이변량 가우시안 Q-함수의 Craig 표현에 대한 기하학적인 유도

정회원 박 승 근*, 종신회원 이 일 규**°

A Geometric Derivation of the Craig Representation for the Two-Dimensional Gaussian Q-Function

Seungkeun Park* *Regular Member*, Il-kyoo Lee**° *Lifelong Member*

요 약

본 논문에서는 기하학적인 관점으로 이변량 가우시안 Q-함수의 Craig 표현에 대한 새롭고 간단한 유도를 제시하고 있다. 또한, 이러한 기하학적인 유도는 이변량 가우시안 Q-함수의 또 다른 Craig 표현 식을 제시하고 있다. 새롭게 유도된 이변량 가우시안 Q-함수의 Craig 식은 2개의 상관 가우시안 잡음에서 직교좌표의 변환으로 생성되는 2개 웨지 영역의 기하학으로부터 새롭게 구한 것이다. 제시된 Craig 형태는 이변량 가우시안 Q-함수로 표현되는 확률을 계산하는데, 중요한 역할을 할 수 있다.

Key Words: Geometric Derivation, Two-dimension, Craig, Gaussian Q-function, Cartesian

ABSTRACT

In this paper, we present a new and simple derivation of the Craig representation for the two-dimensional (2-D) Gaussian Q-function in the viewpoint of geometry. The geometric derivation also leads to an alternative Craig form for the 2-D Gaussian Q-function. The derived Craig form is newly obtained from the geometry of two wedge-shaped regions generated by the rotation of Cartesian coordinates over two correlated Gaussian noises. The presented Craig form can play a important role in computing the probability represented by the 2-D Gaussian Q-function.

Ⅰ. Introduction

It is very important to evaluate error probability performance in designing wireless communication systems. The Craig representation has played a key role when evaluating the error probability performance of digital modulation systems over fading channels by using the moment-generating function(MGF) approach[1-3]. Several derivations of the Craig form for the one-dimensional(1-D) Gaussian Q-function were recently reported[4-5]. The Craig form for the 2-D Gaussian Q-function applied to compute the error probability of M-ary phase shift keying system over various fading channels[6]~[7]. The approximation for the 2-D Gaussian Q-function was presented in terms of the 1-D Gaussian Q-function[8]. The algebraic derivation of the Craig form for the 2-D Gaussian Q-function was presented

※ This work was supported by the IT R&D program of MKE/KCC/KEIT [KI001913, Study on Electromagnetic Environment in Ubiquitous Society.
* 한국전자통신연구원 전자파환경연구팀 (seungpk@etri.re.kr), ** 공주대학교 전기전자제어공학부(leeik@kongju.ac.kr), (° : 교신저자)
논문번호 : KICS2010-11-541, 접수일자 : 2010년 11월 12일, 최종논문접수일자 : 2011년 2월 8일

in terms of the change of variables[9]. However, it is said that there may exist a simple and new derivation of the Craig form.

It is well-known that the geometric derivation is easy to understand. Thus, in this paper, we will present a geometric derivation of the Craig representation by using the rotation of Cartesian coordinates. We hope that the presented derivation is easy and simple to understand the Craig form for the 2-D Gaussian Q-function.

Ⅱ. Problem

Our starting point is the analytical expression of a double integral for the 2-D Gaussian Q-function in the following:

$$Q(x,y;\rho)=\int_x^{\infty}\int_y^{\infty}\frac{\exp\left[-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\right]}{2\pi\sqrt{1-\rho^2}}dudv \quad (1)$$

where ρ represents the correlation coefficient. Chronologically, Simon first derived the Craig form of the 2-D Gaussian Q-function by using the clever change of variables[9]:

$$\theta=\tan^{-1}\left(\frac{\tan\Phi\pm\rho}{\sqrt{1-\rho^2}}\right). \quad (2)$$

It is also known that the Craig representation of the 2-D Gaussian Q-function developed by Simon is given by [9, eq. (10)]

$$Q(x,y;\rho)=\frac{1}{2\pi}\int_0^{\tan^{-1}[(\sqrt{1-\rho^2}x/y)/(1-\rho x/y)]}\exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta+\frac{1}{2\pi}\int_0^{\tan^{-1}[(\sqrt{1-\rho^2}y/x)/(1-\rho y/x)]}\exp\left(-\frac{y^2}{2\sin^2\theta}\right)d\theta; \quad x\ge 0, y\ge 0. \quad (3)$$

where

$$\tan^{-1}\left(\frac{x}{y}\right)=\frac{\pi}{2}[1-sgn(y)]+sgn(y)\tan^{-1}\left(\frac{x}{|y|}\right)$$

in which $sgn(u)=1$ if $u\ge 0$ and $sgn(u)=-1$ if $u<0$. The generic Craig form for the 2-D Gaussian Q-function provided in [10] was derived from the upper limit of (3) and the properties of the 2-D Gaussian Q-function given in [11, eq. (26.3.8) and (26.3.9)]. So far, however, geometric interpretations on the change of variables (1) have not been reported in detail.

It is said that the geometric solution of a problem is easy and simple. Thus, in this letter, motivated by unknown geometric derivation of the Craig form for the 2-D Gaussian Q-function, we present a new derivation for the Craig form on the basis of the geometry of two wedge-shaped regions. The regions are generated by the rotation of Cartesian coordinates.

Ⅲ. Geometric Derivation of the Craig form for the 2-D Gaussian Q-Function

We consider X and Y to be two-dimensional Gaussian random variables (RV) with two zero means, $\mu_X=\mu_Y=0$, two unit variances, $\sigma_X^2=\sigma_Y^2=1$ and a correlation coefficient, ρ_{XY}. Figure 1 shows a graphical representation of the open region, $\Omega=\{(x,y)|X\ge x^*, Y\ge y^*\}$, determined by two constants $x^*, y^*\ge 0$.

Here, as illustrated in Figure 1, we rotate the Cartesian coordinates counterclockwise through the angel $\psi=\tan^{-1}(y^*/x^*)$ about the origin in a way that

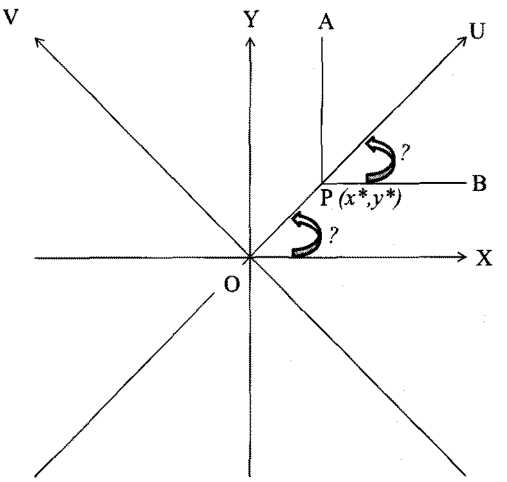

그림 1. 직교좌표 회전에 의해 생성된 2개의 웨지영역에 대한 기하적인 해석
Fig. 1. The geometric interpretation on two wedge-shaped regions generated by the rotation of Cartesian coordinates.

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}. \tag{4}$$

In terms of $U-V$ Cartesian coordinates, the open region, $\Omega=\{(x,y)|X\geq x^*, Y\geq y^*\}$, can be divided into two wedge-shaped regions, $\measuredangle APU$ and $\measuredangle BPU$. The probability of the wedge-shaped region, $\Pr\{\measuredangle APU\}$, is obtained by using (4) and the theory of linear combination of Gaussian RVs as

$$\begin{aligned} \Pr\{\measuredangle APU\} &= \Pr\{X\geq x^*, V\geq 0\} \\ &= Q(x^*,0;\rho_{XV}) \end{aligned} \tag{5}$$

where

$$\rho_{XV}=\frac{-\sin\psi+\rho_{XY}\cos\psi}{\sqrt{1-\rho_{XY}\sin 2\psi}}. \tag{6}$$

Similarly, the probability $\Pr\{\measuredangle BPU\}$ is obtained as

$$\Pr\{\measuredangle BPU\}=\Pr\{Y\geq y^*, V<0\} \tag{7}$$

$$=\int_{y^*}^{\infty}\int_{-\infty}^{0}\frac{\exp\left[-\dfrac{y^2-2\rho_{YV}yv+v^2}{2(1-\rho_{YV}^2)}\right]}{2\pi\sqrt{1-\rho_{YV}^2}}dvdy$$

where

$$\rho_{YV}=\frac{\cos\psi-\rho_{XY}\sin\psi}{\sqrt{1-\rho_{XY}\sin 2\psi}}. \tag{8}$$

Employing [11, eq.(26.3.6)] to (7) gives

$$\Pr\{\measuredangle BPU)\}=Q(y^*,0;-\rho_{YV}). \tag{9}$$

Next, applying [12, eq. (A·5)] to (5) and (9), respectively, and using the trigonometric identity for $\sin^{-1}\phi+\cos^{-1}\phi=\pi/2$ yield the result in the Craig form as

$$\begin{aligned} Q(x^*,y^*;\rho) &= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\rho_{XV}}\exp\left(-\frac{x^{*2}}{2\sin^2\theta}\right)d\theta \\ &+\frac{1}{2\pi}\int_0^{\cos^{-1}\rho_{YV}}\exp\left(-\frac{y^{*2}}{2\sin^2\theta}\right)d\theta; \\ &\qquad x^*\geq 0, y^*\geq 0. \end{aligned} \tag{10}$$

Finally, letting $x^*=x$ and $y^*=y$ and substituting $\cos\psi=x/\sqrt{x^2+y^2}$ and $\sin\psi=y/\sqrt{x^2+y^2}$ into (10) result in an alternative expression for the Craig form presented in (3) as

$$\begin{aligned} Q(x,y;\rho) &= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho x-y}{\sqrt{x^2-2\rho xy+y^2}}\right)}\exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta \\ &+\frac{1}{2\pi}\int_0^{\cos^{-1}\left(\frac{x-\rho y}{\sqrt{x^2-2\rho xy+y^2}}\right)}\exp\left(-\frac{y^2}{2\sin^2\theta}\right)d\theta; \\ &\qquad x\geq 0, y\geq 0. \end{aligned} \tag{11}$$

Note that the alternative expression of (11) do not require the user-defined arc tan function such as the upper limit of (3).

Ⅳ. Conclusion

The main contribution of this paper is the geometric derivation of the Craig form for the 2-D Gaussian Q-function by using the rotation of Cartesian coordinates. The new derivation leads to the alternative Craig representation of the 2-D Gaussian Q-function with geometric interpretation. The derived expression can be applicable to the exact computation of the probability represented by the 2-D Gaussian Q-function.

References

〔1〕 J. W. Craig, "A new, simple, and exact result for calculating the probability of error for two-dimensional signal constellations", *Proc. of IEEE MILCOM*, Oct. 1991, pp.571-575.

〔2〕 M. K. Simon and M. -S. Alouini, "A unified approach to the performance analysis of digital communications over generalized fading channels", *Proc. IEEE, Vol.86, No.9,* pp.1860-1877, Sep. 1998.

〔3〕 M. K. Simon and M. -S. Alouini, "Digital Communication over Fading Channels", *A Unified Approach to Performance Analysis*, New York. Wiley, Aug. 2000.

〔4〕 K. V. Lever, "New derivation of Craig's formula for the Gaussian probability function", *Electronics Letters, Vol.34, No.19*, p.1821, Sep. 1998.

〔5〕 C. Tellambura and A. Annamalai, "Derivation of Craig's formula for Gaussian probability function", *Electronics Letters, Vol.35, No.17*, pp.1424-1425, Aug. 1999.

[6] S. Park and S. H. Cho, "SEP performance of coherent MPSK over fading channels in the presence of phase/quadrature error and I-Q gain mismatch", *IEEE Trans. on Commun., Vol.53, No.7*, pp.1088-1091, Jul. 2005.

[7] J. Park, S. Park, and K. R. Cho, "Performance analysis of symbol error probability for MPSK with an I-Q unbalance over a Rician fading channel", *IEICE Trans. Commun., Vol.E88-B, No.4*, pp.1702-1704, Apr. 2005.

[8] J. Park and S. Park, "Approximation for the two-dimensional Gaussian Q-function with its application", a*ccepted in ETRI Journal*, Feb. 2010.

[9] M. K. Simon, "A simpler form of the Craig representation for the two-dimensional joint Gaussian Q-function", *IEEE Commun. Let., Vol.6, No.2*, pp.49-51, Feb. 2002.

[10] S. Park and U. J. Choi, "A generic Craig form for the two-dimensional Gaussian Q-function", *ETRI Journal, Vol 29, No.4*, Aug. 2007.

[11] M. Abramowiz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York:Dover, 1972.

[12] S. Park B. Kang, and D. Park, "An Extended Pawula F-function for MPSK signal vector perturbed by correlated two-dimensional Gaussian noise", *IEICE Trans. Commun., Vol. E89-B, No.11, pp.3104-3107*, Nov. 2006.

박 승 근 (Seungkeun Park) 정회원

1991년 2월 고려대학교 응용통계학과(이학 학사)

1993년 8월 고려대학교 응용통계학과(이학석사)

2004년 2월 충북대학교 정보통신공학과(공학박사)

1993년 8월~현재 한국전자통신연구원 전파환경연구팀 책임연구원

<관심분야> 통신이론, 디지털 통신, 전자파간섭, 스펙트럼공학

이 일 규 (Il-kyoo Lee) 종신회원

1994년 2월 충남대학교 전자공학과(공학석사)

2003년 2월 충남대학교 전자공학과(공학박사)

1994년 2월~2004년 12월 한국전자통신연구원 선임연구원

2004년 3월~현재 공주대학교 전기전자제어공학부 부교수

<관심분야> RF 시스템, 초고주파회로 설계, 전자파간섭, 스펙트럼공학