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ERROR ESTIMATE OF EXTRAPOLATED DISCONTINUOUS GALERKIN APPROXIMATIONS FOR THE VISCOELASTICITY TYPE EQUATION†

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Abstract. In this paper, we adopt discontinuous Galerkin methods with penalty terms namely symmetric interior penalty Galerkin methods, to solve nonlinear viscoelasticity type equations. We construct finite element spaces and define an appropriate projection of u and prove its optimal convergence. We construct extrapolated fully discrete discontinuous Galerkin approximations for the viscoelasticity type equation and prove $\ell^\infty(L^2)$ optimal error estimates in both spatial direction and temporal direction.

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1. Introduction

Let Ω be an open bounded domain in R^d, $d = 2$ with smooth boundary $\partial\Omega$, and let $0 < T < \infty$ be given. We consider the problem of approximating the solution $u(x,t)$ satisfying the following nonlinear viscoelasticity type equations

$$\begin{aligned} &u_{tt} - \nabla \cdot \{a(u)\nabla u + b(u)\nabla u_t\} = f(u) && \text{in } \Omega \times (0,T] \\ &(a(u)\nabla u + b(u)\nabla u_t)\cdot n = 0 && \text{on } \partial\Omega \times (0,T] \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) && \text{in } \Omega \end{aligned} \tag{1.1}$$

where n denotes the unit outward normal vector to $\partial\Omega$ and $u_0(x)$, $u_1(x)$ are given functions defined on Ω. The initial data $u_0(x)$, $u_1(x)$, f, a and b are assumed to be such that (1.1) admits a solution sufficiently smooth to guarantee the convergence results to be presented below. For details about the physical significance and various properties of existence and uniqueness of viscoelasticity type equations, we refer to [6, 7, 8, 12, 14].

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Discontinuous Galerkin method (DGM) with interior penalty terms has been under rapid progession recently since the DGM have the following advantages over classical finite element method. DG methods are very well suited to handling mesh adaptation, adaptive control of error and high orders of accuracy which are essential in case that the numerical method is adopted to solve differential equations related to the area of natural science and engineering.

Early, Douglas and Dupont [5] and Wheeler [18] introduced discontinuous Galerkin methods with interior penalties for elliptic and parabolic equations. Darlow et al. [4] and Douglas et al. [5] applied DG method to approximate the behavior of the flow in porous media which is not locally mass conservative. And also Oden, Babuska and Baumann [9] adopted a new type of elementwise conservative discontinuous Galerkin method for diffusion problem. On the other hand, Rivière and Wheeler [13] introduced a locally conservative discontinuous Galerkin method to approximate the solution of nonlinear parabolic equations and proved a priori $L^\infty(L^2)$ and $L^2(H^1)$ error estimates. Ohm, Lee and Shin [11] developed a discontinuous Galerkin method with interior penalty terms for nonlinear parabolic equations and obtained an optimal $L^\infty(L^2)$ error estimate. Without using Ritz projection or its modified projection Lin and Zhang [8] proved the global superconvergence of semidiscrete Galerkin approximation of the solution to the Sobolev equation and viscoelasticity type equation.

Recently Sun and Wheeler [17] developed a parabolic lift-technique to approximate the solutions of the reactive transport problems using symmetric interior penalty Galerkin method, nonsymmetric interior penalty Galerkin method and incomplete interior penalty Galerkin methods and analyzed the error estimates. Recently in [15, 16] Sun and Yang adopted the discontinuous Galerkin method to nonlinear Sobolev equations and obtained optimal H^1 error estimates. In this work we shall approximate the solution of (1.1) using a discontinuous symmetric Galerkin method with interior penalty terms for the spatial discretization and extrapolated Crank-Nicolson method for the temporal discretization. To obviate the order reduction phenomenon which occurs when the system involved is nonlinear, we adopt the extrapolated technique and induce the linear systems which can be solved explicitly. To our knowledge this paper appears to be the first trial to construct extrapolated fully discrete approximations of viscoelasticity type equation using discontinuous Galerkin method with symmetric interior penalty terms and obtain the optimal convergence in $\ell^\infty(L^2)$ norm. The rest of this paper is organized as follows. In section 2, we introduce some notations and preliminaries. In section 3, we construct finite element space and introduce a modified Ritz projection $\widetilde{u}$ of the solution u of (1.1) onto finite element spaces. We prove the optimal convergence of $\widetilde{u}$ to u in L^2 normed space and in the Sobolev spaces of higher order. In section 4, we apply extrapolated Crank-Nicolson method to construct fully discrete discontinuous Galerkin approximations and obtain the optimal convergence of approximation in the $\ell^\infty(L^2)$ normed space.

2. Notations and basis assumptions

For an integer $s \geq 0$ and a domain $E \subset \mathbb{R}^d$ with $d = 2$ we let $H^s(E)$ the Sobolev space of order s equipped with the following usual Sobolev norms $\|\cdot\|_{s,E}$

$$\|u\|_{s,E} = \left(\sum_{|k| \leq s} \int_E |D^k u|^2 dx \right)^{\frac{1}{2}}.$$

And we define the following usual Sobolev seminorms $|\cdot|_{s,E}$ such that

$$|u|_{s,E} = \left(\sum_{|k| = s} \int_E |D^k u|^2 dx \right)^{\frac{1}{2}}.$$

If $E = \Omega$ we simply write $\|\cdot\|_s$ instead of $\|\cdot\|_{s,\Omega}$, and also we denote $|\cdot|_s$ instead of $|\cdot|_{s,E}$. And for a fractional number $s > 0$ we also adopt the usual extension of Sobolev space $H^s(E)$ defined on E with a fractional order s.

Let $\mathcal{E}_h = \{E_1, E_2, \cdots, E_{N_h}\}$ be a regular quasi-uniform subdivision of Ω where E_i is a triangle or a quadrilateral. We let $h_i = \mathrm{diam}(E_i)$ be the diameter of E_i and we let $h = \max_{1 \leq i \leq N_h} h_i$. We assume that $\mathcal{E}_h$ satisfies the following regularity condition: there exists a constant $\rho > 0$ such that each E_i contains a ball of radius ρh_i. And also we assume that $\mathcal{E}_h$ satisfies the following quasiuniformity requirement: there is a constant $\gamma > 0$ such that $\dfrac{h}{h_i} \leq \gamma$, $\forall\ i = 1, 2, \cdots, N_h$.

Now we assume that the functions accompanying with the problem (1.1) satisfy the following conditions and the solution u satisfies the following regularity conditions:

1. there exist constants $a_0,\ a^* > 0$ such that $0 < a_0 \leq a(x,u) \leq a^*$, $\forall (x,u) \in \Omega \times \mathbb{R}$ and $0 < a_0 \leq b(x,u) \leq a^*$, $\forall (x,u) \in \Omega \times \mathbb{R}$.
2. $a(x,u)$, $b(x,u)$, $f(x,u)$ are continuously differentiable with respect to each variable and there exists a constant $K > 0$ such that
$$\left|\frac{\partial a}{\partial u}\right|, \quad \left|\frac{\partial b}{\partial u}\right|, \quad \left|\frac{\partial f}{\partial u}\right| \leq K.$$
3. (1.1) has a unique solution satisfying $u \in L^\infty(H^s)$, $u_t \in L^\infty(H^s)$, $u_{tt} \in H^s$, $u_{ttt} \in L^\infty(L^2)$ and $u_0 \in H^s$ for $s \geq \dfrac{d}{2} + 1$.

3. Finite element spaces and an auxiliary projection

For an $s \geq 0$ and a given subdivision $\mathcal{E}_h$, we define the following space

$$H^s(\mathcal{E}_h) = \{v \in L^2(\Omega) \mid v|_{E_i} \in H^s(E_i), \quad i = 1, 2, \cdots, N_h\}.$$

Let the edges of $\mathcal{E}_h$ be denoted by $\{e_1, e_2, \cdots, e_{P_h}, e_{P_h+1}, \cdots, e_{M_h}\}$ where $e_k \subset \Omega$, $1 \leq k \leq P_h$ and $e_k \subset \partial\Omega$ $P_h + 1 \leq k \leq M_h$. With each edge e_k, $1 \leq k \leq P_h$, we associate a unit outward normal vector n_k to E_i if $e_k = \partial E_i \cap \partial E_j$ and $i < j$. For $P_h + 1 \leq k \leq M_h$, we define $n_k = n$ the unit outward normal vector to $\partial\Omega$.

To present the dicontinuous Galerkin scheme, we need define some functions on edges between two elements. For $\phi \in H^s(\mathcal{E}_h)$, $s > \dfrac{1}{2}$ we define the following average function $\{\phi\}$ and the jump function $[\phi]$:

$$\{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}(\phi|_{E_j})|_{e_k}, \quad \forall x \in e_k, \quad 1 \leq k \leq P_h$$
$$[\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}, \quad \forall x \in e_k, \quad 1 \leq k \leq P_h$$

where $e_k = \partial E_i \cap \partial E_j$, $i < j$.

We associate the following broken norms with the space $H^s(\mathcal{E}_h)$, $s \geq 1$

$$|||\phi|||^2 = \sum_{i=1}^{N_h} \|\phi\|_{0,E_i}^2$$
$$|||\phi|||_1^2 = \sum_{i=1}^{N_h} (\|\phi\|_{1,E_i}^2 + h_i^2 \|\nabla^2 \phi\|_{0,E_i}^2) + J_\beta^\sigma(\phi, \phi)$$

where

$$J_\beta^\sigma(\phi, \psi) = \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} [\phi][\psi] ds, \quad \beta \geq 1$$

is an interior penalty term and σ is a discrete positive function that takes the positive constant σ_k on the edge e_k and is bounded below by $\sigma_0 > 0$ and above by $\sigma^* > 0$.

Let r be a positive integer. The finite element space used in this paper is taken to be

$$D_r(\mathcal{E}_h) = \{v \in L^2(\Omega) : v|_{E_j} \in P_r(E_j), \ j = 1, 2, \cdots, N_h\}$$

where $P_r(E_j)$ denotes the set of polynomials of total degree$\leq r$ on E_j.

Throughout this paper C denotes a positive generic constant independent of r and h. We apply the following trace inequalities whose proofs are given in [1].

For each $E_j \in \mathcal{E}_h$, there exists a positive consitive constant C depending only on γ and ρ such that the two following trace inequalities hold:

$$\|\phi\|_{0,e_j}^2 \leq C\left(\frac{1}{h_j}|\phi|_{0,E_j}^2 + h_j|\phi|_{1,E_j}^2\right), \quad \forall \phi \in H^1(E_j)$$
$$\left\|\frac{\partial \phi}{\partial \eta_j}\right\|_{0,e_j}^2 \leq C\left(\frac{1}{h_j}|\phi|_{1,E_j}^2 + h_j|\phi|_{2,E_j}^2\right), \quad \forall \phi \in H^2(E_j)$$

where e_j is an edge of E_j and η_j is the unit outward normal vector to E_j. And also throughout this paper, we need the following well-known *hp*-approximation properties. For the proofs of these properties, we refer to [2, 3].

Let $E_j \in \mathcal{E}_h$ and $\phi \in H^s(E_j)$. Then there exist a positive constant C depending on s, γ, and ρ but independent of ϕ, r and h and a sequence $z_r^h \in P_r(E_j)$,

$r = 1, 2, \cdots$ such that for any $0 \le q \le s$,

$$\|\phi - z_r^h\|_{q,E_j} \le C\frac{h_j^{\mu-q}}{r^{s-q}}\|\phi\|_{s,E_j} \quad s \ge 0,$$
$$\|\phi - z_r^h\|_{0,e_j} \le C\frac{h_j^{\mu-\frac{1}{2}}}{r^{s-\frac{1}{2}}}\|\phi\|_{s,E_j} \quad s > \frac{1}{2},$$
$$\|\phi - z_r^h\|_{1,e_j} \le C\frac{h_j^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}}\|\phi\|_{s,E_j} \quad s > \frac{3}{2}$$

where $\mu = \min(r+1, s)$ and e_j is an edge of E_j.

Now we introduce the following bilinear mappings $A(\rho;\cdot,\cdot)$ and $B(\rho;\cdot,\cdot)$ defined on $H^s(\mathcal{E}_h)\times H^s(\mathcal{E}_h)$

$$\begin{aligned}A(\rho;\phi,\psi) = (a(\rho)\nabla\phi,\nabla\psi) &- \sum_{k=1}^{P_h}\int_{e_k}\{a(\rho)\nabla\phi\cdot n_k\}[\psi] - \sum_{k=1}^{P_h}\int_{e_k}\{a(\rho)\nabla\psi\cdot n_k\}[\phi]\\ &+ J_\beta^\sigma(\phi,\psi)\\ B(\rho;\phi,\psi) = (b(\rho)\nabla\phi,\nabla\psi) &- \sum_{k=1}^{P_h}\int_{e_k}\{b(\rho)\nabla\phi\cdot n_k\}[\psi] - \sum_{k=1}^{P_h}\int_{e_k}\{b(\rho)\nabla\psi\cdot n_k\}[\phi]\\ &+ J_\beta^\sigma(\phi,\psi).\end{aligned}$$

Now we define the following weak formulation of the problem (1.1): Find $u \in H^s(\mathcal{E}_h)$ such that

$$(u_{tt}, v) + A(u;u,v) + B(u;u_t,v) = (f(u),v), \quad \forall v \in H^s(\mathcal{E}_h). \tag{3.1}$$

For a $\lambda > 0$ we define the following bilinear forms $A_\lambda(\rho;\cdot,\cdot)$ and $B_\lambda(\rho;\cdot,\cdot)$ on $H^s(\mathcal{E}_h)\times H^s(\mathcal{E}_h)$ such that

$$A_\lambda(\rho;\phi,\psi) = A(\rho;\phi,\psi) + \lambda(\phi,\psi)$$
$$B_\lambda(\rho;\phi,\psi) = B(\rho;\phi,\psi) + \lambda(\phi,\psi).$$

A_λ and B_λ satisfy the following boundedness and coercivity properties which are proved in [10, 11].

Lemma 3.1. *For a $\lambda > 0$, there exists a constant $C > 0$ satisfying*

$$|A_\lambda(\rho;\phi,\psi)| \le C|||\phi|||_1 |||\psi|||_1, \quad \forall\, \phi,\psi \in H^s(\mathcal{E}_h)$$
$$|B_\lambda(\rho;\phi,\psi)| \le C|||\phi|||_1 |||\psi|||_1, \quad \forall\, \phi,\psi \in H^s(\mathcal{E}_h).$$

Lemma 3.2. *For a $\lambda > 0$ and $\beta \ge \frac{1}{d-1}$, there exists a constant $\widetilde{c} > 0$ satisfying*

$$A_\lambda(\rho;\phi,\phi) \ge \widetilde{c}\ |||\phi|||_1^2, \quad \forall\, \phi \in D_r(\mathcal{E}_h),$$
$$B_\lambda(\rho;\phi,\phi) \ge \widetilde{c}\ |||\phi|||_1^2, \quad \forall\, \phi \in D_r(\mathcal{E}_h).$$

We define the following auxiliary projection by modifying the elliptic type projection which is initiated by Wheeler [19] to approximate the Galerkin approximations to parabolic equations.

Define a projection $\widetilde{u}(t) : [0,T] \to D_r(\mathcal{E}_h)$ such that

$$\begin{cases} A_\lambda(u; u-\widetilde{u}, v) + B_\lambda(u; u_t - \widetilde{u}_t, v) = 0, & \forall\, v \in D_r(\mathcal{E}_h), \quad \forall\, t > 0, \\ \widetilde{u}(0) = \widetilde{u}_0(x), \quad \widetilde{u}_t(0) = \widetilde{u}_1(x) & \forall\, x \in \Omega \end{cases} \tag{3.2}$$

where $\widetilde{u}_0(x)$, $\widetilde{u}_1(x)$ is appropriate projections onto $D_r(\mathcal{E}_h)$ of $u_0(x)$ and $u_1(x)$ respectively satisfying the following approximation properties

$$|\!|\!| u_0(x) - \widetilde{u}_0(x) |\!|\!| + h |\!|\!| u_0(x) - \widetilde{u}_0(x) |\!|\!|_1 \le C h^{r+1} \|u\|_{r+1}$$

and

$$|\!|\!| u_1(x) - \widetilde{u}_1(x) |\!|\!| + h |\!|\!| u_1(x) - \widetilde{u}_1(x) |\!|\!|_1 \le C h^{r+1} \|u\|_{r+1}.$$

Now we let $\eta(x,t) = u(x,t) - \widetilde{u}(x,t)$. We state the following approximations for η whose proofs can be found in [10].

Theorem 3.1. *If $u_t \in L^2(H^s)$ and $u_0 \in H^s$ then there exists a constant C independent of h satisfying*

(i) $|\!|\!| \eta_t |\!|\!| + h |\!|\!| \eta_t |\!|\!|_1 \le C h^\mu (\|u_t\|_{H^s} + \|u_0\|_s)$

(ii) $|\!|\!| \eta |\!|\!| + h |\!|\!| \eta |\!|\!|_1 \le C h^\mu (\|u_t\|_{L^2(H^s)} + \|u_0\|_s)$

where $\mu = \min(r+1, s)$.

Theorem 3.2. *If $u_t \in L^2(H^s)$, $u_{tt} \in H^s$ and $u_0 \in H^s$, then there exists a constant C independent of h such that*

$$|\!|\!| \eta_{tt} |\!|\!| + h |\!|\!| \eta_{tt} |\!|\!|_1 \le C h^\mu (\|u_{tt}\|_{H^s} + \|u_t\|_{L^2(H^s)} + \|u_0\|_s)$$

where $\mu = \min(r+1, s)$.

Throughout this paper ε denotes a generic positive constant which is assumed to be sufficiently small.

4. The convergence of extrapolated discontinuous Galerkin approximations

To define the extrapolated discontinuous Galerkin approximations to the problem (1.1) for a positive integer N we let $\Delta t = \dfrac{T}{N}$, $t^n = n\Delta t$, $t^{n,\theta} = \dfrac{1+\theta}{2} t^{n+1} + \dfrac{1-\theta}{2} t^{n-1}$, $g^n = g(x, t^n)$, $0 \le n \le N$. Now we define

$$g^{j,\theta} = \frac{1+\theta}{2} g^{j+1} + \frac{1-\theta}{2} g^{j-1}, \quad \partial_t g^j = \frac{g^{j+1} - g^{j-1}}{2\Delta t} \text{ for } 1 \le j \le N-1,$$

$$\overline{\partial}_t g^j = \frac{g^j - g^{j-1}}{\Delta t} \text{ for } 1 \le j \le N, \quad \overline{\partial}_t^2 g^{j+1} = \frac{g^{j+1} - 2g^j + g^{j-1}}{(\Delta t)^2} \text{ for } 1 \le j \le N-1.$$

Now we define the following extrapolated fully discrete approximation $\{U^n\}_{n=0}^N$ such that

$$\begin{cases}(\overline{\partial}_t^2 U^{j+1}, v) + A(EU^{j,\theta}; U^{j,\theta}, v) + B(EU^{j,\theta}; \partial_t U^j, v) = (f(U^{j,\theta}), v)\\ U^0 = \widetilde{u}(x,0) = \widetilde{u}_0(x)\\ (\overline{\partial}_t U^1, v) = (V_1, v) + \dfrac{\Delta t}{2}(V_2, v)\end{cases} \tag{4.1}$$

where $EU^{j,\theta} = (1+\theta)U^j - \theta U^{j-1}$, V_1 is L^2-projection of $u_1 = u_t(0)$ and V_2 is L^2-projection of $\nabla \cdot (a(u_0)\nabla u_0 + b(u_0)\nabla u_1) + f(u_0) = u_{tt}(0)$.

By applying Taylor's expansion, we can easily obtain the results of the following Lemmas.

Lemma 4.1. *For $1 \le j \le N-1$, if we let $\gamma^{j,\theta} = \widetilde{u}(t^{j,\theta}) - \widetilde{u}^{j,\theta}$ then*

(i) *for $\theta = 1$, $\|\gamma^{j,\theta}\| = 0$ and $\|\gamma^{j,\theta}\|_1 = 0$*
(ii) *for $\theta = 0$, $\|\gamma^{j,\theta}\| \le C(\Delta t)^2 \|\widetilde{u}_{tt}\|_{L^\infty(L^2)}$ and $\|\gamma^{j,\theta}\|_1 \le C(\Delta t)^2 \|\widetilde{u}_{tt}\|_{L^\infty(\|\cdot\|_1)}$.*

Lemma 4.2. *For $1 \le j \le N-1$ if we let $\rho^{j,\theta} = \widetilde{u}_t(t^{j,\theta}) - \partial_t \widetilde{u}^j$, then*

(i) *for $\theta = 1$, $\|\rho^{j,\theta}\| \le C\Delta t \|\widetilde{u}_{tt}\|_{L^\infty(L^2)}$ and $\|\rho^{j,\theta}\|_1 \le C\Delta t \|\widetilde{u}_{tt}\|_{L^\infty(\|\cdot\|_1)}$*
(ii) *for $\theta = 0$, $\|\rho^{j,\theta}\| \le C(\Delta t)^2 \|\widetilde{u}_{ttt}\|_{L^\infty(L^2)}$ and $\|\rho^{j,\theta}\|_1 \le C(\Delta t)^2 \|\widetilde{u}_{ttt}\|_{L^\infty(\|\cdot\|_1)}$.*

Lemma 4.3. *If we let $\sigma^{j,\theta} = \widetilde{u}_{tt}(t^{j,\theta}) - \overline{\partial}_t^2 \widetilde{u}^{j+1}$ then*

(i) *for $\theta = 1$, $\|\sigma^{j,\theta}\| \le C\Delta t \|\widetilde{u}_{ttt}\|_{L^\infty(L^2)}$,*
(ii) *for $\theta = 0$, $\|\sigma^{j,\theta}\| \le C(\Delta t)^2 \|\widetilde{u}_{tttt}\|_{L^\infty(L^2)}$.*

Lemma 4.4. *For $1 \le j \le N-1$ if we let $\alpha^{j,\theta} = \widetilde{u}(t^{j,\theta}) - E\widetilde{u}^{j,\theta}$ then*

(i) *for $\theta = 1$, $\|\alpha^{j,\theta}\| \le (\Delta t)^2 \|\widetilde{u}_{tt}\|_{L^\infty(L^2)}$,*
(ii) *for $\theta = 0$, $\|\alpha^{j,\theta}\| = 0$ and $\|\alpha^{j,\theta}\|_1 = 0$.*

To proceed the error analysis now we let $e^n = u(x,t^n) - U^n$, $\xi^n = \widetilde{u}(x,t^n) - U^n(x)$.

Theorem 4.1. *If $0 < \lambda < 1$, $\Delta t = O(h)$ and $\beta = \dfrac{1}{d-1}$ then there exists a constant $C > 0$ independent of h and Δt such that for $j = 1, 2, \cdots, N$*

(i) *if $\theta = 1$, $\|u(t_j) - U^j\|_{\ell^\infty(L^2)} \le C(h^\mu + \Delta t)$ and*
(ii) *if $\theta = 0$, $\|u(t_j) - U^j\|_{\ell^\infty(L^2)} \le C(h^\mu + (\Delta t)^2)$,*

where $u = \min(s, r+1)$.

Proof. From (1.1) and (4.1) the following holds

$$\begin{aligned}&(u_{tt}(t^{j,\theta}) - \overline{\partial}_t^2 U^{j+1}, v) + A_\lambda(u(t^{j,\theta}); u(t^{j,\theta}), v) - A_\lambda(EU^{j,\theta}; U^{j,\theta}, v)\\ &\quad + B_\lambda(u(t^{j,\theta}); u_t(t^{j,\theta}), v) - B_\lambda(EU^{j,\theta}; \partial_t U^j, v) \qquad\qquad (4.2)\\ &= (f(u(t^{j,\theta})), v) - (f(EU^{j,\theta}), v) + \lambda(u(t^{j,\theta}) - U^{j,\theta}, v) + \lambda(u_t(t^{j,\theta}) - \partial_t U^j, v).\end{aligned}$$

From the definition of η, γ and ξ we have the followings

$$\begin{aligned}&A_\lambda(u(t^{j,\theta});u(t^{j,\theta}),v)-A_\lambda(EU^{j,\theta};U^{j,\theta},v)\\&=A_\lambda(u(t^{j,\theta});\eta(t^{j,\theta}),v)+A_\lambda(u(t^{j,\theta});\widetilde{u}(t^{j,\theta})-\widetilde{u}^{j,\theta},v)\\&\quad+A_\lambda(u(t^{j,\theta});\widetilde{u}(t^{j,\theta}),v)-A_\lambda(EU^{j,\theta};\widetilde{u}^{j,\theta},v)+A_\lambda(EU^{j,\theta};\xi^{j,\theta},v),\end{aligned}\tag{4.3}$$

$$\begin{aligned}&B_\lambda(u(t^{j,\theta});u(t^{j,\theta}),v)-B_\lambda(EU^{j,\theta};\partial_tU^j,v)\\&=B_\lambda(EU^{j,\theta};\partial_t\xi^j,v)+B_\lambda(u(t^{j,\theta});\eta_t(t^{j,\theta}),v)\\&\quad+B_\lambda(u(t^{j,\theta});\widetilde{u}_t(t^{j,\theta})-\partial_t\widetilde{u}^j,v)+B_\lambda(u(t^{j,\theta});\partial_t\widetilde{u}^j,v)\\&\quad-B_\lambda(EU^{j,\theta};\partial_t\widetilde{u}^j,v).\end{aligned}\tag{4.4}$$

By substituting (4.3) and (4.4) into (4.2), we obtain the following

$$\begin{aligned}&(\overline{\partial}_t^2\xi^{j+1},v)+A_\lambda(EU^{j,\theta};\xi^{j,\theta},v)+B_\lambda(EU^{j,\theta};\partial_t\xi^j,v)\\&=-(\eta_{tt}(t^{j,\theta}),v)-(\sigma^{j,\theta},v)-A_\lambda(u(t^{j,\theta});\gamma^{j,\theta},v)-A_\lambda(u(t^{j,\theta});\widetilde{u}^{j,\theta},v)\\&\quad+A_\lambda(EU^{j,\theta};\widetilde{u}^{j,\theta},v)-B_\lambda(u(t^{j,\theta});\rho^{j,\theta},v)-B_\lambda(u(t^{j,\theta});\partial_t\widetilde{u}^j,v)\\&\quad+B_\lambda(EU^{j,\theta};\partial_t\widetilde{u}^j,v)+(f(u(t^{j,\theta}))-f(EU^{j,\theta}),v)\\&\quad+\lambda(u(t^{j,\theta})-U^{j,\theta},v)+\lambda(u_t(t^{j,\theta})-\partial_tU^j,v).\end{aligned}\tag{4.5}$$

By the definition of A_λ, we have

$$\begin{aligned}&A_\lambda(EU^{j,\theta};\xi^{j,\theta},\partial_t\xi^j)\\&\geq(a(EU^{j,\theta})\nabla\xi^{j,\theta},\nabla\partial_t\xi^j)-\sum_{k=1}^{P_h}\int_{e_k}\{a(EU^{j,\theta})\nabla\xi^{j,\theta}\cdot n_k\}[\partial_t\xi^j]\\&\quad-\sum_{k=1}^{P_h}\int_{e_k}\{a(EU^{j,\theta})\nabla(\partial_t\xi^j)\cdot n_k\}[\xi^{j,\theta}]+\frac{1}{2}\partial_tJ(\xi^j,\xi^j)+\frac{\lambda}{2}\partial_t\|\xi^j\|^2.\end{aligned}$$

By simple computation we have

$$\begin{aligned}(\overline{\partial}_t^2\xi^{j+1},\partial_t\xi^j)&=\frac{1}{2}\overline{\partial}_t\|\overline{\partial}_t\xi^{j+1}\|^2\\B_\lambda(EU^{j,\theta};\partial_t\xi^j,v)&\geq C\|\partial_t\xi^j\|_1^2.\end{aligned}$$

We take $v=\partial_t\xi^j$ in (4.5) to obtain the following error equation

$$\begin{aligned}&\frac{1}{2}\overline{\partial}_t\|\overline{\partial}_t\xi^{j+1}\|^2+\frac{1}{2}\partial_tJ(\xi^j,\xi^j)+\frac{\lambda}{2}\partial_t\|\xi^j\|^2+\widetilde{c}\|\partial_t\xi^j\|_1^2\\&\leq C\Big(-(\eta_{tt}(t^{j,\theta}),\partial_t\xi^j)-(\sigma^{j,\theta},\partial_t\xi^j)-A_\lambda(u(t^{j,\theta});\gamma^{j,\theta},\partial_t\xi^j)\\&\quad-A_\lambda(u(t^{j,\theta});\widetilde{u}^{j,\theta},\partial_t\xi^j)+A_\lambda(EU^{j,\theta};\widetilde{u}^{j,\theta},\partial_t\xi^j)-B_\lambda(u(t^{j,\theta});\rho^{j,\theta},\partial_t\xi^j)\\&\quad-B_\lambda(u(t^{j,\theta});\partial_t\widetilde{u}^j,\partial_t\xi^j)+B_\lambda(EU^{j,\theta};\partial_t\widetilde{u}^j,\partial_t\xi^j)\\&\quad+(f(u(t^{j,\theta}))-f(EU^{j,\theta}),\partial_t\xi^j)+\lambda(u(t^{j,\theta})-U^{j,\theta},\partial_t\xi^j)\end{aligned}$$

$$
\begin{aligned}
&+ \lambda(u_t(t^{j,\theta}) - \partial_t U^j, \partial_t \xi^j) + \sum_{k=1}^{P_h} \int_{e_k} \{a(EU^{j,\theta}) \nabla \xi^{j,\theta} \cdot n_k\}[\partial_t \xi^j] \\
&+ \sum_{k=1}^{P_h} \int_{e_k} \{a(EU^{j,\theta}) \nabla(\partial_t \xi^j) \cdot n_k\}[\xi^{j,\theta}] \\
&+ a^* \|\nabla \xi^{j,\theta}\| \|\nabla \partial_t \xi^j\| \Big)
\end{aligned}
$$

which implies the following

$$
\begin{aligned}
&\overline{\partial}_t \|\overline{\partial}_t \xi^{j+1}\|^2 + \|\xi^j\|^2 + J_\beta^\sigma(\xi^j, \xi^j)) + \|\partial_t \xi^j\|_1^2 \\
&\le C\big(- (\eta_{tt}(t^{j,\theta}) + \sigma^{j,\theta}, \partial_t \xi^j) \\
&\quad - A_\lambda(u(t^{j,\theta}); \gamma^{j,\theta}, \partial_t \xi^j) \\
&\quad - A_\lambda(u(t^{j,\theta}); \widetilde{w}^{j,\theta}, \partial_t \xi^j) + A_\lambda(EU^{j,\theta}; \widetilde{w}^{j,\theta}, \partial_t \xi^j)\big) \\
&\quad - B_\lambda(u(t^{j,\theta}); \rho^{j,\theta}, \partial_t \xi^j) \\
&\quad - \big(B_\lambda(u(t^{j,\theta}); \partial_t \widetilde{w}^j, \partial_t \xi^j) + B_\lambda(EU^{j,\theta}; \partial_t \widetilde{w}^j, \partial_t \xi^j)\big) \qquad (4.6) \\
&\quad + (f(u(t^{j,\theta})) - f(EU^{j,\theta}), \partial_t \xi^j) \\
&\quad + \lambda\big((\eta(t^{j,\theta}) + \gamma^{j,\theta} + \xi^{j,\theta} + \eta_t(t^{j,\theta}) + \rho^{j,\theta} + \partial_t \xi^j, \partial_t \xi^j)\big) \\
&\quad + \sum_{k=1}^{P_h} \int_{e_k} \{a(EU^{j,\theta}) \nabla \xi^{j,\theta} \cdot n_k\}[\partial_t \xi^j] \\
&\quad + \sum_{k=1}^{P_h} \int_{e_k} \{a(EU^{j,\theta}) \nabla(\partial_t \xi^j) \cdot n_k\}[\xi^{j,\theta}] \\
&\quad + a^* \|\nabla \xi^{j,\theta}\| \|\nabla \partial_t \xi^j\| \Big) \\
&:= \sum_{i=1}^{10} I_i.
\end{aligned}
$$

Obviously, for sufficiently small $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$
\begin{aligned}
|I_1| &\le C(\|\eta_{tt}(t^{j,\theta})\| + \|\sigma^{j,\theta}\|)\|\partial_t \xi^j\| \\
&\le C(\|\eta_{tt}(t^{j,\theta})\|^2 + \|\sigma^{j,\theta}\|^2) + \varepsilon \|\partial_t \xi^j\|_1^2 \\
|I_2| &= C|A_\lambda(u(t^{j,\theta}); \gamma^{j,\theta}, \partial_t \xi^j)| \le C\|\gamma^{j,\theta}\|_1^2 + \varepsilon \|\partial_t \xi^j\|_1^2.
\end{aligned}
$$

We can separate I_3 as follows

$$
\begin{aligned}
|I_3| = &|((a(u(t^{j,\theta})) - a(EU^{j,\theta}))\nabla \widetilde{w}^{j,\theta}, \nabla(\partial_t \xi^j))| \\
&+ \left| \sum_{k=1}^{P_h} \{(a(u(t^{j,\theta})) - a(EU^{j,\theta}))\nabla \widetilde{w}^{j,\theta} \cdot n_k\}[\partial_t \xi^j] \right|
\end{aligned}
$$

$$+\left|\sum_{k=1}^{P_h}\{(a(u(t^{j,\theta}))-a(EU^{j,\theta}))\nabla(\partial_t\xi^j)\cdot n_k\}[\widetilde{w}^{j,\theta}]\right|$$
$$=\sum_{j=1}^{3} I_{3j}$$

and estimate I_{3i}, $1\le i\le 3$ in the followings

$$\begin{aligned}
I_{31} &\le C\|\nabla\widetilde{w}^{j,\theta}\|_\infty\Big(\|\eta(t^{j,\theta})\|+\|\alpha^{j,\theta}\|+\|\xi^j\|+\|\xi^{j-1}\|\Big)\|\nabla(\partial_t\xi^j)\|\\
&\le C\Big(\|\eta(t^{j,\theta})\|^2+\|\alpha^{j,\theta}\|^2+\|\xi^j\|^2+\|\xi^{j-1}\|^2\Big)+\varepsilon\|\partial_t\xi^j\|_1^2\\
I_{32} &\le C\sum_{k=1}^{P_h}\|\nabla\widetilde{w}^{j,\theta}\|_{\infty,e_k}\Big(\|\eta(t^{j,\theta})\|_{0,e_k}+\|\alpha^{j,\theta}\|_{0,e_k}+\|\xi^j\|_{0,e_k}+\|\xi^{j-1}\|_{0,e_k}\Big)\\
&\quad\cdot\|[\partial_t\xi^j]\|_{0,e_k}\\
&\le C\sum_{i=1}^{N_h}\|\nabla\widetilde{w}^{j,\theta}\|_{\infty,E_i}h^{-1/2}\Big(\|\eta(t^{j,\theta})\|_{0,E_i}+h\|\nabla\eta(t^{j,\theta})\|_{0,E_i}+\|\alpha^{j,\theta}\|_{0,E_i}\\
&\quad+\|\xi^j\|_{0,E_i}+\|\xi^{j-1}\|_{0,E_i}\Big)h^{1/2}\|\partial_t\xi^j\|_1\\
&\le C\Big(\|\eta(t^{j,\theta})\|^2+h^2\|\nabla\eta(t^{j,\theta})\|^2+\|\alpha^{j,\theta}\|^2+\|\xi^j\|^2+\|\xi^{j-1}\|^2\Big)+\varepsilon\|\partial_t\xi^j\|_1^2\\
I_{33} &\le C\sum_{k=1}^{P_h}\|\nabla(\partial_t\xi^j)\|_{\infty,e_k}\Big(\|\eta(t^{j,\theta})\|_{0,e_k}+\|\alpha^{j,\theta}\|_{0,e_k}+\|\xi^j\|_{0,e_k}+\|\xi^{j-1}\|_{0,e_k}\Big)\\
&\quad\cdot\|[\eta^{j,\theta}]\|_{0,e_k}\\
&\le C\sum_{i=1}^{N_h}\|\nabla(\partial_t\xi^j)\|_{\infty,E_i}h^{-1/2}\Big(\|\eta(t^{j,\theta})\|_{0,E_i}+h\|\nabla\eta(t^{j,\theta})\|_{0,E_i}+\|\alpha^{j,\theta}\|_{0,E_i}\\
&\quad+\|\xi^j\|_{0,E_i}+\|\xi^{j-1}\|_{0,E_i}\Big)\cdot h^{-1/2}\Big(\|\eta^{j,\theta}\|_{0,E_i}+h\|\nabla\eta^{j,\theta}\|_{0,E_i}\Big)\\
&\le Ch^{-\frac{d}{2}-1}\sum_{i=1}^{N_h}\|\nabla(\partial_t\xi^j)\|_{0,E_i}\Big(\|\eta(t^{j,\theta})\|_{0,E_i}+h\|\nabla\eta(t^{j,\theta})\|_{0,E_i}+\|\alpha^{j,\theta}\|_{0,E_i}\\
&\quad+\|\xi^j\|_{0,E_i}+\|\xi^{j-1}\|\Big)\cdot h^{\frac{d}{2}+1}\|u(t^{j,\theta})\|_{\frac{d}{2}+1}\\
&\le C\Big(\|\eta(t^{j,\theta})\|^2+h^2\|\nabla\eta(t^{j,\theta})\|^2+\|\alpha^{j,\theta}\|^2+\|\xi^j\|^2+\|\xi^{j-1}\|^2\Big)+\varepsilon\|\partial_t\xi^j\|_1^2.
\end{aligned}$$

Therefore we obtain

$$|I_3|\le C\Big(\|\eta(t^{j,\theta})\|^2+h^2\|\nabla\eta(t^{j,\theta})\|^2+\|\alpha^{j,\theta}\|^2+\|\xi^j\|^2+\|\xi^{j-1}\|^2\Big)+3\varepsilon\|\partial_t\xi^j\|_1^2.$$

And

$$|I_4|\le C\|\rho^{j,\theta}\|_1^2+\varepsilon\|\partial_t\xi^j\|_1^2.$$

To estimate I_5, we split I_5 into the following 3 terms

$$\begin{aligned}|I_5| &\le |((b(u(t^{j,\theta})) - b(EU^{j,\theta}))\nabla(\partial_t\widetilde{u}^j), \nabla\partial_t\xi^j)| \\ &\quad + \left|\sum_{k=1}^{P_h}\int_{e_k}\{(b(u(t^{j,\theta})) - b(EU^{j,\theta}))\nabla(\partial_t\widetilde{u}^j)\cdot n_k\}[\partial_t\xi^j]\right| \\ &\quad + \left|\sum_{k=1}^{P_h}\int_{e_k}\{(b(u(t^{j,\theta})) - b(EU^{j,\theta}))\nabla(\partial_t\xi^j)\cdot n_k\}[\partial_t\widetilde{u}^j]\right| \\ &= \sum_{j=1}^{3} I_{5j}.\end{aligned}$$

Now we estimate I_{5i}, $1 \le i \le 3$ in the followings

$$\begin{aligned}|I_{51}| &\le C\|\nabla(\partial_t\widetilde{u}^j)\|_\infty\Big(\|\eta(t^{j,\theta})\| + \|\alpha^{j,\theta}\| + \|\xi^j\| + \|\xi^{j-1}\|\Big)\|\partial_t\xi^j\|_1 \\ &\le C\Big(\|\eta(t^{j,\theta})\|^2 + \|\alpha^{j,\theta}\|^2 + \|\xi^j\|^2 + \|\xi^{j-1}\|^2\Big) + \varepsilon\|\partial_t\xi^j\|_1^2, \\ |I_{52}| &\le C\sum_{k=1}^{P_h}\|\nabla(\partial_t\widetilde{u}^j)\|_{\infty,e_k}\Big(\|\eta(t^{j,\theta})\|_{0,e_k} + \|\alpha^{j,\theta}\|_{0,e_k} + \|\xi^j\|_{0,e_k} + \|\xi^{j-1}\|_{0,e_k}\Big) \\ &\quad\cdot\|[\partial_t\xi^j]\|_{0,e_k} \\ &\le C\Big(\|\eta(t^{j,\theta})\|^2 + h^2\|\nabla\eta(t^{j,\theta})\|^2 + \|\alpha^{j,\theta}\|^2 + \|\xi^j\|^2 + \|\xi^{j-1}\|^2\Big) + \varepsilon\|\partial_t\xi^j\|_1^2, \\ |I_{53}| &\le C\sum_{k=1}^{P_h}\|\nabla(\partial_t\xi^j)\|_{\infty,e_k}\Big(\|\eta(t^{j,\theta})\|_{0,e_k} + \|\alpha^{j,\theta}\|_{0,e_k} + \|\xi^j\|_{0,e_k} + \|\xi^{j-1}\|_{0,e_k}\Big) \\ &\quad\cdot\|[\partial_t\widetilde{u}^j]\|_{0,e_k} \\ &\le C\sum_{i=1}^{N_h}\|\nabla(\partial_t\xi^j)\|_{\infty,E_i}h^{-1}\Big(\|\eta(t^{j,\theta})\|_{0,E_i} + h\|\nabla\eta(t^{j,\theta})\|_{0,E_i} + \|\alpha^{j,\theta}\|_{0,E_i} \\ &\quad + \|\xi^j\|_{0,E_i} + \|\xi^{j-1}\|_{0,E_i}\Big)\cdot\Big(\|\eta_t(t^{j,\theta})\|_{0,E_i} + h\|\nabla\eta_t(t^{j,\theta})\|_{0,E_i} + h\|\rho^{j,\theta}\|_1\Big) \\ &\le Ch^{-\frac{d}{2}-1}\sum_{i=1}^{N_h}\|\nabla(\partial_t\xi^j)\|_{0,E_i}\Big(\|\eta(t^{j,\theta})\|_{0,E_i} + h\|\nabla\eta(t^{j,\theta})\|_{0,E_i} + \|\alpha^{j,\theta}\|_{0,E_i} \\ &\quad + \|\xi^j\|_{0,E_i} + \|\xi^{j-1}\|_{0,E_i}\Big)\cdot\Big(h^{\frac{d}{2}+1}\|u_t(t^{j,\theta})\|_{\frac{d}{2}+1} + h\|\rho^{j,\theta}\|_1\Big) \\ &\le C\Big(\|\eta(t^{j,\theta})\|^2 + h^2\|\nabla\eta(t^{j,\theta})\|^2 + \|\alpha^{j,\theta}\|^2 + \|\xi^j\|^2 + \|\xi^{j-1}\|^2\Big) + \varepsilon\|\partial_t\xi^j\|_1^2.\end{aligned}$$

Summing the estimations of I_{5i}, $1 \le i \le 3$ then gives

$$|I_5| \le C\Big(\|\eta(t^{j,\theta})\|^2 + h^2\|\nabla\eta(t^{j,\theta})\|^2 + \|\alpha^{j,\theta}\|^2 + \|\xi^j\|^2 + \|\xi^{j-1}\|^2\Big) + 3\varepsilon\|\partial_t\xi^j\|_1^2.$$

Similarly we obtain the estimations of $I_6 \sim I_{10}$ as follows

$$
\begin{aligned}
|I_6| &\le C\Big(\|\eta(t^{j,\theta})\|^2 + \|\alpha^{j,\theta}\|^2 + \|\xi^j\|^2 + \|\xi^{j-1}\|^2\Big) + \varepsilon\|\partial_t \xi^j\|_1^2 \\
|I_7| &\le C\Big(\|\eta(t^{j,\theta})\|^2 + \|\gamma^{j,\theta}\|^2 + \|\xi^{j,\theta}\|^2 + \|\eta_t(t^{j,\theta})\|^2 + \|\rho^{j,\theta}\|^2 + \|\overline{\partial}_t \xi^{j+1}\|^2 \\
&\qquad + \|\overline{\partial}_t \xi^j\|^2\Big) + \varepsilon\|\partial_t \xi^j\|_1^2 \\
|I_8| &\le C\sum_{k=1}^{P_h} \|\nabla \xi^{j,\theta}\|_{0,e_k} \|[\partial_t \xi^j]\|_{0,e_k} \le C\|\nabla \xi^{j,\theta}\|^2 + \varepsilon\|\partial_t \xi^j\|_1^2 \\
|I_9| &\le C\sum_{k=1}^{P_h} \|\nabla(\partial_t \xi^j)\|_{0,e_k} \|[\xi^{j,\theta}]\|_{0,e_k} \le \; C\sum_{i=1}^{N_h} \|\nabla(\partial_t \xi^j)\|_{0,E_i} J(\xi^{j,\theta}, \xi^{j,\theta})^{1/2} \\
&\le C J_\beta^\sigma(\xi^{j,\theta}, \xi^{j,\theta}) + \varepsilon\|\partial_t \xi^j\|_1^2 \\
|I_{10}| &\le a^*\|\nabla \xi^{j,\theta}\|\|\nabla \partial_t \xi^j\| \le C\|\nabla \xi^{j,\theta}\|^2 + \varepsilon\|\partial_t \xi^j\|_1^2 \\
&\le C\{\|\nabla \xi^{j+1}\|^2 + \|\nabla \xi^{j-1}\|^2\} + \varepsilon\|\partial_t \xi^j\|_1^2 \\
&\le C\Delta t\sum_{\ell=1}^{j} \|\partial_t \xi^\ell\|^2 + \|\nabla \xi^1\| + \varepsilon\|\partial_t \xi^j\|_1^2.
\end{aligned}
$$

Substituting the estimations of I_i, $1 \le i \le 10$ into (4.6), we get

$$
\begin{aligned}
&\overline{\partial}_t \|\overline{\partial}_t \xi^{j+1}\|^2 + \partial_t\Big(\|\xi^j\|^2 + J_\beta^\sigma(\xi^j, \xi^j)\Big) + \|\partial_t \xi^j\|_1^2 \\
&\le C\Big\{\|\eta_{tt}(t^{j,\theta})\|^2 + \|\sigma^{j,\theta}\|^2 + \|\gamma^{j,\theta}\|_1^2 + \|\eta(t^{j,\theta})\|^2 + h^2\|\nabla \eta(t^{j,\theta})\|^2 + \|\alpha^{j,\theta}\|^2 \\
&\quad + \|\xi^j\|^2 + \|\xi^{j-1}\|^2 + \|\rho^{j,\theta}\|_1^2 + \|\eta_t(t^{j,\theta})\|^2 + \|\overline{\partial}_t \xi^{j+1}\|^2 + \|\overline{\partial}_t \xi^j\|^2 \\
&\quad + \|\xi^{j+1}\|^2 + \|\nabla \xi^{j+1}\|^2 + \|\nabla \xi^{j-1}\|^2 + J_\beta^\sigma(\xi^{j+1}, \xi^{j+1}) + J_\beta^\sigma(\xi^{j-1}, \xi^{j-1}) \\
&\quad + \Delta t\sum_{\ell=1}^{j} \|\partial_t \xi^\ell\|^2 + \|\nabla \xi^1\|\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\Delta t}\Big[(\|\overline{\partial}_t \xi^{j+1}\|^2 - \|\overline{\partial}_t \xi^j\|^2) + \frac{1}{2}(\|\xi^{j+1}\|^2 - \|\xi^{j-1}\|^2) \\
&\quad + \frac{1}{2}(J_\beta^\sigma(\xi^{j+1}, \xi^{j+1}) - J_\beta^\sigma(\xi^{j-1}, \xi^{j-1}))\Big] + \|\partial_t \xi^j\|_1^2 \\
&\le C\Big\{\|\eta(t^{j,\theta})\|^2 + h^2\|\nabla \eta(t^{j,\theta})\|^2 + \|\eta_t(t^{j,\theta})\|^2 + \|\eta_{tt}(t^{j,\theta})\|^2 + \|\sigma^{j,\theta}\|^2 \qquad (4.7) \\
&\quad + \|\gamma^{j,\theta}\|_1^2 + \|\alpha^{j,\theta}\|^2 + \|\rho^{j,\theta}\|_1^2 + \|\overline{\partial}_t \xi^{j+1}\|^2 + \|\overline{\partial}_t \xi^j\|^2 + \|\xi^{j-1}\|^2 + \|\xi^j\|^2 \\
&\quad + \|\xi^{j+1}\|^2 + \|\nabla \xi^{j+1}\|^2 + \|\nabla \xi^{j-1}\|^2 + J_\beta^\sigma(\xi^{j+1}, \xi^{j+1}) + J_\beta^\sigma(\xi^{j-1}, \xi^{j-1}) \\
&\quad + \Delta t\sum_{\ell=1}^{j} \|\partial_t \xi^\ell\|^2 + \|\nabla \xi^1\|\Big\}.
\end{aligned}
$$

By summing the both sides of (4.7) from $j = 1$ to $N - 1$, we conclude

$$\begin{aligned}
&\|\overline{\partial}_t\xi^N\|^2 - \|\overline{\partial}_t\xi^1\|^2 + \frac{1}{2}\Big(\|\xi^N\|^2 + \|\xi^{N-1}\|^2 - \|\xi^0\|^2 - \|\xi^1\|^2) + J^\sigma_\beta(\xi^N,\xi^N) \\
&+ J^\sigma_\beta(\xi^{N-1},\xi^{N-1}) - J^\sigma_\beta(\xi^0,\xi^0) - J^\sigma_\beta(\xi^1,\xi^1)\Big) + \Delta t\sum_{j=1}^{N-1}\|\partial_t\xi^j\|_1^2 \\
\leq\ & C\bigg((\Delta t)\sum_{j=1}^{N-1}\Big(\|\eta(t^{j,\theta})\|^2 + h^2\|\nabla\eta(t^{j,\theta})\|^2 + \|\eta_t(t^{j,\theta})\|^2 + \|\eta_{tt}(t^{j,\theta})\|^2 + \|\sigma^{j,\theta}\|^2 \\
&+ \|\gamma^{j,\theta}\|_1^2 + \|\alpha^{j,\theta}\|^2 + \|\rho^{j,\theta}\|_1^2\Big) + \|\nabla\xi^1\| + (\Delta t)\sum_{j=1}^{N}\|\overline{\partial}_t\xi^j\|^2 \\
&+ \Delta t\sum_{j=0}^{N}\Big(\|\xi^j\|^2 + \|\nabla\xi^j\|^2 + J^\sigma_\beta(\xi^j,\xi^j)\Big) + \Delta t\sum_{j=1}^{N-1}\Delta t\sum_{\ell=1}^{j}\|\partial_t\xi^\ell\|^2\bigg),
\end{aligned}$$

which implies

$$\begin{aligned}
&\|\overline{\partial}_t\xi^N\|^2 + C\big(\|\xi^N\|^2 + \|\nabla\xi^N\|^2 + J^\sigma_\beta(\xi^N,\xi^N)\big) + \Delta t\sum_{j=1}^{N-1}\|\partial_t\xi^j\|_1^2 \\
\leq\ & \|\overline{\partial}_t\xi^1\|^2 + C\big(\|\xi^0\|^2 + \|\xi^1\|^2 + \|\nabla\xi^1\|^2 + J^\sigma_\beta(\xi^0,\xi^0) + J^\sigma_\beta(\xi^1,\xi^1) + \|\nabla\xi^0\|^2\big) \\
&+ C\Delta t\sum_{j=1}^{N-1}\Big(\|\eta(t^{j,\theta})\|^2 + h^2\|\nabla\eta(t^{j,\theta})\||^2 + \|\eta_t(t^{j,\theta})\|^2 + \|\eta_{tt}(t^{j,\theta})\|^2 \\
&+ \|\sigma^{j,\theta}\|^2 + \|\gamma^{j,\theta}\|_1^2 + \|\alpha^{j,\theta}\|^2 + \|\rho^{j,\theta}\|_1^2\Big) + C\Delta t\sum_{j=1}^{N}\|\overline{\partial}_t\xi^j\|^2 \\
&+ C\Delta t\sum_{j=0}^{N}\Big(\|\xi^j\|^2 + \|\nabla\xi^j\|^2 + J^\sigma_\beta(\xi^j,\xi^j)\Big) + C\Delta t\sum_{j=0}^{N-1}\Delta t\sum_{\ell=1}^{j}\|\partial_t\xi^\ell\|^2.
\end{aligned}$$

By applying the discrete type Gronwall's Lemma we have

$$\begin{aligned}
&\|\overline{\partial}_t\xi^N\|^2 + \frac{1}{2}\big(\|\xi^N\|^2 + \|\nabla\xi^N\|^2 + J^\sigma_\beta(\xi^N,\xi^N)\big) + \Delta t\sum_{j=1}^{N-1}\|\partial_t\xi^j\|_1^2 \\
\leq\ & \|\overline{\partial}_t\xi^1\|^2 + \frac{1}{2}\big(\|\xi^1\|^2 + \|\nabla\xi^1\|^2 + J^\sigma_\beta(\xi^1,\xi^1)\big) + C\Delta t\sum_{j=1}^{N-1}\Big\{\|\eta(t^{j,\theta})\|^2 \\
&+ h^2\|\nabla\eta(t^{j,\theta})\|^2 + \|\eta_t(t^{j,\theta})\|^2 + \|\eta_{tt}(t^{j,\theta})\|^2 + \|\sigma^{j,\theta}\|^2 + \|\gamma^{j,\theta}\|_1^2 \\
&+ \|\alpha^{j,\theta}\|^2 + \|\rho^{j,\theta}\|_1^2\Big\}.
\end{aligned}$$

Since $(\overline{\partial}_t U^1, v) = (V_1, v) + \dfrac{\Delta t}{2}(V_2, v)$ and (4.1) we have

$$(\overline{\partial}_t \xi^1, v) = (\overline{\partial}_t \widetilde{u}^1, v) - (\overline{\partial}_t U^1, v) = (\overline{\partial}_t \widetilde{u}^1, v) - (V_1, v) - \frac{\Delta t}{2}(u_{tt}(0), v). \tag{4.8}$$

Now we let $v = \overline{\partial}_t \xi^1$ in (4.8) to get

$$\begin{aligned}
\|\overline{\partial}_t \xi^1\|^2 &= \Big(\overline{\partial}_t \widetilde{u}^1 - u_t(0) - \frac{\Delta t}{2} u_{tt}(0), \overline{\partial}_t \xi^1\Big) \\
&= \Big(\frac{\widetilde{u}^1 - \widetilde{u}^0}{\Delta t} - u_t(0) - \frac{\Delta t}{2} u_{tt}(0), \overline{\partial}_t \xi^1\Big) \\
&= \Big(\widetilde{u}_t(0) + \frac{\Delta t}{2} \widetilde{u}_{tt}(0) - u_t(0) - \frac{\Delta t}{2} u_{tt}(0) + \frac{(\Delta t)^2}{2} u_{ttt}(t_0^*), \overline{\partial}_t \xi^1\Big) \\
&= \Big(- \Big(\eta_t(0) + \frac{\Delta t}{2} \eta_{tt}(0)\Big) + \frac{(\Delta t)^2}{2} u_{ttt}(t_0^*), \overline{\partial}_t \xi^1\Big),
\end{aligned}$$

which implies the following

$$\|\overline{\partial}_t \xi^1\| \le C\Big(\|\eta_t(0)\| + \Delta t \|\eta_{tt}(0)\| + (\Delta t)^2\Big)$$

$$\|\xi^1\| = \|\Delta t \overline{\partial}_t \xi^1\| \le C\Delta t\Big(\|\eta_t(0)\| + \Delta t \|\eta_{tt}(0)\| + (\Delta t)^2\Big) \tag{4.9}$$

$$\begin{aligned}
\|\nabla \xi^1\| = Ch^{-1}\|\xi^1\| &\le \ Ch^{-1}(\Delta t)\Big(\|\eta_t(0)\| + \Delta t \|\eta_{tt}(0)\| + (\Delta t)^2\Big) \\
&\le C\Big(\|\eta_t(0)\| + \Delta t \|\eta_{tt}(0)\| + (\Delta t)^2\Big).
\end{aligned}$$

By combining the results of (4.9), the definition of J_β^σ and the condition that $\beta = \dfrac{1}{d-1}$ we have

$$J_\beta^\sigma(\xi^1, \xi^1) \le C\|\xi^1\|_1^2 \le C\Big(\|\eta_t(0)\|^2 + (\Delta t)^2 \|\eta_{tt}(0)\|^2 + (\Delta t)^4\Big).$$

Therefore we have the estimation of ξ^1 as follows

$$\begin{aligned}
&\|\xi^1\|^2 + \|\nabla \xi^1\|^2 + J_\beta^\sigma(\xi^1, \xi^1) + \|\overline{\partial}_t \xi^1\|^2 \\
&\le C\Big(\|\eta_t(0)\|^2 + (\Delta t)^2 \|\eta_{tt}(0)\|^2 + (\Delta t)^4\Big).
\end{aligned} \tag{4.10}$$

Consequently if $\theta = 1$, then

$$\|\overline{\partial}_t \xi^N\|^2 + \|\xi^N\|^2 + \|\nabla \xi^N\|^2 + J_\beta^\sigma(\xi^N, \xi^N) + \Delta t \sum_{j=1}^{N-1} \|\partial_t \xi^j\|_1^2 \le C(h^{2s} + (\Delta t)^2)$$

and if $\theta = 0$, then

$$\begin{aligned}
&\|\overline{\partial}_t \xi^N\|^2 + \|\xi^N\|^2 + \|\nabla \xi^N\|^2 + J_\beta^\sigma(\xi^N, \xi^N) + \Delta t \sum_{j=1}^{N-1} \|\partial_t \xi^j\|_1^2 \\
&\le C(h^{2s} + (\Delta t)^4).
\end{aligned}$$

Thererfore we have the following results

$$\|e\|_{\ell^\infty(L^2)} \leq \begin{cases} C(h^\mu + \Delta t) & \text{if } \theta = 1 \\ C(h^\mu + (\Delta t)^2) & \text{if } \theta = 0 \end{cases}$$

where $\mu = \min(r+1, s)$. □

References

1. D. N. Arnold, *An interior penalty finite element method with discontinuous elements*, SIAM J. Numer. Anal. 19 (1982), 724–760.
2. I. Babuška and M. Suri, *The h-p version of the finite element method with quasi-uniform meshes*, RAIRO Modél. Math. Anal. Numer. 21 (1987), 199–238.
3. I. Babuška and M. Suri, *The optimal convergence rates of the p-version of the finite element method*, SIAM J. Numer. Anal. 24 (1987), 750–776.
4. B. Darlow, R. Ewing and M. F. Wheeler, *Mixed finite element methods for miscible displacement problems in porous media*, Soc. Pet. Erg. Report SPE 10500 (1982).
5. J. Douglas and T. Dupont, *Interior penalty procedures for elliptic and parabolic Galerkin methods*, Lecture Notes in Physics 58 (1976), 207–216.
6. R. E. Ewing, *Time-stepping Galerkin methods for nonlinear Sobolev partial differential equations*, SIAM J. Numer. Anal. 15 (1978), 1125–1150.
7. H. Koch and S. Antman, *Stability and Hopf bifurcation for fully nonlinear parabolic hyperbolic equations*, SIAM J. Math. Anal. 32(2) (2000), 360–384.
8. Q. Lin and S. Zhang, *A direct global superconvergence analysis for Sobolev and viscoelasticity type equations*, Applications of Mathematics 42(1) (1997), 23–34.
9. J. T. Oden, I. Babuška and C. E. Baumann, *A discontinuous hp finite element method for diffusion problems*, J. Compu. Phys. 146 (1998), 491–519.
10. M. R. Ohm, H. Y. Lee and J. Y. Shin, *L^2-error estimates of the extrapolated Crank-Nicolson discontinuous Galerkin approximations for nonlinear Sobolev equations*, Jour. of Inequalities and Applications, vol. 5(100) (2010), 1–17.
11. M. R. Ohm, H. Y. Lee and J. Y. Shin, *Error estimates for a discontinuous Galerkin method for nonlinear parabolic equations*, Jour. of Math. Anal. and Appli. 315 (2006), 132–143.
12. B. Rivière and S. Shaw, *Discontinuous Galerkin finite element approximation of nonlinear non-Fickian diffusion in viscoelastic polymers*, SIAM J. Numer. Anal. 44(6) (2006), 2650–2670.
13. B. Rivière and M. F. Wheeler, *Nonconforming methods for transport with nonlinear reaction*, Contemporary Mathematics 295 (2002), 421–432.
14. S. Shaw and J. Whiteman, *Numerical solution of linear quasistatic hereditary viscoelasticity problems*, SIAM J. Numer. Anal., 38(1) (2000), 80–97.
15. T. Sun and D. Yang, *A priori error estimates for interior penalty discontinuous Galerkin method applied to nonlinear Sobolev equations*, Applied Mathematics and Computation 200 (2008), 147–159.
16. T. Sun and D. Yang, *Error estimates for a discontinuous Galerkin method with interior penalties applied to nonlinear Sobolev equations*, Numerical Methods Partial Differential Equations 24(3) (2008), 879–896.
17. S. Sun and M. F. Wheeler, *Symmetric and nonsymmetric discontinuous Galerkin methods for reactive transport in porous media*, SIAM J. on Num. Anal. 43(1) (2005), 195–219.
18. M. F. Wheeler, *An elliptic collocation finite element method with interior penalties*, SIAM J. Numer. Anal. 15 (1978), 152–161.

19. M. F. Wheeler, *A priori error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal. 10 (1973), 723–759.

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