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Certain Polynomials Related to Chebyshev Polynomials

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Abstract

Bae and Kim displayed a sequence of 4th degree self-reciprocal polynomials whose maximal zeros are related in a very nice and far from obvious way. The auxiliary polynomials in their results that parametrize their coefficients are of significant independent interest. In this note we show that such auxiliary polynomials are related to Chebyshev polynomials.

Key words : Polynomials, Chebyshev Polynomials, Sequences.

There are infinitely many sequences of monic integral polynomials $\{p_n(z)\}$ whose largest (in modulus) zero of $p_{k+1}(z)$ is $\alpha\beta$ where α and β are the first two largest (in modulus) zeros of $p_{k+1}(z)$. An example is taken when $p_1(z)$ is the minimal polynomial of a Salem number in which case we can take $p_k(z)=p_1(z)$ for all $k \geq 1$ because a Salem number is a real algebraic integer >1 all of whose conjugates lie inside or on the unit circle, and at least one of these conjugates has modulus exactly . It is known that there are infinitely many Salem numbers. It does not seem obvious how to find such a sequence of distinct polynomials each of which has the same degree. Bae and Kim[1] displayed a sequence of th degree polynomials whose maximal zeros are related in a very nice and far from obvious way. In fact, they proved.

Theorem 1. For each real number $n > 6$, there is a sequence $\{p_k(n,z)\}_{k=1}^{\infty}$ of fourth degree self-reciprocal polynomials such that the zeros of $p_k(n, z)$ are all simple and real, and every $p_{k+1}(n, z)$ has the largest (in modulus) zero $\alpha\beta$ where α and β are the first and the second largest (in modulus) zeros of $p_k(n, z)$, respectively. One such sequence is given by $p_k(n, z)$ so that

$$p_k(n,z) = z^4 - q_{k-1}(n)z^3 + (q_k(n)+2)z^2 - q_{k-1}(n)z + 1\ ,$$

Where $q_0(n)$=1 and other $q_k(n)$'s are polynomials in n defined by the severely nonlinear recurrence

$$4q_{2m-1}(n) = q_{2m-2}^2(n) - (4n+1)\prod_{j=0}^{m-2} q_{2j}^2(n)$$

$$4q_{2m}(n) = q_{2m-1}^2(n) - (n-2)(n-6)\prod_{j=0}^{m-2} q_{2j+1}^2(n)$$

for $m \geq 1$, with the usual empty product conventions, i.e. $\prod_{j=0}^{-1} b_j 1$.

For notational convenience, write q_k instead of $q_k(n)$ in the theorem. The auxiliary polynomials q_k that parametrize their coefficients are of significant independent interest and they satisfy good recurrence relations (see Lemma 3 (i) of [1])

$$q_{k+1} = q_{k-1}^2 - 2q_k - 4\ ,\quad k \geq 1 \tag{1}$$

The purpose of this note is to show that the polynomials q_k are some sort of Chebyshev polynomials. For this, we define the following.

Definition 2. Define

$$H(n)\text{: } 2T_{2^n}\left(\frac{1}{2}\sqrt{4-x}\right),$$

Where $T_n(x)$ is the th degree Chebyshev polynomial of

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the first kind.

Using an identity $T_{2n}(x)=2T_n^2(x)-1$ followed from double angle formula, we may compute

$$H(n+1)=H(n)^2-2 \tag{2}$$

One can then verify

Proposition 3. The sequence

$$H(1)^2,\ 2H(2),\ H(2)^2,\ 2H(3),\ H(3)^2,\ 2H(4),... \tag{3}$$

Comes from the recurrence

$$Q_{k+1}=Q_{k-1}^2-2Q_k-4 \tag{4}$$

With initial conditions

$$\{Q_0,\ Q_1\}=\{0,-x\}$$

Where the sequence (3) begins with Q_3.

Proof. We use induction on $k\geq 3$. For $k=3$,

$$Q_3=Q_1^2-2Q_2-4=x^2-2(2x-4)-4=(x-2)^2$$

and

$$H(1)=2T_2\left(\frac{1}{2}\sqrt{4-x}\right)=2\left[2\left(\frac{1}{2}\sqrt{4-x}\right)^2-1\right]=2-x.$$

So $H(1)^2=Q_3$. Assume the result holds for $3,4,...,k$, i.e., Qj($3\leq j\leq k$) corresponds to

$$\begin{cases}2H(j/2), & j \text{ even},\\ H^2((j-1)/2), & j \text{ odd},\end{cases}$$

If is even, by (2)

$$\begin{aligned}Q_{k+1}&=Q_{k-1}^2-2Q_k-4=H^4(k/2-1)-2\cdot 2H(k/2)-4\\ &=H^4(k/2-1)-4[H^2(k/2-1)-2]-4\\ &=(H^2(k/2-1)-2)^2\\ &=H^2(k/2)\end{aligned}$$

and if k is odd, by (2)

$$\begin{aligned}Q_{k+1}&=Q_{k-1}^2-2Q_k-4=(2H((k+1)/2-1))^2\\ &\quad -2H^2((k+1)/2-1)-4\\ &=2[H^2((k+1)/2-1)-2]\\ &=2H((k+1)/2),\end{aligned}$$

Which completes the proof.

Remark The recurrence (4) for polynomials Q_k is in the same form as (1) for polynomials q_k. Proposition 3 asserts that the polynomials q_k are some sort of Chebyshev polynomials.

Aknowlegement

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References

[1] J. Bae and S. H. Kim, “Self-reciprocal polynomials with related maximal zeros”, in revision.