

# An Improved Binomial Method using Cell Averages for Option Pricing

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**Abstract.** We present an improved binomial method for pricing financial derivatives by using cell averages. After non-overlapping cells are introduced around each node in the binomial tree, the proposed method calculates cell averages of payoffs at expiry and then performs the backward valuation process. The price of the derivative and its hedging parameters such as Greeks on the valuation date are then computed using the compact scheme and Richardson extrapolation. The simulation results for European and American barrier options show that the proposed method gives much more accurate price and Greeks than other recent lattice methods with less computational effort.

**Keywords:** Option Pricing, Binomial Method, Barrier Options, Cell Averages

## 1. INTRODUCTION

Under classical Black-Scholes model, asset prices are described by stochastic differential equations and option prices are modeled by either partial differential equations or expectations under risk neutral measure. There have been different approaches of pricing options (Clewlow and Strickland 1998, Higham 2004, Kwok 1998, Lyuu 2002, Wilmott *et al.* 1995). However still analytical approach to the American options is very limited due to early exercise before the maturity and the estimation of its price or hedging parameters mainly relies on computational simulation.

Among several methods for option pricing, binomial method is a very popular computational scheme for valuation of derivatives due to its ease of implementation and wide application to exotic options. After Cox *et al.* (1979) introduced the binomial method to value American options, the scheme has been generalized by many researchers including Boyle (1988), Boyle *et al.* (1989), Kamrad and Ritchken (1991), Kwok (1998), Lyuu (2002) and its convergence is proved by (Amin and Khanna 1994). However, if there exists a discontinuity or non-smoothness in the payoff as in a barrier option, the convergence of the binomial method may be very oscillatory. The accuracy of the binomial method is poor

for such an option as shown in Boyle and Lau (1994) or Ritchken (1995), especially when the parameter values are not placed above layers of nodes and the binomial method in such a case requires a large number of time steps to ensure that the price error is small. In order to cure this oscillatory behavior, there have been many studies of modification of the binomial method. Boyle and Lau (1994) improved the binomial method for the barrier option by placing the barrier above a layer of horizontal nodes in the tree. After Boyle (1988) studied trinomial methods for two underlying assets, Ritchken (1995) improved the trinomial method for barrier options by making sure that the layers of the lattice coincide with the barrier. Broadie and Detemple (1996) introduced the Binomial Black and Scholes method with Richardson extrapolation (BBSR) scheme, which includes the Black-Scholes formula in the binomial tree method one step before expiry and applies the Richardson extrapolation. Later Cheuk and Vorst (1996) developed a new trinomial tree model for barrier options, and Gaudenzi and Pressacco (2003) presented an efficient binomial interpolation method with Richardson extrapolation, BIR.

The purpose of this work is to introduce an improved binomial method, which reduces errors of the standard lattice methods occurring when option parameters are

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represented by nodes in the tree. The proposed method defines and updates the cell average of option values around each node in the binomial tree and cell averaging has an effect to decrease those errors. In this sense, this proposed scheme has similarities with the finite volume method in computational fluid dynamics. In order to be more accurate, we apply the compact scheme from Lele (1992) and Richardson extrapolation from Richardson (1927) on the valuation date.

In Section 2, a simple option pricing model and the standard binomial method are introduced. The proposed method is presented in Section 3. The simulation results in Section 4 show that the proposed method gives more accurate result than many recent lattice methods with less computational effort and the conclusions are summarized in Section 6.

## 2. STANDARD APPROACHES

### 2.1 Option Pricing Model

Let us consider the price of the underlying asset as a stochastic process  $\{S_t\}_{t \in [0, T]}$  on a suitable probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ . Let us assume that the evolution of the underlying satisfies the following stochastic differential equation:

$$dS(t) = \mu S dt + \sigma S dW(t), \quad 0 < t < T \tag{1}$$

where  $\mu$  is an expected rate of return,  $\sigma$  is a volatility,  $T$  is an expiration date, and  $W(t)$  is a Brownian motion. Let us define a Gaussian  $X(t) \equiv \ln(S(t))$ . From the Ito for-

mula in ksandal (1998),  $X(t)$  satisfies

$$X(t) = X(0) + (\mu - \sigma^2 / 2)t + \sigma W(t), \quad t > 0.$$

In the risk-neutral world, the value of the European option, which gives the holder the right to buy or sell the underlying asset at the expiration date, can be computed by the discounted conditional expectation of the terminal payoff,

$$V(x, t) = e^{-r(T-t)} E[\Lambda(X(T)) | X(t) = x],$$

where  $\Lambda(X(T))$  is the payoff at  $t = T$ . Without loss of generality, we denote again the risk neutral process to be  $X(t)$  with drift rate equal to the risk-free interest rate  $r$ , instead of  $\mu$  in (1). If we consider a continuous dividend yield  $q$ , the drift rate becomes  $r - q$ .

### 2.2 The Binomial Model

Let us partition the interval  $[0, T]$  into  $N$  cells of uniform length  $\Delta t = T / N$ ,  $0 = t_0 < t_1 < \dots < t_N = T$ . The binomial method by Cox *et al.* in Cox *et al.* (1979), Kwok (1998), Lyuu (2002) assumes that the asset price  $S(t_n)$  at  $t = t_n$  moves either up to  $uS(t_n)$  for  $u = \exp(\sigma\sqrt{\Delta t}) > 1$  or down to  $dS(t_n) = S(t_n)/u$  for  $\Delta t = t_{n+1} - t_n$  for  $n = 0, 1, \dots, N - 1$  with probabilities  $p = (\exp(r\Delta t) - d) / (u - d)$  or  $1 - p$ , respectively. Let  $X_j^n = X_0 + (2j - n)h$  denote the value at  $t = t_n = n\Delta t$  for, where  $X(0) = X_0$  and  $h = \ln u$ . Then the standard binomial method calculates the payoffs of the option at expiry,

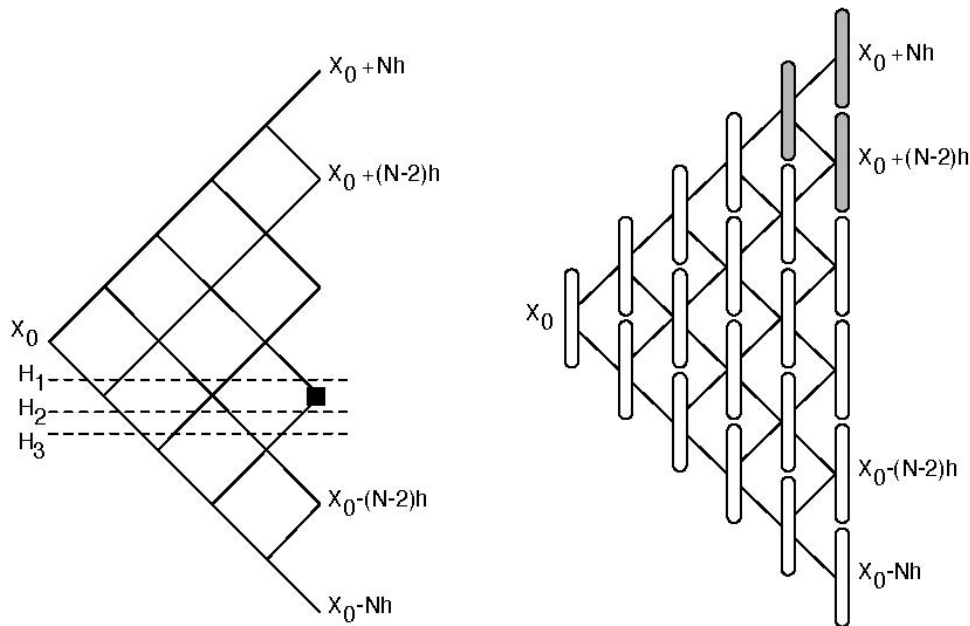


Figure 1. (Left) The tree of the standard binomial method and (Right) the tree of the binomial method using cell averages.

$V_j^N = \Lambda(X_j^N)$  for  $j = 0, 1, \dots, N$ , and computes the option price  $V_0^0 = V(X_0, 0)$  by backward averaging,

$$V(x, t) = e^{-r\Delta t} (pV(x+h, t+\Delta t) + (1-p)V(x-h, t+\Delta t)) \quad (2)$$

where  $x = X_j^n$ ,  $j = 0, \dots, n$ , and  $n = N-1, N-2, \dots, 0$

Let us point out a weakness of the standard binomial method using the European barrier option as an example. Barrier options are similar to European vanilla options except that the option is knocked out or in if the underlying asset price hits the barrier level before expiration date. As Derman *et al.* (1995) pointed out, there are two sources of inaccuracy in pricing options on a lattice. One type of error is caused by the existence of discrete lattice, which quantizes the underlying asset price and the instants in time at which it can be observed. Once a lattice is chosen, the underlying asset is allowed to take the values of only those points on the lattice and such a lack of continuity is called the quantization error. The other type of inaccuracy, called the specification error, occurs due to the inability of the lattice to represent the terms of the option accurately. For example, when the down-and-out European barrier call option is considered on a lattice in Figure 1 (Left), the available prices of the underlying are fixed. If the barrier level is given by  $H_1$ , the barrier at expiry does not coincide with one of the available prices but falls between two available nodes. Thus, the specified barrier is moved to the closest underlying price available (called the effective barrier), which is the value at the black square in case of the expiry. Also, whether the barrier level is given by  $H_2$  or  $H_3$  in Figure 1 (Left), the standard binomial method will result in the same option price while the exact price of the option won't. As explained in Boyle and Lau (1994) and Ritchken (1995), it is well known that the accuracy of the lattice method for the barrier option is poor when the barrier is not placed above a layer of nodes and that the binomial method requires a large number of time steps to ensure that the price error of a barrier option is small. An improved binomial method is suggested in Section 3.

### 3. IMPROVED BINOMIAL METHOD

The proposed method below will be derived using the same parameters as those for the standard binomial method. Let us first divide the interval  $[X_0 - (N+1)h, X_0 + (N+1)h]$  into  $N+1$  non-overlapping equidistant cells of length  $2h$  centered at points  $X_0 + jh$ ,  $j = -N, -N+2, \dots, N$ , and compute average option prices on each cell at  $t = t_N$ ,

$$\bar{V}_j^N \equiv \frac{1}{2h} \int_{X_{j-1/2}^N}^{X_{j+1/2}^N} \Lambda(\xi) d\xi, \quad j = 0, \dots, N, \quad (3)$$

where  $\Lambda(\cdot)$  is the payoff function at expiry and  $X_j^N = X_0 + (2j - N)h$ . If (2) is satisfied at every point  $\xi^j$  in the cell  $\xi \in [X_j^n - h, X_j^n + h]$  at time  $t_n$ , then the average option price  $\bar{V}_j^n = \frac{1}{2h} \int_{X_{j-1/2}^n}^{X_{j+1/2}^n} V(\xi, t_n) d\xi$  satisfies

$$\bar{V}_j^n = e^{-r\Delta t} (p\bar{V}_{j+1}^{n+1} + (1-p)\bar{V}_j^{n+1}) \quad (4)$$

because from the relation (2)

$$\begin{aligned} \bar{V}_j^n &= \frac{1}{2h} \int_{X_j^n-h}^{X_j^n+h} e^{-r\Delta t} \left( pV(\xi+h, t_n+\Delta t) + (1-p)V(\xi-h, t_n+\Delta t) \right) d\xi \\ &= e^{-r\Delta t} \left( \frac{p}{2h} \int_{X_j^n}^{X_j^n+2h} V(\xi, t_n+\Delta t) d\xi + \frac{1-p}{2h} \int_{X_j^n-2h}^{X_j^n} V(\xi, t_n+\Delta t) d\xi \right) \\ &= e^{-r\Delta t} (p\bar{V}(X_j^n+h, t_n+\Delta t) + (1-p)\bar{V}(X_j^n-h, t_n+\Delta t)) \end{aligned}$$

See Figure 1 (Right). That is, an equation similar to (2) is also satisfied for the cell averages. Thus, cell averages of the option values at expiry from (3) can be updated iteratively using (4), which eventually leads to the average of the option price  $\bar{V}_0^0$  on  $[X_0 - h, X_0 + h]$  at  $t = 0$ . Then we calculate  $V_0^0$  and Greeks on the valuation date from averaged values by applying the following Proposition 3.1.

**Proposition 3.1.** *If  $v(x)$  is smooth on an interval  $I = [m, M]$  and  $m < x_0 - 3h < x_0 < x_0 + 3h < M$  for some  $x_0 \in I$  and  $h > 0$ ,  $v(x_0)$  and  $v'(x_0)$  have the following approximate representations with respect to  $\bar{v}_j \equiv \frac{1}{2h} \int_{x_0+(j-1)h}^{x_0+(j+1)h} v(\xi) d\xi$  based on the compact scheme Lele (1992):*

$$v(x_0) = -\frac{1}{24} \bar{v}_{-1} + \frac{13}{12} \bar{v}_0 - \frac{1}{24} \bar{v}_1 \quad (5)$$

$$\text{and } v'(x_0) = \frac{\bar{v}_1 - \bar{v}_{-1}}{4h}.$$

*proof.* Suppose that a function  $f(x)$  is continuously differentiable on  $I$ . The compact scheme from Lele (1992) shows that

$$f_0' = \frac{9}{8} \frac{f_1 - f_{-1}}{2h} - \frac{1}{8} \frac{f_3 - f_{-3}}{6h} \quad (6)$$

and

$$f_0'' = -\frac{1}{8} \frac{f_1 - 2f_0 + f_{-1}}{h^2} + \frac{9}{8} \frac{f_3 - 2f_0 + f_{-3}}{9h^2} \quad (7)$$

are 4th and 2nd order accurate representations of  $f'(x_0)$  and  $f''(x_0)$ , respectively, where  $f_i = f(x_0 + ih)$ ,  $f_i' = f'(x_0 + ih)$  and  $f_i'' = f''(x_0 + ih)$ . Let  $f(x) \equiv \int_{x_0-3h}^x v(\xi) d\xi$ . Then,  $f(x_0 + h) - f(x_0 - h) = 2h\bar{v}_0$  and  $f(x_0 + 3h) - f(x_0 - 3h) = 2h(\bar{v}_{-1} + \bar{v}_0 + \bar{v}_1)$ . Since  $f'(x_0) = v(x_0)$ , the equation (6) for the first derivative implies

$$v(x_0) = \frac{9}{8} \left( \frac{2h\bar{v}_0}{2h} \right) - \frac{1}{8} \left( \frac{2h(\bar{v}_{-1} + \bar{v}_0 + \bar{v}_1)}{6h} \right) \\ = \frac{9\bar{v}_0}{8} - \frac{\bar{v}_{-1} + \bar{v}_0 + \bar{v}_1}{24} = -\frac{1}{24} \bar{v}_{-1} + \frac{13}{12} \bar{v}_0 - \frac{1}{24} \bar{v}_1.$$

In a similar way, the equation (7) leads to the following representation for  $v'(x_0)$ ,

$$v'(x_0) = \frac{1}{h} \left( -\frac{1}{4} (2\bar{v}_{-1} + \bar{v}_0) + \frac{1}{4} (\bar{v}_{-1} + \bar{v}_0 + \bar{v}_1) \right) \\ = \frac{\bar{v}_1 - \bar{v}_{-1}}{4h}.$$

Equations (5) in Proposition 3.1 motivate the following approximations for the option price and the Delta ( $\Delta$ , sensitivity measure of option value with respect to changes in the underlying asset's price)

$$V(X_0, 0) = -\frac{1}{24} \bar{V}_{-1}^0 + \frac{13}{12} \bar{V}_0^0 - \frac{1}{24} \bar{V}_1^0 \quad (8)$$

$$\text{and } \Delta(X_0, 0) = \frac{\bar{V}_1^0 - \bar{V}_{-1}^0}{4h}.$$

**Algorithm 1** Binomial Tree method using Cell Averages (BTCA)

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for  $j = -1$  to  $N + 1$  do
    compute the cell averages at expiry,  $\bar{V}_j^N$  in (3)
end for
for  $n = N - 1$  to  $0$  do
    for  $j = -1$  to  $n + 1$  do
        compute the backward iteration for  $\bar{V}_j^n$  in (4)
    end for
end for
compute the option price and the Delta in (8)
perform Richardson extrapolation
    
```

In fact, the price evaluated by the binomial method may not be of class  $C^1$  in  $x$ , but the simulations show that these approximations work well. Then Richardson extrapolation from Richardson (1927) is used to improve the accuracy. The proposed algorithm of the Binomial Tree method using Cell Averages (termed BTCA) for the European vanilla option can be summarized as in Algorithm 1. Note that when cell averages are taken, the specification error seems to cancel out, and this may help the BTCA reduce total errors.

### 3.1 Barrier Options

Let us apply the BTCA method to the European barrier option as a benchmark test, then extend it to the American barrier option. Even though the proposed scheme is tested on barrier options, it can be applied to other types of options as well. When the backward iteration is performed for the barrier option, it is worthwhile to point out that (4) can be used in all cells but one. In fact, when a cell contains the barrier, note that the assumption for (4) is not satisfied and we need to modify it. Equations (9) and (10) below show an example for the European down-and-out barrier call option and similar modification can be easily derived for other types of options.

Given a cell  $[x - h, x + h]$  at  $t$ , if  $x - h < H < x$ , the average option price of European down-and-out barrier call option satisfies

$$\bar{V}_j^n = e^{-r\Delta t} \left\{ p \frac{x + h - H}{2h} \bar{V}_{j+1}^{n+1} + (1 - p) \bar{V}_j^{n+1} \right\} \quad (9) \\ + \frac{(H - x + h)R}{2h} (1 - e^{-r\Delta t} (1 - p)),$$

with a cash rebate  $R \geq 0$ . If  $x \leq H < x + h$ ,  $\bar{V}_j^n$  satisfies

$$\bar{V}_j^n = \frac{(H - x + h)R}{2h} \quad (10) \\ + e^{-r\Delta t} \frac{x + h - H}{2h} (p \bar{V}_{j+1}^{n+1} + (1 - p)R).$$

In fact, approximations (9) and (10) are motivated by the following reasoning. For  $x - h < H < x$ , since  $V(x, t) = R$  for  $x \leq H$ ,

$$\int_{x-h}^{x+h} V(\xi, t) d\xi = \int_{x-h}^H V(\xi, t) d\xi + \int_H^{x+h} V(\xi, t) d\xi \\ = R(H - x + h) \\ + e^{-r\Delta t} \left( p \int_{H+h}^{x+2h} V(\xi, t + \Delta t) d\xi \right. \\ \left. + (1 - p) \int_{H-h}^x V(\xi, t + \Delta t) d\xi \right).$$

If  $h$  is sufficiently small, the variation of  $V$  can be assumed to be small and the first integral inside the parenthesis has the approximation

$$\int_{H+h}^{x+2h} V(\xi, t + \Delta t) d\xi \approx \frac{x+h-H}{2h} \int_x^{x+2h} V(\xi, t + \Delta t) d\xi.$$

The second integral can be computed by

$$\begin{aligned} & \int_{H-h}^x V(\xi, t + \Delta t) d\xi \\ &= \int_{x-2h}^x V(\xi, t + \Delta t) d\xi - \int_{x-2h}^{H-h} V(\xi, t + \Delta t) d\xi \\ &= \int_{x-2h}^x V(\xi, t + \Delta t) d\xi - R(H-x+h), \end{aligned}$$

which results in (9). Similarly for  $x < H < x+h$ ,

$$\begin{aligned} & \int_{x-h}^{x+h} V(\xi, t) d\xi \\ &= \int_{x-h}^H V(\xi, t) d\xi + \int_H^{x+h} V(\xi, t) d\xi \\ &= R(H-x+h) \\ &+ \int_H^{x+h} e^{-r\Delta t} \left( pV(\xi+h, t + \Delta t) + (1-p)V(\xi-h, t + \Delta t) \right) d\xi. \end{aligned}$$

Thus, we have (10) for small  $h$ , using

$$\begin{aligned} & \int_H^{x+h} V(\xi+h, t + \Delta t) d\xi \\ & \approx \frac{x+h-H}{2h} \int_x^{x+2h} V(\xi, t + \Delta t) d\xi. \end{aligned}$$

### 3.2 American Options

The American option allows early exercise of the option and the update algorithm needs appropriate modification. In case of the American put option, at each time  $t = t_n$ , there is a unknown boundary  $S_n^* = S_n^*(t_n)$  such that

$$\max\{V(x, t_n), \Lambda(x, t_n)\} = \begin{cases} V(x, t_n) & \text{if } S_n^* \leq x \\ \Lambda(x, t_n) & \text{if } x \leq S_n^* \end{cases},$$

where  $V(x, t_n)$  denotes the option price from the backward process of the binomial method and  $\Lambda(x, t_n)$  denotes the exercise price. Thus, we have

$$\begin{aligned} & \int_{X_j^n-h}^{X_j^n+h} \max\{V(\xi, t_n), \Lambda(\xi, t_n)\} d\xi \\ &= \begin{cases} \int_{X_j^n-h}^{X_j^n+h} V(\xi, t_n) d\xi & \text{if } S_n^* \leq X_j^n-h \\ \int_{X_j^n-h}^{X_j^n+h} \Lambda(\xi, t_n) d\xi & \text{if } X_j^n+h \leq S_n^* \\ \int_{X_j^n-h}^{S_n^*} \Lambda(\xi, t_n) d\xi + \int_{S_n^*}^{X_j^n+h} V(\xi, t_n) d\xi & \text{if } X_j^n-h < S_n^* < X_j^n+h \\ \max\left\{ \int_{X_j^n-h}^{X_j^n+h} V(\xi, t_n) d\xi, \int_{X_j^n-h}^{X_j^n+h} \Lambda(\xi, t_n) d\xi \right\} & \text{if } S_n^* \leq X_j^n-h \text{ or } X_j^n+h \leq S_n^* \\ \int_{X_j^n-h}^{S_n^*} \Lambda(\xi, t_n) d\xi + \int_{S_n^*}^{X_j^n+h} V(\xi, t_n) d\xi & \text{if } X_j^n-h < S_n^* < X_j^n+h \end{cases} \end{aligned}$$

Then, we can show that the error,

$$\begin{aligned} & \int_{X_j^n-h}^{X_j^n+h} \max\{V(\xi, t_n), \Lambda(\xi, t_n)\} d\xi \\ & - \max\left\{ \int_{X_j^n-h}^{X_j^n+h} V(\xi, t_n) d\xi, \int_{X_j^n-h}^{X_j^n+h} \Lambda(\xi, t_n) d\xi \right\}, \end{aligned}$$

is identically 0 if the cell  $[X_j^n-h, X_j^n+h]$  does not contain  $S_n^*$ , and that the error is not 0 but converges to 0 as  $h$  decreases to 0 if the cell contains  $S_n^*$ . Thus, if

$$\bar{\Lambda}_j^n \text{ denotes } \bar{\Lambda}_j^n = \frac{1}{2h} \int_{X_j^n-h}^{X_j^n+h} \Lambda(\xi, t_n) d\xi,$$

we need to add the update procedure,

$$\bar{V}_j^n = \max\{\bar{V}_j^n, \bar{\Lambda}_j^n\}, \tag{11}$$

at the end of each time step in Algorithm 1 to derive an algorithm for the American option.

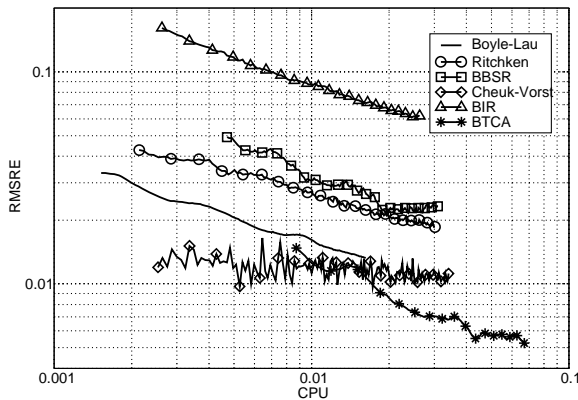
## 4. SIMULATIONS

We use a random sample of 5,000 options following the same distributions as those used by Broadie and Detemple(1996): risk free interest rate  $r$  is uniform between 0 and 0.1; volatility  $\sigma$  is uniform between 0.1 and 0.6; strike  $K$  is uniform between 70 and 130; time to maturity (years)  $T$  is uniform between 0.1 and 1 with probability 0.75 and uniform between 1 and 5 with probability 0.25; The initial price of the underlying  $S_0$  is 100. We consider the European down-and-out barrier call option whose barrier  $H$  is uniform between 55 and 85,

and the American up-and-out barrier put option whose barrier  $H$  is uniform between 115 and 145. The rebate  $R$  for both barrier options is  $R = 0$ . The exact formula in Haug (1997) and Reimer and Rubinstein (1991) is used for the European option. For the American option, the benchmark price and hedging parameters are computed by the binomial model with 20,000 steps. The error measure we use is the root mean squared relative error (RMSRE),

$$\text{RMSRE} = \sqrt{\frac{1}{n} \sum_{i=1}^n e_i^2},$$

Where  $e_i = (\hat{V}_i - V_i) / V_i$  is the relative error,  $V_i$  is the true value, and  $\hat{V}_i$  is the computed value. The summation is taken over options satisfying  $V_i \geq 0.5$  and  $H \leq K$  for European options ( $K \leq H$  for American options) as in (Broadie and Detemple 1996). Out of 5,000 options, 4,474 European options and 4,225 American options satisfied the criterion.

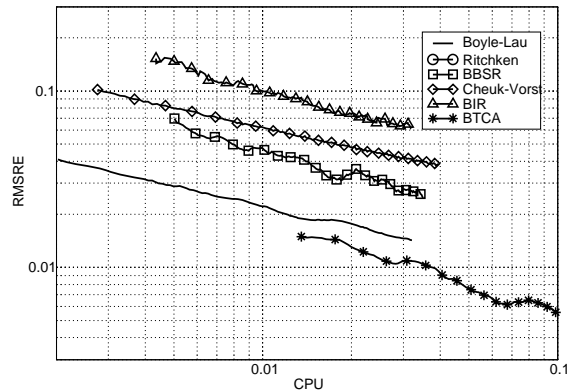


**Figure 2.** The RMSRE of the option price vs the computational time from the Boyle-Lau (solid), Ritchken (circle), BBSR (square), Cheuk-Vorst (rhombus), BIR (triangle) and BTCA (star) methods for the European down-and-out barrier call option.

Figure 2 shows the accuracy of the option price measured by RMSRE vs the computational cost (measured by the CPU time, obtained with Intel Core 2 Duo processor at 2.4 GHz). Some papers consider the convergence error of the computational scheme in terms of the number of time steps. Since different schemes take differ-

ent computation time even with the same number of time steps as seen in Table 1 below, the CPU time is used as the measure of the computational cost in this study. The result of BTCA method is compared to the binomial method by Boyle and Lau (1994), the trinomial method by Ritchken (1995), the BBSR method by Broadie and Detemple (1996), another trinomial method by Cheuk and Vorst (1996), and the BIR method by (Gaudenzi and Pressacco 2003). When the number of time step is small so that the CPU time is small, both the trinomial method by Cheuk and Vorst and the proposed BTCA method result in smallest errors. As the number of time steps increases, superiority of the proposed method can be observed.

Table 1 compares the error with respect to the number of time steps. When the number of time steps  $N = 100$ , the trinomial method by Cheuk and Vorst and the proposed BTCA method result in smallest errors as pointed out above. For  $N \geq 200$ , the proposed BTCA method results in the smallest error.

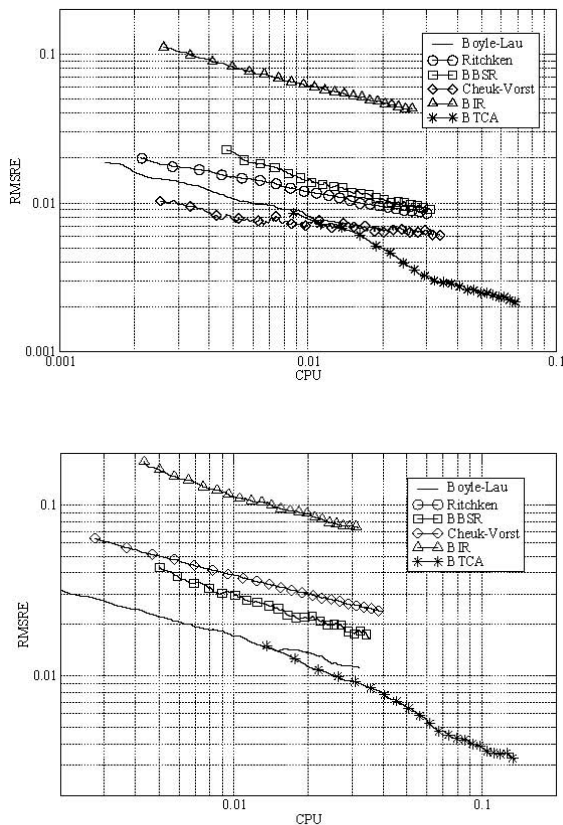


**Figure 3.** The RMSRE of the option price vs the computational time from the Boyle-Lau (solid), Ritchken (circle), BBSR (square), Cheuk-Vorst (rhombus), BIR (triangle) and BTCA (star) methods for the American up-and-out barrier put option.

Figure 3 shows the accuracy for the American up-and-out barrier put option. For the American option, the method by Boyle and Lau shows quite good result but still the proposed BTCA method results in far better performance than the other schemes including even the Boyle-Lau scheme.

**Table 1.** The RMSRE of the option price for the European down-and-out barrier call option as the number of steps increases. The numbers in the parenthesis are the CPU time in seconds.

N	Boyle-Lau	Ritchken	BBSR	Cheuk-Vorst	BIR	BTCA
100	0.03328(0.00152)	0.04278(0.00214)	0.04919(0.00470)	0.01197(0.00254)	0.16077(0.00262)	0.01476(0.00873)
200	0.02412(0.00327)	0.03441(0.00502)	0.03839(0.00791)	0.00998(0.00583)	0.11419(0.00540)	0.00960(0.01770)
300	0.01978(0.00536)	0.02856(0.00866)	0.02922(0.01162)	0.01123(0.01000)	0.09140(0.00873)	0.00699(0.02728)
400	0.01701(0.00779)	0.02354(0.01300)	0.02819(0.01591)	0.01225(0.01491)	0.08109(0.01253)	0.00689(0.03717)
500	0.01554(0.01046)	0.02120(0.01815)	0.02264(0.02064)	0.01065(0.02071)	0.07216(0.01686)	0.00584(0.04760)
600	0.01414(0.01342)	0.01965(0.02419)	0.02191(0.02583)	0.01030(0.02740)	0.06503(0.02162)	0.00558(0.05835)



**Figure 4.** The RMSRE of Delta vs the computational time from the Boyle-Lau (solid), Ritchken (circle), BBSR (square), Cheuk-Vorst (rhombus), BIR (triangle) and BTCA (star) methods for (Left) the European down-and-out barrier call option and (Right) the American up-and-out barrier put option.

Figure 4 compares the results for Delta. Note that similar results to those for the option price are obtained and that the errors from the proposed BTCA method are smaller than those from the other schemes in all cases.

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## 6. CONCLUSIONS

A new method for option pricing based on the binomial method is introduced. Instead of calculating option prices at each node of the binomial tree, the proposed scheme estimates the cell average of prices around each node similarly to the finite volume method

in fluid dynamics. The option price and Delta are then obtained by the compact scheme and Richardson extrapolation. Cell averaging reduces errors due to over- and under-estimation of parameter values for the standard lattice method and thus improves the accuracy. In-depth simulation leads to the consistent results.

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