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Restrictions on the Entries of the Maps in Free Resolutions and SC_r-condition

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Abstract

We discuss an application of 'restrictions on the entries of the maps in the minimal free resolution' and 'SC_r-condition of modules', and give an alternative proof of the following result of Foxby: Let M be a finitely generated module of dimension over a Noetherian local ring (A,m). Suppose that $\hat{A}$ has no embedded primes. If A is not Gorenstein, then $\mu_i(m,A) \geq 2$ for all $i \geq \dim A$.

Key words : Bass Numbers, SC_r-condition, Cohen-Macaulay Rings; Gorenstein Rings, Minimal Injective Resolution

1. Introduction

All rings (A,m) we consider in this paper are assumed to be Noetherian local, and all modules are assumed to be unital.

Hochster ([3]) dened the notion for the nite module to have small conite ir-reducibles: We say that a nite A-module M has small conite irreducible (*SCI*) if for each $q \geq 1$, there is an irreducible submodule N_q of M such that $N_q \subseteq m^q M$ and $l(M/N_q) < \infty$. He also called a ring A an approximately Gorenstein ring if A has *SCI*.

Melkersson ([6]) extended the denition of having *SCI* to modules, which are not assumed to be nite, and then gave generalizations of Hochster's results (see [3]). It is known that the following three conditions are equivalent: (a) M has *SCI*, (b) for each $q \geq 1$, there is a submodule N_q of M such that $N_q \subseteq m^q M$ and M/N_q is embedded in E (E denotes the injective hull of the residue eld A/m), and (c) for each $f \in Hom_A(M,E^t)$ $(t \geq 1)$, there is $g \in Hom_A(M,E)$ such that $Ker\ g \subseteq Ker\ f$.

Using the third condition (c), may be naturally extended as follows:

Denition. ([4]) Let (A,m) be a Noetherian local ring and M an A-module (not necessarily nitely generated). Let $r \geq 1$ be an integer. M is said to have the condition SC_r if for $f \in Hom_A(M,E^t)$ $(t \geq 1)$, there is $g \in Hom_A(M,E^r)$ such that $Ker\ g \subseteq Ker\ f$.

In Section 1, we study the properties of the SC_r-condition dened above. Recently, it was proved by Koh and Lee ([5]) that there are certain restrictions on the entries of the maps in the minimal free resolutions of finitely generated modules of innite projective dimension over Noetherian local rings. From these restrictions, some previously known results in commutative ring theory are slightly improved; for examples, Herzog's extension of Kunz's result to a characterization of modules of nite projective and injective dimensions in characteristic $p > 0$ (see [5, Corollary 2.8]), and Eisenbud's and Dutta's results on the existence of free summands in syzygy modules ([see [5, Proposition 2.2]). The new proof of Foxby's result could be another good application of these restrictions.

Using the properties of the SC_r-condition and the restrictions on the maps in the minimal free resolutions, we prove the fact: for a 1-dimensional local domain (A,m,k), if φ: $K^s \rightarrow E(k)^r$ is an A-linear homomorphism with a finitely generated kernel where K is a field of quotients of A, then $s \leq r$. From this result, we have an alternative proof of Foxby theorem: If $ht(q/p)=1$, *then* $\mu_s(p,M) \leq \mu_i(q,M)$ where $\mu_i(p,M)$ is i-th Bass number of M_p. At the end of this article, we also prove, in a slightly different way, the known theorem that if a completion $\hat{A}$ of A has no embedded primes, then $\mu_i(m,A) \neq 1$ for all $i > d$.

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1. SC_r Condition and Applications

In this section, we study the notion of small conite irreducible condition (SC_r) dened by Koh, and the restriction on the maps of a minimal free resolution (see [5]) to give their application.

We restate the denition of the condition (SC_r) stated in introduction, and give the proposition proved by Koh ([4]).

Denition 1.1. ([4]) Let (A,m) be a Noetherian local ring and M an A-module (not necessarily finitely generated). Let $r \geq 1$ be an integer. M is said to have the condition (SC_r) if for each $f \in Hom_A(M,E^t)$ $(t \geq 1)$, there is $g \in Hom_A(M,E^r)$ such that $Ker g \subseteq Ker\ f$. (E denotes the injective hull of the residue field A/m.)

Proposition 1.2. ([4, Proposition 1.5.(i)]) Let (A,m) be a Cohen-Macaulay local ring. Suppose A has an ideal isomorphic to the canonical module of A. If A^s satises (SC_r), then for each $q \geq 1$, there is a homomorphism j: $A^s \rightarrow A^r$ such that $Ker\ \varphi_q \subseteq m^q A^s$.

It was proved by Koh and Lee that there are certain restrictions on the entries of the maps in the minimal free resolutions of finitely generated modules of infinite projective dimension over Noetherian local rings A. We state the fact in a slightly different way, which is easy to quot.

Theorem 1.3. ([5, Theorem 1.7]) Let (A,m) be a Cohen-Macaulay local ring of dimension d. There is a positive integer $\rho = \rho(A)$ such that if $(F_\cdot,\delta_\cdot)$ is a minimal free resolution of a finitely generated module M of infinite projective dimension, then $Ker \delta_i \not\subset m^\rho F_i$ for all $i \geq d$.

Using the above theorem, Corollary 1.4 below was also obtained (see [4, Remark 1.6]), but we give its proof for reader's convenience.

Corollary 1.4. Let A be as in Proposition 1.2. If dim $A = 1$, *then* $s \leq r$.

Proof. For each $q \geq 1$, there is a map $\varphi_q : A^s \rightarrow A^r$ such that $Ker \varphi_q \not\subset m^q A^s$ by Proposition 1.2, and each φ_q may be the first map of a minimal free resolution of Coker φ_q since dim $A = 1$. For $\rho = \rho(A)$ in Theorem 1.3, we must have $Ker\ \varphi_p = 0$. If not, Coker φ_p is of infinite projective dimension and thus $Ker \varphi_\rho \not\subset m^\rho A^S$ by Theorem 1.3, which is a contradiction. Hence φ_p is injective, and so $s \leq r$.

Theorem 1.5. Let (A,m,k) be a 1-dimensional local domain, and φ: $K^s \rightarrow E(k)^r$ be an A-linear homomorphism with a finitely generated kernel, where K is a field of quotients of A. Then we have $s \leq r$.

Proof. Since $Ker\ \varphi$ is finitely generated, we have $Ker \varphi \subseteq \frac{1}{x} A^s K^s$ for some x in m. For each $q \geq 1$, let us define g_q by a composite of maps

$$g_q\colon\ A^s \xrightarrow{\sim} \frac{1}{x^{q+1}} A^s \hookrightarrow K^s \xrightarrow{\varphi} E(k)^r$$

Then $Ker g_q \subseteq m^q A^s$ f since $Ker \varphi \subseteq \frac{1}{x} A^s$. To show that A^s satisfies (SC_r), let $0 \neq f \in Hom_A(A^s,E^t)$ for a positive integer t.

Since $f(A^s) \supseteq mf(A^s) \supseteq m^2 f(A^s) \supseteq \cdots$ is a descending chain in a finitely generated Artinian module $f(A^s)(\subseteq E^t)$, there is a positive integer q such that $m^q f(A^s) = 0$ by Nakayama lemma, which implies $m^q A^s \subseteq Ker f$, i.e., $Ker\ g \subseteq Ker\ f$. Thus A^s satisfies (SC_r).

Since A^s satisfies (SC_r) if and only if $\hat{A}^s$ satisfies (SC_r) (see [4]), we may assume that A is complete so that A has a canonical module. Since A is a domain, A is generically Gorenstein, i.e., A_p is Gorenstein for all minimal prime ideals p of A. Therefore, A has an ideal isomorphic to the canonical module of A (see [1, Proposition3.3.18]), and thus $s \leq r$ by Corollary 1.4.

We recall the definition of Bass numbers of modules: Let (A,m,k) be a Noetherian local ring and M a finite A-module of dimension d. For a prime ideal p, the i-th Bass number of M at p is defined by $\dim_k Ext^i_{A_p}(k(p),M_p)$, and denoted $\mu_i(p,M)$. In particular, the d-th Bass number $\mu_d(m,M)$ is called the type of M.

It is known that $\mu_i(p,M)$ is equal to the number of $E(A/p)$ copies of which appear in I^i as a direct summand, where $I^\bullet$ is the minimal injective resolution of M and $E(A/p)$ denote the injective hull of A/p, i.e., if

$$(I^\bullet,\phi^\bullet) : 0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \rightarrow I^i \rightarrow \cdots,$$

is a minimal injective resolution of M, then each I^i is $\oplus_{p \in \mathrm{Supp}(M)} E(A/p)^{\mu_i(p,M)}$.

Using Theorem 1.5, we can get the old theorem of

Foxby in a diffrent way.

Corollary 1.6. ([2]) Let M be a finitely generated A-module. Suppose $ht(q/p) = 1$ for prime ideals p and q. Then we have

$$\mu_i(p,M) \leq \mu_{i+1}(q,M)$$

We close this section by giving the alternative proof of another Foxby theorem using Local Duality, hoping that the proof is shorter if not any simpler, even though the suggested proof is similar to its original proof.

First we prove the following:

Proposition 1.7. Let (A,m,k) be a Noetherian local ring of dimension 0. If $\mu_i(m,A) \leq 1$ for some $i \geq 0$, then A is Gorenstein.

Proof. $\mu_i(m,A) = 0$ implies that A is Gorenstein since injdim A = sup {i : $\mu_i(m,A) \neq 0$ }. We assume that $\mu_i(m,A) = 1$. Let $(I^\bullet,\phi^\bullet)$ be a minimal injective resolution of A;

$$(I^\bullet,\phi^\bullet): 0 \to A \to E^{\mu_0} \xrightarrow{\phi^0} E^{\mu_1} \to \cdots \to E^{\mu_{i-1}} \xrightarrow{\phi^{i-1}} E \xrightarrow{\phi^i} E^{\mu_{i+1}} \to \cdots$$

where $E = E(A/m)$, the injective hull of A/m. Then we have an exact sequence, $0 \to A \to E^{\mu 0} \to \cdots \to E^{\mu_{i-1}} \to Im\phi_{i-1} \to 0$. Since $l(Im\phi_{i-1}) \leq l(E) = l(E^\vee) = l(A) < \infty$, we have $l(A) - l(E^{\mu_0}) + \cdots + (-1)^i l(Im\phi_{i-1}) = 0$, and hence $l(Im\phi_{i-1}) = tl(E)$ for some nonnegative integer t. This implies that $Im\phi_{i-1}$ is either E or zero-module. Either case says that $I^\bullet$ is a finite resolution. Hence A is Gorenstein.

The following lemma is well-known, and we omit its proof.

Lemma 1.8. Let M_i be A-modules such that $l(M_i) < \infty$ for all $0 \leq i \leq s$. For a complex of A-modules $C.$: $0 \to M_s \to \cdots \to M_0 \to 0$,

$$\sum_{i=0}^{s}(-1)^i l(M_i) = \sum_{i=0}^{s}(-1)^i l(H_i(C.))$$

Finally, we prove the second theorem in the abstract: we recall that a ring A is said to have no embedded primes if every associated prime is a minimal prime.

Theorem 1.9. ([2]) Let (A,m,k) be a Noetherian local ring of dimension d. Suppose that $\hat{A}$ has no embedded primes. Then $\mu_i(m,A) \neq 1$ for all $i \geq d$. Equivalently, if A is not Gorenstein, then $\mu_i(m,A) \geq 2$ for all $i \geq d$.

Proof. We may assume that A is complete since $\mu_i(m) = \mu_i(\hat{m})$. Suppose to the contrary that $\mu_i(m) = 1$ for some $i \geq d$. It is true by Proposition 1.7 that A_p is Gorenstein for all minimal prime ideals p since $i\text{-}ht(m/p) > 0 = \dim A_p$ and $\mu_{i-ht(h/p)}(A_p) \leq 1$ by Corollary 1.6.

Let $I^\bullet$ be a minimal injective resolution of A. Consider a complex of free modules $Hom_A(H_m^0(I^\bullet)E(A/m))$, simply denoted by $F.$:

$$F_\bullet : \cdots \to A^{\mu_{i+1}(m)} \xrightarrow{g} A \xrightarrow{f} A^{\mu_{i-1}(m)} \to \cdots \to A^{\mu_1(m)} \to A^{\mu_0(m)} \to 0$$

We note that the i-th homology of $F.$ is $H_m^i(A)^\vee$, where $(-)^\vee$ denotes the Matlis dual. Since A is complete, $A \cong S/J$ such that (S,m_s) is a Gorenstein local ring, J is an ideal of S and $\dim S = \dim A$. Hence by local duality

$$H_m^i(A)^\vee \cong H_{ms}^i(A)^\vee \cong Ext_s^{d-i}(A,S)$$

For a minimal prime ideal p of A, since A_p is Gorenstein (and hence Cohen-Macaulay) and S_p is Gorenstein, it follows that

$$\begin{aligned} Ext_s^j(A,S) \otimes S_p &\cong Ext_{S_p}^j(A_p,S_p) = 0 \quad \text{if } j \neq \dim S_p - \dim A_p = \dim S_p \\ &\cong \omega_{A_p} \quad \text{if } j = \dim S_p \end{aligned}$$

where ω_{A_p} is a canonical module of A_p. Since A_p is Gorenstein, its canonical module ω_p is isomorphic to A_p. Thus it is obtained that

$$(*) \quad \begin{aligned} H_i(F_\bullet \otimes A_p) &= 0 \quad \text{if } i \neq d - \dim S_p \\ &\cong A_p \quad \text{if } i = d - \dim S_p, \end{aligned}$$

since

$$H_i(F_\bullet \otimes A_p) \cong H_i(F_\bullet) \otimes A_p \cong H_m^i(A)^\vee \otimes A_p \cong Ext_S^{d-i}(A,S) \otimes S_p \cong Ext_{S_p}^{d-i}(A_p,S_p)$$

We will show that either $g = 0$ or $f = 0$. If then, we have a minimal exact sequence $\cdots \to A^{\mu_{i+2}(m)} \to A^{\mu_{i+1}(m)} \xrightarrow{g} 0$, or $\cdots \to A^{\mu_{i+1}(m)} \xrightarrow{g} A \xrightarrow{f} 0$, which is a contradiction to the minimality of sequence.

Let p be a minimal prime ideal and $t = d\text{–}\dim S_p$ for simplicity. Consider the following complex:

$$G_{\bullet} : 0 \to A_p \xrightarrow{f_p} A_p^{\mu_{i-1}(m)} \to \cdots \xrightarrow{\varphi_{t+1}}$$
$$A_p^{\mu_t(m)} \xrightarrow{\varphi_t} \cdots \to A_p^{\mu_0(m)} \to 0$$

Note that $H_i(G_{\bullet}) = Ker\, f_p = Im\, g_p$. By Lemma 1.8 and $(*)$,

$$(**) \quad \sum_{j=0}^{i} (-1)^j \mu_j(m) \ell(A_p)$$
$$= (-1)^i \ell(H_i(G_{\bullet})) + (-1)^t \ell(H_t(G_{\bullet}))$$
$$= (-1)^i \ell(Im\, g_p) + (-1)^t \ell(A_p)$$

which implies that $l(Im\, g_p)$ is a non-negative integer multiple of $l(A_p)$, and thus $Im\, g_p = A_p$, or since $Im\, g_p \subseteq A_p$. Therefore, either $f_p = 0$ or $g_p = 0$. We claim that either $f_p = 0$ for all minimal primes p, or $g_p = 0$ for all minimal primes p. This is clear from $(**)$ since $\sum_{j=1}^{i} (-1)^j \mu_j(m)$ is constant, and so is independent on primes p. Since Min(A) = Ass(A) by assumption, we have either $g = 0$ or $f = 0$. This completes the proof.

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