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ON SPECIAL CONFORMALLY FLAT SPACES WITH WARPED PRODUCT METRICS

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Abstract. In 1973, B. Y. Chen and K. Yano introduced the special conformally flat space for the generalization of a subprojective space. The typical example is a canal hypersurface of a Euclidean space. In this paper, we study the conditions for the base space B to be special conformally flat in the conharmonically flat warped product space $B^n \times_f R^1$. Moreover, we study the special conformally flat warped product space $B^n \times_f F^p$ and characterize the geometric structure of $B^n \times_f F^p$.

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1. Introduction

The conformal transformation on the Riemannian manifold is characterized by the conformal change of the Riemannian metric and does not change the angle between two vectors at a point. The Weyl conformal curvatures C(see(2.1)) and 3-tensor D (see(2.2)) are conformal invariants.

The Riemannian manifold (M, g) is called conformally flat if, for each x in M, there exist a neighborhood V of x and a C^∞ function f on V such that $(V, e^{2f}g)$ is flat ([1,3]). It is well known that ([1,3]) M is conformally flat if and only if $C = 0$ for $m > 3$, $D = 0$ for $m = 3$, where m is the dimension of M.

In 1930, B. Kagan([6,7]) introduced the subprojective space which is a kind of conformally flat space, and in 1973, B.Y. Chen and K. Yano([4]) introduced a special conformally flat space which is a generalization of the subprojective space. Every conformally flat hypersurface of a Euclidean space and canal hypersurface of a Euclidean space are examples of a special conformally flat space([4]).

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The conharmonic transformation is a conformal transformation preserving the harmonicity of a certain function. The conharmonic curvature tensor T (see (2.4))is invariant under the conharmonic transformation.

In [9], present authors proved that if the warped product space $B^n \times_f R^1 (n > 3, K > 0)$ is conharmonically flat, then B is a special conformally flat space. But we can easily see that the warped product of a special conformally flat space and R^1 is not conharmonically flat, in general.

In this point of a view, it is natural to consider the best condition of the warped product of a special conformally flat space and R^1 to be conharmonically flat. For these problems, we investigate the necessary and sufficient conditions of $B^n \times_f R^1$ to be conharmonically flat. Moreover we extend our result of $B^n \times_f R^1$ to $B^n \times_f F^p$ for the general fibre F and characterize the base space and each fibre when $B^n \times_f F^p$ is conharmonically flat.

2. Special conformally flat spaces

A conformal transformation between two Riemannian manifolds (M, g) and (M', g') is a diffeomorphism preserving angle measured by the metrics g and g' respectively. It is characterized by $g' = e^{2\rho} g$, where ρ is a scalar function. In this case g and g' are said to be conformally equivalent. If the function ρ is constant, then the conformal transformation is said to be homothetic([1,3]). The Weyl conformal curvature tensor C which is conformally invariant in an m-dimensional Riemannian manifold M is defined by

$$\begin{aligned}(2.1)\qquad C(X,Y)Z = R(X,Y)Z - \tfrac{1}{m-2}\{S(Y,Z)X - g(X,Z)QY \\ + g(Y,Z)QX - S(X,Z)Y\} + \tfrac{K}{(m-1)(m-2)}\{g(Y,Z)X - g(X,Z)Y\},\end{aligned}$$

where R, S and K are curvature tensor, Ricci curvature tensor and scalar curvature of M respectively and $g(QX, Y) = S(X, Y)$. The Weyl conformal curvature 3-tensor D is also conformally invariant and defined by

$$(2.2)\qquad D(X,Y)Z = \nabla_X L(Y,Z) - \nabla_Y L(X,Z),$$

where we have put

$$(2.3)\qquad L(X,Y) = -\frac{S(X,Y)}{m-2} + \frac{K}{2(m-1)(m-2)} g(X,Y).$$

The conharmonic curvature tensor T is defined by

$$\begin{aligned}(2.4)\qquad T(X,Y)Z = R(X,Y)Z \\ - \tfrac{1}{m-2}\{S(Y,Z)X - g(X,Z)QY + g(Y,Z)QX - S(X,Z)Y\},\end{aligned}$$

which is invariant under conharmonic transformation([5]).

On a conformally flat Riemannian manifold N, if there exist two functions α and β such that α is positive and

$$(2.5)\qquad L(X,Y) = -\frac{\alpha^2}{2} g(X,Y) + \beta(X\alpha)(Y\alpha),$$

then N is called a special conformally flat space([4]). In particular, if β is a function of α, then the special conformally flat space N is called a subprojective space([4,6,7]).

3. Conharmonically flat warped product spaces

Let (B, g) be a n-dimensional Riemannian manifold with Riemannian metric g and let $M = B^n \times_f R^1$ be a warped product manifold, where $f : B \to R^+$ is a warping function. Then the curvature tensors $\tilde{R}$ and R of M and B respectively are given by

(3.1) $\tilde{R}_{dcb}{}^{a} = R_{dcb}{}^{a}, \qquad \tilde{R}_{d1b}{}^{1} = \frac{1}{f}\nabla_d f_b$

and the others are zero, where $f_b = \nabla_b f$ and the range of indices a,b,c,$\cdots$ is $\{2, 3, \cdots, n+1\}$. Hence the Ricci curvature tensors $\tilde{S}$ and S for M and B respectively are given by

(3.2) $\tilde{S}_{cb} = S_{cb} - \frac{1}{f}(\nabla_c f_b), \qquad \tilde{S}_{c1} = 0, \qquad \tilde{S}_{11} = -f(\triangle f)$

where $\triangle f$ is the Laplacian of f with respect to g. The scalar curvatures $\tilde{K}$ and K for M and B respectively are related by

$$\tilde{K} = K - \frac{2\triangle f}{f}. \tag{3.3}$$

If $M = B^n \times_f R^1$ is conharmonically flat warped product space, then, from (2.4), the Riemannian curvature tensor $\tilde{R}$ on M is given by

$$\tilde{R}_{kji}{}^{h} = \frac{1}{n-1}\left(\tilde{S}_{ji}\delta_k^h - \tilde{S}_{ki}\delta_j^h + \tilde{S}_k{}^{h}\tilde{g}_{ji} - \tilde{S}_j{}^{h}\tilde{g}_{ki}\right), \tag{3.4}$$

where the range of indices i, j, k,$\cdots$ is $\{1, \cdots, n, n+1 = m\}$.

Using (3.1), (3.2), (3.3) and (3.4), we get

$$\begin{aligned} R_{dcb}{}^{a} \;\; = \;\; & \frac{1}{n-1}\left(S_{cb}\delta_d^a - S_{db}\delta_c^a + S_d{}^{a}g_{cb} - S_c{}^{a}g_{db}\right) \\ & -\frac{1}{(n-1)f}\left(\delta_d^a\nabla_c f_b - \delta_c^a\nabla_d f_b + g_{cb}\nabla_d f^a - g_{db}\nabla_c f^a\right), \end{aligned} \tag{3.5}$$

$$S_{cb} = Kg_{cb} - \frac{n-2}{f}\nabla_c f_b - \frac{\triangle f}{f}g_{cb}, \tag{3.6}$$

$$K = \frac{2\triangle f}{f}, \tag{3.7}$$

where $f^a = f_b g^{ba}$. Thus we obtain the following theorem([9]).

THEOREM 1. *Let $M = B \times_f R$ be a conharmonically flat warped product space with $n > 3$ and $K_c \neq 0$. If $K > 0$, then B is a special conformally flat space.*

If $M^m (m > 3)$ is conformally flat, from (2.1) and (2.3), then we see that

$$R_{kji}{}^{h} = \delta_j^h L_{ki} - \delta_k^h L_{ji} + L_j^h g_{ki} - L_k^h g_{ji}. \tag{3.8}$$

Moreover if M is special conformally flat, using (2.5) and (3.8), then we have

$$R_{kji}{}^{h} = \alpha^2(g_{ji}\delta_k^h - g_{ki}\delta_j^h) + \beta(\alpha_k\alpha_i\delta_j^h + \alpha_j\alpha^h g_{ki} - \alpha_j\alpha_i\delta_k^h - \alpha_k\alpha^h g_{ji}), \tag{3.9}$$

where α and β are $C^\infty-$functions on M and α is positive.

Conversely, if M has the curvature of the form (3.9), then we see that

$$S_{ji} = \alpha^2(m-1)g_{ji} + \beta\{(2-m)\alpha_j\alpha_i - \|\alpha_h\|^2 g_{ji}\}, \tag{3.10}$$

$$K = (m-1)(m\alpha^2 - 2\beta\|\alpha_h\|^2). \tag{3.11}$$

Using (2.3), (3.10) and (3.11), we get

$$L_{ij} = -\frac{\alpha^2}{2}g_{ij} + \beta\alpha_i\alpha_j. \tag{3.12}$$

Also, using (2.1), (3.9) and (3.10), we see that $C = 0$. Thus we have

Theorem 2. *A Riemannian manifold M^m is special conformally flat if and only if*

$$R_{kji}{}^h = \alpha^2(g_{ji}\delta_k^h - g_{ki}\delta_j^h) + \beta(\alpha_k\alpha_i\delta_j^h + \alpha_j\alpha^h g_{ki} - \alpha_j\alpha_i\delta_k^h - \alpha_k\alpha^h g_{ji}),$$

where $m > 3$, α and β are $C^\infty-$functions on M and α is positive.

Let $M = B^n \times_f R^1$ be conharmonically flat. Then B is special conformally flat by Theorem 1. Thus B is conformally flat. Using (2.4), (3.6) and (3.7), we get

$$S_{cb} = \frac{K}{2}g_{cb} - \frac{n-2}{f}\nabla_c f_b. \tag{3.13}$$

The non-zero components of $\widetilde{T}$ in (2.4) are given by

$$\begin{aligned} \widetilde{T}_{dcb}{}^a \quad = \quad & R_{dcb}{}^a - \tfrac{1}{n-1}\{(S_{cb} - \tfrac{1}{f}\nabla_c f_b)\delta_d^a + (S_d^a - \tfrac{1}{f}\nabla_d f^a)g_{cb} \\ & -(S_{db} - \tfrac{1}{f}\nabla_d f_b)\delta_c^a - (S_c^a - \tfrac{1}{f}\nabla_c f^a)g_{db}\}, \end{aligned} \tag{3.14}$$

$$\widetilde{T}_{d1b}{}^1 = \frac{n-2}{f(n-1)}\nabla_d f_b + \frac{1}{n-1}(S_{db} - \frac{\triangle f}{f}g_{db}) \tag{3.15}$$

and the others are zero, where $\triangle f = \nabla_a f^a$. Therefore, we can see that

Theorem 3. *On $M = B^n \times_f R^1$, the followings are equivalent:*
(1) *M is conharmonically flat.*
(2) *B is conformally flat and* (3.13) *holds.*
(3) *B is special conformally flat and* (3.13) *holds,*
where $n > 3$ and $K > 0$.

Proof. By the above arguments, we see that (1) implies (3) and trivially (3) implies (2). Let B be conformally flat and satisfying the equation (3.13). Using (3.13), we see that all components of $\widetilde{T}$ vanishes. Thus M is conharmocally flat, that is (2) implies (1). Hence the proof is completed.

Using Theorem 3 and known examples of the special conformally flat space, we can construct new examples of the conharmonically flat space.

4. Special conformally flat warped product spaces

Let (B, g) and $(F, \bar{g})$ be Riemannian manifolds of dimensions n and p respectively, and f be a positive smooth function on B. Then the warped product space $M = B \times_f F$ is defined by the Riemannian metric $\tilde{g} = \pi^*(g) + (f \circ \pi)^2 \sigma^*(\bar{g})$, where π and σ the projections of $B \times F$ onto B and F respectively. In this case, the nonzero components of the Riemannian curvature tensor $\tilde{R}$ of M are given by

$$\begin{aligned} \tilde{R}_{dcb}{}^{a} &= R_{dcb}{}^{a}, \\ \tilde{R}_{dxb}{}^{y} &= \tfrac{1}{f}(\nabla_d f_b)\delta_x^y, \\ \tilde{R}_{xyz}{}^{w} &= \bar{R}_{xyz}{}^{w} - \|f_e\|^2(\delta_x^w \bar{g}_{yz} - \delta_y^w \bar{g}_{xz}), \end{aligned} \tag{4.1}$$

where the range of indices $a, b, c, \cdots$ is $\{1, 2, \cdots, n\}$, and $x, y, z, \cdots$ is $\{n+1, n+2, \cdots, n+p = m\}$.

The components of Ricci tensor $\tilde{S}$ of M are given by ([2,8])

$$\begin{aligned} \tilde{S}_{cb} &= S_{cb} - \tfrac{p}{f}(\nabla_c f_b)\ , \\ \tilde{S}_{cx} &= 0\ , \\ \tilde{S}_{yx} &= \bar{S}_{yx} - (p-1)\|f_e\|^2 \bar{g}_{yx} - f(\triangle f)\bar{g}_{yx}. \end{aligned} \tag{4.2}$$

Let $\tilde{K}$, K and $\bar{K}$ be the scalar curvatures of M, B and F respectively, then we have

$$\tilde{K} = K + \frac{1}{f^2}\bar{K} - \frac{2p(\triangle f)}{f} - \frac{p(p-1)}{f^2}\|f_e\|^2. \tag{4.3}$$

Assume that $M = B \times_f F$ is conharmonically flat, then we get

$$\tilde{R}_{kji}{}^{h} = \tfrac{1}{m-2}\left(\tilde{S}_{ji}\delta_k^h - \tilde{S}_{ki}\delta_j^h + \tilde{S}_k{}^h \tilde{g}_{ji} - \tilde{S}_j{}^h \tilde{g}_{ki}\right). \tag{4.4}$$

Using (4.1), (4.2) and (4.4), we have

$$\begin{aligned} R_{dcb}{}^{a} = \ & \tfrac{1}{m-2}\left(S_{cb}\delta_d^a - S_{db}\delta_c^a + S_d{}^a g_{cb} - S_c{}^a g_{db}\right) \\ & - \tfrac{p}{(m-2)f}\left(\delta_d^a \nabla_c f_b - \delta_c^a \nabla_d f_b + g_{cb}\nabla_d f^a - g_{db}\nabla_c f^a\right). \end{aligned} \tag{4.5}$$

Contracting (4.5) with respect to a and d, we obtain

$$S_{cb} = \frac{K}{p} g_{cb} - \frac{n-2}{f}\nabla_c f_b - \frac{\triangle f}{f} g_{cb}, \tag{4.6}$$

and that

$$(n-p)fK = 2p(n-1)\triangle f. \tag{4.7}$$

Suppose that $n \neq p$, then we obtain

$$K = \frac{2p(n-1)}{f(n-p)}\triangle f. \tag{4.8}$$

Using (4.5), (4.6) and (4.8), we see that

$$\begin{aligned} R_{dcb}{}^{a} &= \tfrac{1}{n-2}\left(S_{cb}\delta_d^a - S_{db}\delta_c^a + S_d{}^{a}g_{cb} - S_c{}^{a}g_{db}\right) \\ &\quad - \tfrac{K}{(n-1)(n-2)}\left(g_{cb}\delta_d^a - g_{db}\delta_c^a\right) , \end{aligned} \tag{4.9}$$

that is B is conformally flat.

Using (2.3), (4.6) and (4.8), L on B is reduced to

$$L_{cb} = -\frac{g_{cb}}{2p(n-1)}K + \frac{1}{f}\nabla_c f_b \ . \tag{4.10}$$

If we put

$$\alpha = \sqrt{\frac{K}{p(n-1)}}, \qquad \beta = \frac{4p(n-1)K}{fK_cK_b}\nabla_c f_b$$

and considering (4.9) and (4.10), then we see that B is special conformally flat if $K_c \neq 0$. Hence, we have the following theorem.

THEOREM 4. *Let $M = B \times_f F$ be a conharmonically flat warped product space with $n > 3$, $K_c \neq 0$ and $n \neq p$. If $K > 0$, then B is special conformally flat .*

Let us consider the case of $n = p$. Then we get $\triangle f = 0$ from (4.7). Hence, if B is compact then f is constant by Hopf Theorem. Thus M is a Riemannian product manifold.

PROPOSITION 5. *Let $M^m = B^n \times_f F^p$ be a conharmonically flat warped product space and $n = p(> 1)$. If B is compact, then M is a Riemannian product manifold.*

Since $\triangle f = 0$, we get, using (4.6),

$$S_{cb} = \frac{K}{n}g_{cb} - \frac{n-2}{f}\nabla_c f_b. \tag{4.11}$$

Then we can see that B is conformally flat by use of (4.5) and (4.11). In this case L is reduced to

$$L_{cb} = -\frac{K}{2n(n-1)}g_{cb} + \frac{1}{f}\nabla_c f_b. \tag{4.12}$$

If we put $\alpha = \sqrt{\frac{K}{n(n-1)}}, \beta = \frac{4n(n-1)K}{fK_cK_b}\nabla_c f_b$ and considering (4.12), then we see that B is special conformally flat if $K_c \neq 0$. Hence, we have

THEOREM 6. *Let $M = B \times_f F$ be a conharmonically flat warped product space with $n > 3$, $K_c \neq 0$ and $n = p$. If $K > 0$, then B is special conformally flat.*

If we combine Theorems 4 and 6, then we have

THEOREM 7. *If $M = B \times_f F$ is a conharmonically flat warped product space with $K > 0$, $n > 3$ and $K_c \neq 0$, then B is special conformally flat.*

Using (4.1), (4.2) and (4.4), we obtain

$$\bar{R}_{xyz}{}^{w} = \|f_e\|^2(\delta_x^w \bar{g}_{yz} - \delta_y^w \bar{g}_{xz}) + \tfrac{1}{m-2}\{\bar{S}_{yz}\delta_x^w - \bar{S}_{xz}\delta_y^w + \bar{S}_x{}^{w}\bar{g}_{yz} - \bar{S}_y{}^{w}\bar{g}_{xz} - 2(p-1)\|f_e\|^2\bar{g}_{yz}\delta_x^w + 2(p-1)\|f_e\|^2\bar{g}_{xz}\delta_y^w - 2f\Delta f\bar{g}_{yz}\delta_x^w + 2f\Delta f\bar{g}_{xz}\delta_y^w\}. \tag{4.13}$$

Contracting (4.13) with respect to x and w, we obtain

$$\bar{S}_{yz} = \frac{(n-p)(p-1)}{n}\|f_e\|^2\bar{g}_{yz} + \frac{\bar{K}}{n}\bar{g}_{yz} + \frac{2(1-p)}{n}f\Delta f\bar{g}_{yz}, \tag{4.14}$$

which implies

$$f\Delta f = \frac{n-p}{2p(1-p)}\bar{K} + \frac{n-p}{2}\|f_e\|^2 \tag{4.15}$$

and that

$$\bar{S}_{yz} = \frac{\bar{K}}{p}\bar{g}_{yz}. \tag{4.16}$$

Thus F is Einstein if $\bar{K}$ is constant. Using (4.13) and (4.16), we see that

$$\bar{R}_{xyz}{}^{w} = \frac{\bar{K}}{p(p-1)}(\bar{g}_{yz}\delta_x^w - \bar{g}_{xz}\delta_y^w) \tag{4.17}$$

where $p \neq 1$. Thus F is a space of constant curvature. Therefore we have the following theorem.

THEOREM 8. *Let $M = B \times_f F$ be a conharmonically flat warped product space with $p \neq 1$. Then F is a space of constant curvature if $\bar{K}$ is constant. The condition $\bar{K}$ is constant is necessary only for $p = 2$.*

Let M^m be a space of constant curvature with $m \geq 3$. Then M is conformally flat and Einstein, that is $S_{ij} = \frac{\widetilde{K}}{m}g_{ij}$. Hence L on M^m is reduced to

$$L_{ij} = -\frac{\widetilde{K}}{2m(m-1)}g_{ij}. \tag{4.18}$$

If we put $\alpha = \sqrt{\frac{\widetilde{K}}{m(m-1)}}$ and considering (2.5) and (4.18), then we see that M is special conformally flat with $\beta = 0$.

LEMMA 9. *Let M be a space of constant curvature with $\widetilde{K} > 0$, then M is special conformally flat.*

Finally, if we consider Theorem 8 and Lemma 9, then we have

THEOREM 10. *Let $M = B \times_f F$ be a conharmonically flat warped product space with $p \geq 3$. If $\bar{K} > 0$, then F is special conformally flat .*

References

1. A. Besse, *Einstein manifolds*, Springer-Verlag, Berlin(1987).
2. R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. AMS., 145(1969), 1-49.
3. B. Y. Chen, *Geometry of submanifolds*, Marcel Dekker, New York,(1973).
4. B. Y. Chen and K. Yano, *Special conformally flat spaces and canal hypersurfaces*, Tohoku Math.J., 25(1973), 177-184.
5. Y. Ishi, *On conharmonic transformation*, Tensor N.S 7(1957), 73-80.
6. B. Kagan, *Sur les èspaces sour-projectifs*, C. R., 191(1930), 548-550.
7. B. Kagan, *Über ėine Erweiterung des Begriffes vom projektiven Räume und dem zugehörigen Absolut, Abh. des sem für Vektor und Tensor* analysis, Moskau, 1 (1933), 12-101.
8. B. H. Kim, *Warped products with critical Riemannian metric*, Proc. Japan Academy, 71(1995), 117-118.
9. B. H. Kim, J. H. Choi and Y. O. Lee, *Remarks on conformal transformation on Riemannian manifold*, J. Applied Math. and Informations, 27 (2009), 857-864.

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