviously, $a \in I(\mathbb{R}^+)$(see[4-9]).

Definition 2.5. If $\bar{a} = [a^-, a^+], \bar{b} = [b^-, b^+] \in I(\mathbb{R}^+)$ and $k \in \mathbb{R}^+$, then we define the following operations:
(1) $\bar{a} + \bar{b} = [a^- + b^-, a^+ + b^+]$.
(2) $k\bar{a} = [ka^-, ka^+]$.
(3) $\bar{a}\bar{b} = [a^- b^-, a^+ b^+]$.
(4) $\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+]$.
(5) $\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+]$.
(6) $\bar{a} \leq \bar{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$.
(7) $\bar{a} < \bar{b}$ if and only if $\bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$.
(8) $\bar{a} \subset \bar{b}$ if and only if $b^- \leq a^-$ and $a^+ \leq b^+$.

Theorem 2.6. For $\bar{a}, \bar{b}, \bar{c} \in I(\mathbb{R}^+)$, we have
(1) idempotent law: $\bar{a} \wedge \bar{a} = \bar{a}, \bar{a} \vee \bar{a} = \bar{a}$,
(2) commutative law: $\bar{a} \wedge \bar{b} = \bar{b} \wedge \bar{a}, \bar{a} \vee \bar{b} = \bar{b} \vee \bar{a}$,
(3) associative law: $(\bar{a} \wedge \bar{b}) \wedge \bar{c} = \bar{a} \wedge (\bar{b} \wedge \bar{c})$,
(4) absorption law: $\bar{a} \wedge (\bar{a} \vee \bar{b}) = \bar{a} \vee (\bar{a} \wedge \bar{b}) = \bar{a}$,
(5) distributive law: $\bar{a} \wedge (\bar{b} \vee \bar{c}) = (\bar{a} \wedge (\bar{b}) \vee (\bar{a} \wedge \bar{c}), \bar{a} \vee (\bar{b} \wedge \bar{c}) = (\bar{a} \vee (\bar{b}) \wedge (\bar{a} \vee \bar{c})$.

We note that a set function $d_H : I(\mathbb{R}^+) \times I(\mathbb{R}^+) \longrightarrow \bar{\mathbb{R}}^+$ is called the Hausdorff metric if

$$d_H(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{y \in B} \inf_{x \in A} |x - y|\},$$

for all $A, B \in I(\mathbb{R}^+)$. Then it is easily to see that for any $\bar{a} = [a^-, a^+], \bar{b} = [b^-, b^+] \in I(\mathbb{R}^+)$, we have

$$d_H(\bar{a}, \bar{b}) = \max\{|a^- - b^-|, |a^+ - b^+|\}.$$

We recall that for a sequence of closed intervals $\{\bar{a}_n\}$ converges to $\bar{a}$, in symbols, $d_H - \lim_{n\to\infty} \bar{a}_n = \bar{a}$ if $\lim_{n\to\infty} d_H(\bar{a}_n, \bar{a}) = 0$ and that $d_H - \lim_{n\to\infty} \bar{a}_n = \bar{a}$ if and only if $\lim_{n\to\infty} a_n^- = a^-$ and $\lim_{n\to\infty} a_n^+ = a^+$. We introduce fuzzy numbers and some operations on them which are used in the next sections.

Definition 2.7. A fuzzy set $\widetilde{u}$ on $\mathbb{R}^+$ is called a fuzzy number if it satisfies the following conditions;
(i) (normality) $\widetilde{u}(x) = 1$ for some $x \in \mathbb{R}^+$,
(ii) (fuzzy convexity) for every $\lambda \in (0, 1]$, $\widetilde{u}_\lambda = \{x \in \mathbb{R}^+ | \widetilde{u}(x) \geq \lambda\} \in I(\mathbb{R}^+)$, where $\widetilde{u}_\lambda$ is the level set of $\widetilde{u}$.

Let $FN(\mathbb{R}^+)$ denote the class of all fuzzy numbers. We define the following basic operations on $FN(\mathbb{R}^+)$(see[8,12]); for every $\widetilde{u}, \widetilde{v} \in FN(\mathbb{R}^+)$ and $k \in \mathbb{R}^+$,
$(\widetilde{u} + \widetilde{v})_\lambda = \widetilde{u}_\lambda + \widetilde{v}_\lambda$,
$(k\widetilde{u})_\lambda = k\widetilde{u}_\lambda$,
$(\widetilde{u}\widetilde{v})_\lambda = \widetilde{u}_\lambda \widetilde{v}_\lambda$,
$\widetilde{u} \leq \widetilde{v}$ if and only if $\widetilde{u}_\lambda \leq \widetilde{v}_\lambda$, for all $\lambda \in (0, 1]$,
$\widetilde{u} < \widetilde{v}$ if and only if $\widetilde{u} \leq \widetilde{v}$ and $\widetilde{u} \neq \widetilde{v}$,
$\widetilde{u} \subset \widetilde{v}$ if and only if $\widetilde{u}_\lambda \subset \widetilde{v}_\lambda$, for all $\lambda \in (0, 1]$.

3. Fuzzy complex numbers and interval-valued fuzzy measures

In this section, we consider a fuzzy complex number and interval-valued fuzzy measures(see [1-3, 14]).

Definition 3.1. Let $\widetilde{a}, \widetilde{b} \in FN(\mathbb{R}^+)$. We define a double ordered fuzzy numbers $(\widetilde{a}, \widetilde{b})$ as follows:

$$(\widetilde{a}, \widetilde{b}) : \mathbb{C}^+ \longrightarrow [0, 1]$$

$$z = x + yi \longmapsto (\widetilde{a}, \widetilde{b})(z) = \widetilde{a}(x) \wedge \widetilde{b}(y),$$

where $\mathbb{C}^+ = \{x + yi | x, y \in \mathbb{R}^+\}$. Then the mapping $(\widetilde{a}, \widetilde{b})$ determines a fuzzy complex number, where $\widetilde{a}$ and $\widetilde{b}$ is called a real part and an imaginary part of $(\widetilde{a}, \widetilde{b})$, respectively.

We note that if we put $C = (\widetilde{a}, \widetilde{b})$, then $\widetilde{a} = ReC$ and $\widetilde{b} = ImC$. Let $FCN(\mathbb{C}^+)$ be the class of all fuzzy complex numbers on $\mathbb{C}^+$, writing

$$C \equiv \widetilde{a} + \widetilde{b}i.$$

Note that if $c = a + bi$ is a nonnegative complex number, then its membership function is defined by

$$c(z) = \begin{cases} 1 & \text{if } x = a \text{ and } y = b \\ 0 & \text{otherwise.} \end{cases}$$

where $z = x + yi \in \mathbb{C}^+$. Clearly, $c \in FCN(\mathbb{C}^+)$, i.e., a fuzzy complex number is also a generalization of an ordinary complex number. In [1], we recall that if $C_1, C_2 \in FCN(\mathbb{C}^+)$ and we define

$$C_1 * C_2 = (ReC_1 * ReCC_2, ImC_1 * ImC_2)$$

for an operation $* \in \{+, -, \times, \wedge, \vee\}$, then clearly we have $C_1 * C_2 \in FCN(\mathbb{C}^+)$.

Definition 3.2. Let $C_1, C_2 \in FCN(\mathbb{C}^+)$. Then we define the following order and equality operations:

(1) $C_1 \leq C_2$ if and only if $ReC_1 \leq ReC_2$ and $ImC_1 \leq ImC_2$.
(2) $C_1 < C_2$ if and only if $C_1 \leq C_2$ and $C_1 \neq C_2$.
(3) $C_1 = C_2$ if and only if $C_1 \leq C_2$ and $C_2 \leq C_1$.
(4) $C_1 \vee C_2 = (ReC_1 \vee ReC_2, ImC_1 \vee ImC_2$.
(5) $C_1 \wedge C_2 = (ReC_1 \wedge ReC_2, ImC_1 \wedge ImC_2$.

From Definition 3.2, it is easy to see that if we define λ-cut set $C_\lambda = \{z = x + yi \in \mathbb{C}^+ | (ReC)(x) \geq \lambda \text{ and } (ImC)(y) \geq \lambda\}$, then it is a closed rectangle region in $\mathbb{C}^+$.

Definition 3.3. A mapping $D : FNC(\mathbb{C}^+) \times FNC(\mathbb{C}^+) \longrightarrow \bar{\mathbb{R}}^+$ is defined by

$$D(C_1, C_2) = \max\{\triangle(ReC_1, ReC_2), \triangle(ImC_1, ImC_2)\},$$

where $\triangle(ReC_1, ReC_2) = \sup_{\lambda \in (0,1]} d_H((ReC_1)_\lambda, (ReC_2)_\lambda)$ and $\triangle(ImC_1, ImC_2) = \sup_{\lambda \in (0,1]} d_H((ImC_1)_\lambda, (ImC_2)_\lambda)$.

Then, we can see that $(FNC(\mathbb{C}^+), D)$ is a metric space. By using the metric D, we define the concept of convergence of a sequence in the metric space $(FNC(\mathbb{C}^+), D)$ and introduce an interval-valued fuzzy measure on Ω.

Definition 3.4. Let $\{C_n\} \subset FNC(\mathbb{C}^+)$ be a sequence and $C \in FNC(\mathbb{C}^+)$. A sequence $\{C_n\}$ converges to C, in symbol, $D - \lim_{n\to\infty} C_n = C$ if

$$\lim_{n\to\infty} D(C_n, C) = 0.$$

Definition 3.5. A mapping $\bar{\mu} : \Omega \longrightarrow I(\mathbb{R}^+)$ on a measurable space (X, Ω) is called an interval-valued fuzzy measure if it satisfies the following conditions:

(i) $\bar{\mu}(\emptyset) = \bar{0}$,
(ii) $\bar{\mu}(A) \leq \bar{\mu}(B)$, whenever $A, B \in \Omega$ and $A \subset B$.
(iii) for every increasing sequence $\{A_n\}$ of measurable sets,

$$\bar{\mu}(\cup_{n=1}^{\infty} A_n) = d_H - \lim_{n\to\infty} \bar{\mu}(A_n),$$

(iv) for every decreasing sequence $\{A_n\}$ of measurable sets and $\bar{\mu}(A_1) < \infty$,

$$\bar{\mu}(\cap_{n=1}^{\infty} A_n) = d_H - \lim_{n\to\infty} \bar{\mu}(A_n).$$

4. Fuzzy complex valued fuzzy measures and Choquet integrals

In this section, we consider a fuzzy complex valued fuzzy measure and the Choquet integral with respect to a fuzzy complex valued measure of a real-valued measurable function.

Definition 4.1. A mapping $\widetilde{\mu} : \Omega \longrightarrow FCN(\mathbb{C}^+)$ on a measurable space (X, Ω) is called a fuzzy complex valued fuzzy measure if it satisfies the following conditions:

(i) $\widetilde{\mu}(\emptyset) = (\bar{0}, \bar{0})$,
(ii) $\widetilde{\mu}(A) \leq \widetilde{\mu}(B)$, whenever $A, B \in \Omega$ and $A \subset B$.
(iii) for every increasing sequence $\{A_n\}$ of measurable sets,

$$\widetilde{\mu}(\cup_{n=1}^{\infty} A_n) = D - \lim_{n\to\infty} \widetilde{\mu}(A_n),$$

(iv) for every decreasing sequence $\{A_n\}$ of measurable sets and $\widetilde{\mu}(A_1) \in FCN(\mathbb{C}^+)$,

$$\widetilde{\mu}(\cap_{n=1}^{\infty} A_n) = D - \lim_{n\to\infty} \widetilde{\mu}(A_n).$$

Definition 4.2. Let $\widetilde{\mu}$ be a fuzzy complex valued fuzzy measure and $f : X \longrightarrow \mathbb{R}^+$ a real-valued measurable function.

(1) The Choquet integral with respect to $\widetilde{\mu}$ of f is defined by

$$(C)\int_A f d\widetilde{\mu} = \left((C)\int_A f d\widetilde{\mu}_R, (C)\int_A f d\widetilde{\mu}_I\right)$$

where $\left((C)\int_A f d\widetilde{\mu}_R\right)_\lambda = (C)\int_A f d\widetilde{(\mu_R)}_\lambda = \left[(C)\int_A f d\widetilde{(\mu_R)}_\lambda^-, (C)\int_A f d\widetilde{(\mu_R)}_\lambda^+\right]$ and $\left((C)\int_A f d\widetilde{\mu}_I\right)_\lambda = (C)\int_A f d\widetilde{(\mu_I)}_\lambda = \left[(C)\int_A f d\widetilde{(\mu_I)}_\lambda^-, (C)\int_A f d\widetilde{(\mu_I)}_\lambda^+\right]$ for all $\lambda \in (0, 1]$.

(2) A real-valued measurable function f is $\widetilde{\mu}$-Choquet integrable if $(C)\int_A f d\widetilde{\mu} \in FCN(\mathbb{C}^+)$.

From Definition 4.1 and Definition 3.5, we note that $\widetilde{\mu}$ is a fuzzy complex valued measure if and only if $(\widetilde{\mu}_R)_\lambda$ and $(\widetilde{\mu}_I)_\lambda$ are interval-valued fuzzy measures for all $\lambda \in (0,1]$. Let $\mathfrak{M}$ be the class of all fuzzy measures and $FC\mathfrak{M}$ the class of all fuzzy complex valued fuzzy measures. We remark that $\widetilde{\mu} \in FC\mathfrak{M}$ if and only if $(\widetilde{(\mu_R)}^-_\lambda, \widetilde{(\mu_R)}^+_\lambda, \widetilde{(\mu_I)}^-_\lambda, \widetilde{(\mu_I)}^+_\lambda \in \mathfrak{M}$ for all $\lambda \in (0,1]$.

Theorem 4.3. If a real-valued measurable function f is $\widetilde{\mu}$-Choquet integrable, then $(C)\int fd(\widetilde{\mu}_R)^-_\lambda$, $(C)\int fd(\widetilde{\mu}_R)^+_\lambda$, $(C)\int fd(\widetilde{\mu}_I)^-_\lambda$, $(C)\int fd(\widetilde{\mu}_I)^+_\lambda$ are finite.

But, in general, we see that the converse of Theorem 4.3 does not hold. In order to prove the converse of Theorem 4.3, we need to prove that the four integrals of Theorem 4.3 are satisfied with the following lemma.

Lemma 4.4. ([6,10]) Let $\{[a_\lambda, b_\lambda] | \lambda \in (0,1]\}$ be a family of nonempty closed intervals in $I(\mathbb{R}^+)$. If (i) for all $0 < \lambda_1 \le \lambda_2$, $[a_{\lambda_1}, b_{\lambda_1}] \supset [a_{\lambda_2}, b_{\lambda_2}]$ and (ii) for any nonincreasing sequence $\{\lambda_k\}$ in $(0,1]$ is converging to λ, $[a_\lambda, b_\lambda] = \cap_{k=1}^\infty [a_{\lambda_k}, b_{\lambda_k}]$. Then there exists a unique fuzzy number $\widetilde{u} \in FN(\mathbb{R}^+)$ such that the family $[a_\lambda, b_\lambda]$ represents the λ-level sets of a fuzzy number $\widetilde{u}$.

Conversely, if $[a_\lambda, b_\lambda]$ are the λ-level sets of a fuzzy number $\widetilde{u} \in FN(\mathbb{R}^+)$, then the conditions (i) and (ii) are satisfied.

By Definition 3.5(iii), we gent the following lemma.

Lemma 4.5. If $\{\lambda_k\}$ is a nonincreasing sequence in $(0,1]$ converging to λ, then for all $\alpha \in \mathbb{R}^+$,

(i) $\lim_{k\to\infty}(\widetilde{\mu}_R)^-_{\lambda_k}(\{x|f(x) > \alpha\}) = (\widetilde{\mu}_R)^-_\lambda(\{f(x) > \alpha\})$,

(ii) $\lim_{k\to\infty}(\widetilde{\mu}_R)^+_{\lambda_k}(\{x|f(x) > \alpha\}) = (\widetilde{\mu}_R)^+_\lambda(\{f(x) > \alpha\})$,

(iii) $\lim_{k\to\infty}(\widetilde{\mu}_I)^-_{\lambda_k}(\{x|f(x) > \alpha\}) = (\widetilde{\mu}_I)^-_\lambda(\{f(x) > \alpha\})$,

(iv) $\lim_{k\to\infty}(\widetilde{\mu}_I)^+_{\lambda_k}(\{x|f(x) > \alpha\}) = (\widetilde{\mu}_I)^+_\lambda(\{f(x) > \alpha\})$.

Lemma 4.6. (1) For all $0 < \lambda_1 \le \lambda_2$ and $A \in \Omega$, we have

(i) $(C)\int_A fd(\widetilde{\mu}_R)^-_{\lambda_1} \ge (C)\int_A fd(\widetilde{\mu}_R)^-_{\lambda_2}$,

(ii) $(C)\int_A fd(\widetilde{\mu}_R)^+_{\lambda_1} \ge (C)\int_A fd(\widetilde{\mu}_R)^+_{\lambda_2}$,

(iii) $(C)\int_A fd(\widetilde{\mu}_I)^-_{\lambda_1} \ge (C)\int_A fd(\widetilde{\mu}_I)^-_{\lambda_2}$,

(iv) $(C)\int_A fd(\widetilde{\mu}_I)^+_{\lambda_1} \ge (C)\int_A fd(\widetilde{\mu}_I)^+_{\lambda_2}$.

(2) If $\{\lambda_k\}$ is a nonincreasing sequence in $(0,1]$ converging to λ, then for all $\alpha \in \mathbb{R}^+$,

(i) $(C)\int_A fd(\widetilde{\mu}_R)^-_\lambda = \cap_{k=1}^\infty \big((C)\int_A fd(\widetilde{\mu}_R)^-_{\lambda_k}\big)$,

(ii) $(C)\int_A fd(\widetilde{\mu}_R)^+_\lambda = \cap_{k=1}^\infty \big((C)\int_A fd(\widetilde{\mu}_R)^+_{\lambda_k}\big)$,

(iii) $(C)\int_A fd(\widetilde{\mu}_I)^-_\lambda = \cap_{k=1}^\infty \big((C)\int_A fd(\widetilde{\mu}_I)^-_{\lambda_k}\big)$,

(iv) $(C)\int_A fd(\widetilde{\mu}_I)^+_\lambda = \cap_{k=1}^\infty \big((C)\int_A fd(\widetilde{\mu}_I)^+_{\lambda_k}\big)$.

By Lemma 4.5 and Lemma 4.6, we obtain the following theorem.

Theorem 4.7. Let $\widetilde{\mu}$ be a fuzzy complex valued fuzzy measure and $f : X \longrightarrow \mathbb{R}^+$ a real-valued measurable function. If $A \in \Omega$ and $(C)\int_A fd(\widetilde{\mu}_R)^-_{\lambda_1}$, $(C)\int_A fd(\widetilde{\mu}_R)^+_{\lambda_1}$, $(C)\int_A fd(\widetilde{\mu}_I)^-_{\lambda_1}$, $(C)\int_A fd(\widetilde{\mu}_I)^+_{\lambda_1}$ are finite for all $\lambda \in \Omega$, then we have

(i) $\big((C)\int_A fd\widetilde{\mu}\big)_{\lambda_1} \ge \big((C)\int_A fd\widetilde{\mu}\big)_{\lambda_2}$, and

(ii) for every incerasing sequence $\{\lambda_k\}$ in $(0,1]$ converging to λ,

$$\left((C)\int_A fd\widetilde{\mu}\right)_\lambda = \cap_{k=1}^\infty \left((C)\int_A fd\widetilde{\mu}\right)_{\lambda_k}.$$

By Theorem 4.7, we obtain the following theorem which is the converse of Theorem 4.3.

Theorem 4.8. Let $\widetilde{\mu}$ be a fuzzy complex valued fuzzy measure and $f : X \longrightarrow \mathbb{R}^+$ a real-valued measurable function. If $A \in \Omega$ and $(C)\int_A fd(\widetilde{\mu}_R)^-_{\lambda_1}$, $(C)\int_A fd(\widetilde{\mu}_R)^+_{\lambda_1}$, $(C)\int_A fd(\widetilde{\mu}_I)^-_{\lambda_1}$, $(C)\int_A fd(\widetilde{\mu}_I)^+_{\lambda_1}$ are finite for all $\lambda \in \Omega$, then there are uniquely $U \in FCN(\mathbb{C}^+)$ such that $U = (C)\int_A fd\widetilde{\mu}$.

From Definition 4.1 and Theorem 4.4 and Theorem 4.8, we get the following basic properties of the Choquet integral with respect to a fuzzy complex valued fuzzy measure.

Theorem 4.9. Let $\widetilde{\mu}$ be a fuzzy complex valued fuzzy measure and f and g be $\widetilde{\mu}$-Choquet integrable functions. Then we have

(1) if $f \le g$, then $(C)\int fd\widetilde{\mu} \le (C)\int gd\widetilde{\mu}$,

(2) $E_1 \subset E_2$ and $E_1, E_2 \in \Omega$, then $(C)\int_{E_1} fd\widetilde{\mu} \le (C)\int_{E_2} fd\widetilde{\mu}$,

(3) if $f \sim g$ and $a, b \in \mathbb{R}^+$, then

$$(C)\int (af + bg)d\widetilde{\mu} = a(C)\int fd\widetilde{\mu} + b(C)\int gd\widetilde{\mu},$$

(4) if we define $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in X$, then

$$(C)\int f \vee gd\widetilde{\mu} \ge (C)\int fd\widetilde{\mu} \vee (C)\int gd\widetilde{\mu}$$

and

$$(C)\int f \wedge gd\widetilde{\mu} \le (C)\int fd\widetilde{\mu} \wedge (C)\int gd\widetilde{\mu}.$$

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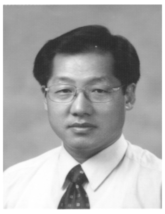

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