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EIGENVALUES OF SECOND-ORDER VECTOR EQUATIONS ON TIME SCALES WITH BOUNDARY VALUE CONDITIONS

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ABSTRACT. This paper is concerned with eigenvalues of second-order vector equations on time scales with boundary value conditions. Properties of eigenvalues and matrix-valued solutions are studied. Relationships between eigenvalues of different boundary value problems are discussed.

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1. Introduction

In order to unify continuous and discrete analysis, Higler [9] in 1988 introduced the theory of time scales, which eliminates obscurity from both to some degree. The investigation on various aspect of this theory has been expounded; see Bohner and Peterson [7], Agarwal et al. [3], Atici and Guseinov [5], Amster, Rogers, and Tisdell [4], Sun [14] and references cited therein.

A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$. An introduction to the time scales calculus can be found in [1, 2, 3, 4, 5, 7, 10]. We lay out the terms and notation needed later in the discussion.

Let $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ be the forward and backward jump functions in $\mathbb{T}$, respectively. The point $t \in \mathbb{T}$ is left-scattered and right-scattered if $\rho(t) < t$ and $\sigma(t) > t$, respectively. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. We abbreviate $f^\sigma(t) = f(\sigma(t))$. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T} = \mathbb{T}^k$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T} = \mathbb{T}_k$. The forward graininess is $\mu(t) := \sigma(t) - t$.

For $f : \mathbb{T} \to \mathbb{C}$ and $t \in \mathbb{T}^k$, the delta derivative of f at t, denoted $f^\Delta(t)$, is the number (provided it exists) with the property that give any $\varepsilon > 0$, there is a

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neighborhood U of t such that $|f^\sigma(t)-f(s)-f^\Delta(t)[\sigma(t)-s]| \le \varepsilon|\sigma(t)-s|$ for all $s\in U$. For $\mathbb{T}=\mathbb{R}$, we have $f^\Delta = f'$ and for $\mathbb{T}=\mathbb{Z}$, we have $f^\Delta(t) = f(t+1)-f(t)$.

A function $f:\mathbb{T}\to\mathbb{C}$ is called right-dense continuous or rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

It is known from [7, Theorem 1.74] that if f is rd-continuous, then there is a function F such that $F^\Delta(t) = f(t)$. In this case, we define $\int_a^b f(t)\Delta t = F(b)-F(a)$.

Consider the following second-order vector equation on time scales

$$-(P(t)x^\Delta(t))^\Delta + Q(t)x^\sigma(t) = \lambda w(t)x^\sigma(t), \quad t\in\mathbb{I} := [\rho(a), b]\cap\mathbb{T}, \tag{1}$$

with the separated boundary value conditions

$$R_1x(\rho(a)) - R_2P(\rho(a))x^\Delta(\rho(a)) = 0 \qquad S_1x(b) + S_2P(b)x^\Delta(b) = 0, \tag{2}$$

where $a,b\in\mathbb{T}$, $a<b$; Q and w are $n\times n$-matrix-valued continuous functions; P is nonsingular; w, P, and Q are Hermitian with $w>0$ for all $t\in\mathbb{I}$; λ is a complex-valued parameter; and R_1, R_2, S_1 and S_2 are $n\times n$ matrices with

$$\operatorname{rank}(R_1, R_2) = \operatorname{rank}(S_1, S_2) = n, \tag{3}$$

and

$$R_1R_2^* = R_2R_1^*, \qquad S_1S_2^* = S_2S_1^*. \tag{4}$$

Denote vector-valued and $n\times n$-matrix-valued solutions by small letters and capital letters, respectively. We always assume that $\mathbb{I}$ contains at least three points.

In this paper, we will discuss the properties of eigenvalues of the problem (1) with (2). Especially, if we set $S_1 = I_n$, $S_2 = 0_n$, and $S_1 = 0_n$, $S_2 = P^{-1}(b)$, (2) will become

$$R_1x(\rho(a)) - R_2P(\rho(a))x^\Delta(\rho(a)) = 0 \qquad x(b) = 0 \tag{5}$$

and

$$R_1x(\rho(a)) - R_2P(\rho(a))x^\Delta(\rho(a)) = 0 \qquad x^\Delta(b) = 0, \tag{6}$$

respectively, where I_n and 0_n denote the $n\times n-$identity and zero matrices. We will discuss the relationships between eigenvalues of the different problems (1) with (5) and (1) with (6).

In fact, Agarwal, Bohner, and Wong [2, Theorem 7] got a beautiful comparison theorem between eigenvalues (denoted by λ_k) of the problem

$$y^{\Delta\Delta} + qy^\sigma = -\lambda y^\sigma \tag{7}$$

with the boundary value conditions

$$\alpha y(\rho(a)) + \beta y^\Delta(\rho(a)) = 0 \quad \gamma y(b) + \delta y^\Delta(b) = 0,$$

and eigenvalues (denoted by λ_k^*) for (7) and another boundary value conditions

$$\alpha y(\rho(a)) + \beta y^\Delta(\rho(a)) = 0 \quad y(b) = 0,$$

with corresponding normalized eigenfunctions, where $\alpha, \beta, \gamma, \delta$ are real number, and q is a real continuous function. They proved that $\lambda_k \le \lambda_k^* < \lambda_{k+1}$ if $k \in \mathbb{N}_0$ with $\lambda_k^* < \infty$.

However, it seems to us that very little is known regarding the relations between eigenvalues of these two different boundary problems (1) with (5) and (1) with (6). A big difficulty lies in that in scalar cases, eigenvalues of the separated boundary value problems are simple while multiple eigenvalues will appear in higher dimensional cases. Nevertheless, we (Theorem 1) can prove that λ is an eigenvalue of the problem (1) with (2) if and only if λ is a zero of $\det \Lambda(b, \lambda)$, where

$$\Lambda(t, \lambda) := S_1 X(t, \lambda) + S_2 P(t) X^{\Delta}(t, \lambda)$$

and $X(\cdot, \lambda)$ is the matrix-valued solution of (1) satisfying the initial value conditions

$$X(\rho(a), \lambda) = R_2^* \quad and \quad X^{\Delta}(\rho(a), \lambda) = P^{-1}(\rho(a)) R_1^*. \tag{8}$$

It follows that λ is an eigenvalue of the problem (1) with (5) and the problem (1) with (6) if and only if λ is a zero of $\det X(b, \lambda)$ and $\det X^{\Delta}(b, \lambda)$, respectively. We will extend the result [2, Theorem 7] to zeros of $\det X(b, \lambda)$ and $\det X^{\Delta}(b, \lambda)$. Our conclusion can be regarded as the higher dimensional case of [2, Theorem 7].

We remark that we use another way which is different from [2] in which functional analysis and a oscillation theorem of eigenfunctions to generalized zeros are used. Motivated by Atkinson [6] and Shi [13], we will mainly use a Cayley transformation. Atkinson [6, Section 6.7] researched the oscillation with respect to the parameter λ of the matrix recurrence relation

$$Y_{n+1}(\lambda) = (\lambda A_n + B_n) Y_n(\lambda) - Y_{n-1}(\lambda), \quad n = 0, 1, ...,$$

where $k \times k$ matrices $A_n > 0$ and B_n are Hermitian. Shi [13] extended his results to the vector difference equation:

$$-\nabla(C_n \Delta x_n) + B_n x_n = \lambda w_n x_n, \quad n \in [1, +\infty),$$

where C_n, B_n, w_n are $d \times d$ Hermitian matrices, C_n is nonsingular, $w > 0$, and Δ and ∇ are forward and backward difference operators. They both used a Cayley transformation to discuss the oscillation and to prove separation theorems for matrix-valued solutions.

The setup of this paper is as follows. Section 2 collects several useful lemmas. In Section 3, the properties of eigenvalues and zeros of $\det \Lambda(b, \lambda)$ and the relations between zeros of $\det X(b, \lambda)$ and $\det X^{\Delta}(b, \lambda)$ are presented.

2. Preliminaries

In this paper, we suppose that $Lx := -(P(t)x^{\Delta})^{\Delta} + Q(t)x^{\sigma}$. The following Lemmas 1, 2, and 3 are well known. Their proofs can be found in [1, Lemma 3], [7, Theorem 5.41], [8, Lemma 2], and [12, Corollary 3.1.3].

Lemma 1. *If x, y are solutions of (1), then*

$$W(x,y)(\tau)|_{\rho(a)}^{t} = \int_{\rho(a)}^{t} [x^{*\sigma}(\tau)(Ly)(\tau) - (Lx)^{*}(\tau)y^{\sigma}(\tau)]\Delta\tau,$$

for all $t \in \mathbb{I}$, where $W(x,y) := (Px^{\Delta})^{*}y - x^{*}Py^{\Delta}$.

Lemma 2. *For each $\lambda \in \mathbb{R}$, if x and y are solutions of (1), then $W(x,y)$ is constant, for $t \in \mathbb{I}$.*

Lemma 3. *If (3) and (4) hold, then $ker(R_1, R_2) = Im \begin{pmatrix} R_2^{*} \\ -R_1^{*} \end{pmatrix}$, $ker(S_1, S_2) = Im \begin{pmatrix} S_2^{*} \\ -S_1^{*} \end{pmatrix}$, where "Im" denotes a image set.*

The following lemma is a direct consequence of Lemma 3.

Lemma 4. *Assume that (2), (3), and (4) hold. Then there exist vectors ξ and $\eta \in \mathbb{C}^n$, such that*

$$\begin{pmatrix} x(\rho(a)) \\ -P(\rho(a))x^{\Delta}(\rho(a)) \end{pmatrix} = \begin{pmatrix} R_2^{*} \\ -R_1^{*} \end{pmatrix} \xi$$

and

$$\begin{pmatrix} x(b) \\ P(b)x^{\Delta}(b) \end{pmatrix} = \begin{pmatrix} S_2^{*} \\ -S_1^{*} \end{pmatrix} \eta.$$

Remark 1. The sign "Im" always denotes a imaginary part in this paper except Lemma 3.

The following Theorem 1, Lemma 5, and Lemma 6 are similar to [8, Corollary 1], [8, formula (7)], and [8, Lemma 4]. We give their proofs for completeness.

Theorem 1. *Let $X(\cdot, \lambda)$ be the matrix-valued solution of (1) and (8). Then λ is an eigenvalue of the boundary value problem (1) with (2) if and only if* $\det \Lambda(b, \lambda) = 0$.

Proof. If $\det \Lambda(b, \lambda) = 0$, then there is a vector $\xi \in \mathbb{C}^n$ with $\xi \neq 0$ such that $\Lambda(t, \lambda)\xi = 0$. Let $x(t, \lambda) = X(t, \lambda)\xi$. It follows from (8) that (2) holds. Therefore, λ is an eigenvalue of the boundary value problem (1) and (2).

Conversely, suppose that λ is an eigenvalue of (1) and (2), and $x(\cdot, \lambda)$ is an eigenfunction respect to λ. Thus there is a unique vector $\eta \in \mathbb{C}^n$ with $\eta \neq 0$, such that $x(t, \lambda) = X(t, \lambda)\eta$. That is the homogeneous linear system $\Lambda(b, \lambda)\eta = 0$ has a nontrivial solution η. So $\det \Lambda(b, \lambda) = 0$. This completes the proof. $\square$

Lemma 5. *For every $\lambda, \mu \in \mathbb{C}$, let $x(\cdot, \lambda)$ and $y(\cdot, \mu)$ be solutions of (1) respect to λ and μ, respectively. Then*

$$W[x(\tau,\lambda), y(\tau,\mu)]_{\rho(a)}^{t} = (\lambda - \bar{\mu}) \int_{\rho(a)}^{t} x^{*\sigma}(\tau,\lambda) w(\tau) y^{\sigma}(\tau,\mu)\Delta t,$$

for all $t \in \mathbb{I}$. Especially,

$$W[x(\tau,\lambda), x(\tau,\lambda)]_{\rho(a)}^{t} = 2iIm\lambda \int_{\rho(a)}^{t} x^{*\sigma}(\tau,\lambda) w(\tau) x^{\sigma}(\tau,\lambda)\Delta t. \tag{9}$$

Proof. From Lemma 1,

$$\begin{aligned}
& W[x(\tau,\lambda), y(\tau,\mu)]_{\rho(a)}^{t} \\
= & \int_{\rho(a)}^{t} [x^{*\sigma}(\tau,\lambda) Ly(\tau,\mu) - (Lx)^{*}(\tau,\lambda) y^{\sigma}(\tau,\mu)]\Delta t \\
= & \int_{\rho(a)}^{t} [x^{*\sigma}(\tau,\lambda) \lambda w(\tau) y^{\sigma}(\tau,\mu) - x^{*\sigma}(\tau,\lambda)\bar{\mu} w(\tau) y^{\sigma}(\tau,\mu)]\Delta t \\
= & (\lambda - \bar{\mu}) \int_{\rho(a)}^{t} x^{*\sigma}(\tau,\lambda) w(\tau) y^{\sigma}(\tau,\mu)\Delta t.
\end{aligned}$$

Especially, when $\lambda = \mu$, (9) holds. This completes the proof. □

From [14], we have known that $x(\cdot, \lambda)$ is an entire function of λ. So, we get the following results.

Lemma 6. *For each $\lambda \in \mathbb{R}$ and all $t \in \mathbb{I}$, if $x(\cdot, \lambda)$ is the solution of (1) and*

$$x(\rho(a), \lambda) = R_2^{*}\xi, \quad x^{\Delta}(\rho(a), \lambda) = P^{-1}(\rho(a)) R_1^{*}\xi, \tag{10}$$

where $\xi \in \mathbb{C}^n$, then

$$\begin{aligned}
& \int_{\rho(a)}^{t} x^{*\sigma}(\tau,\lambda) w(\tau) x(\tau,\lambda)\Delta\tau \\
& = \frac{\partial}{\partial\lambda} x^{\Delta *}(t,\lambda) P(t) x(t,\lambda) - \frac{\partial}{\partial\lambda} x^{*}(t,\lambda) P(t) x^{\Delta}(t,\lambda) \qquad (11) \\
& = x^{\Delta *}(t,\lambda) P(t) \frac{\partial}{\partial\lambda} x(t,\lambda) - x^{*}(t,\lambda) P(t) \frac{\partial}{\partial\lambda} x^{\Delta}(t,\lambda). \qquad (12)
\end{aligned}$$

Proof. Given $\lambda \in \mathbb{R}$ and a small $\delta \in \mathbb{R}$ with $\delta \neq 0$, by Lemma 2, from (4) and (10), we have

$$\begin{aligned}
& (P(t)x^{\Delta}(t,\lambda))^{*} x(t,\lambda) - x^{*}(t,\lambda) P(t) x^{\Delta}(t,\lambda) \\
& = (P(\rho(a))x^{\Delta}(\rho(a),\lambda))^{*} x(\rho(a),\lambda) - x^{*}(\rho(a),\lambda) P(\rho(a)) x^{\Delta}(\rho(a),\lambda) = 0.
\end{aligned}$$

By using of Lemma 5, we get

$$\begin{aligned}
&\int_{\rho(a)}^{t} x^{*\sigma}(\tau,\lambda+\delta)w(\tau)x(\tau,\lambda)\Delta\tau \\
= &\ \frac{1}{\delta}\{[P(t)x^{\Delta}(t,\lambda+\delta)]^{*}x(t,\lambda) - x^{*}(t,\lambda+\delta)P(t)x^{\Delta}(t,\lambda) + h\} \\
= &\ \frac{1}{\delta}\{[x^{\Delta *}(t,\lambda+\delta) - x^{\Delta *}(t,\lambda)]P(t)x(t,\lambda) \\
&-[x^{*}(t,\lambda+\delta) - x^{*}(t,\lambda)]P(t)x^{\Delta}(t,\lambda) + h\},
\end{aligned}$$

where

$$h = -[P(\rho(a))x^{\Delta}(\rho(a),\lambda+\delta)]^{*}x(\rho(a),\lambda) + x^{*}(\rho(a),\lambda+\delta)P(\rho(a))x^{\Delta}(\rho(a),\lambda).$$

Since $x(\cdot,\lambda)$ is an entire function of λ (see [14]), (11) is derived from the above relations by letting $\delta \to 0$. We can prove (12) in a similar way. The proof is completed. $\square$

Lemma 7. *For every $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{C}^n$ with $\xi \neq 0$, if $x(\cdot,\lambda)$ is the solution of (1) and (10), then $x(\cdot,\lambda)$ is a nontrivial solution.*

Proof. Assume the contrary that there exist $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{C}^n$ with $\xi \neq 0$ such that $x(\cdot,\lambda) \equiv 0$ satisfied (10). Then, by the uniqueness of the initial value problem,

$$0 = \begin{pmatrix} x(\rho(a),\lambda) \\ x^{\Delta}(\rho(a),\lambda) \end{pmatrix} = \begin{pmatrix} R_2^{*} \\ P^{-1}(\rho(a))R_1^{*} \end{pmatrix}\xi.$$

So $R_2^{*}\xi = P^{-1}(\rho(a))R_1^{*}\xi = 0$. This implies that $\begin{pmatrix} R_2^{*} \\ R_1^{*} \end{pmatrix}\xi = 0$. Then $\xi = 0$ from (3). This is a contradiction and the proof is completed. $\square$

3. Main Results

In this section, we always assume that $x(\cdot,\lambda)$ and $X(\cdot,\lambda)$ are the vector-valued and the matrix-valued solutions of the problems (1) with (10) and (1) with (8), respectively.

3.1. Properties of Eigenvalues and Zeros of $\det\Lambda(b,\lambda)$

Theorem 2. *Eigenvalues of the boundary value problem (1.1) and (1.2) are real.*

Theorem 2 is well known. See [8, Remark 2(iii)]. The following result is a direct consequence of Theorems 2 and 1.

Theorem 3. *Zeros of $\det\Lambda(b,\lambda)$ are all real.*

The following result implies the property of multiplicity of zeros of $\det\Lambda(b,\lambda)$.

Theorem 4. *Let* λ_0 *be any zero of* $\det \Lambda(b, \lambda)$ *. Then* $\frac{\partial}{\partial \lambda}\Lambda(b, \lambda_0) \neq 0$.

Proof. Assume the contrary. There exists $\lambda_0 \in \mathbb{C}$ such that $\det \Lambda(b, \lambda_0) = 0$ and $\frac{\partial}{\partial \lambda}\Lambda(b, \lambda_0) = 0$. Then λ_0 is real by Theorem 3 and there exists $\xi \in \mathbb{C}^n$ with $\xi \neq 0$ such that

$$\Lambda(b, \lambda_0)\xi = 0. \tag{13}$$

Set $x(t, \lambda_0) = X(t, \lambda_0)\xi$. Since $X(t, \lambda_0)$ satisfies (8), $x(t, \lambda_0)$ is the solution of (1) and (10). For $\xi \neq 0$, $x(t, \lambda_0)$ is a nontrivial solution by using of Lemma 2.7. So $x(t, \lambda_0)$ is an eigenfunction of the boundary value problem (1) and (2). Since

$$\frac{\partial}{\partial \lambda}\Lambda(b, \lambda_0)\xi = S_1 \frac{\partial}{\partial \lambda}x(b, \lambda_0) + S_2 P(b)\frac{\partial}{\partial \lambda}x^{\Delta}(b, \lambda_0) = 0,$$

it follows that, from (13) and by Lemma 4, there exist $\eta_1, \eta_2 \in \mathbb{C}^n$ such that

$$\begin{array}{ll} x(b, \lambda_0) = S_2^*\eta_1 & P(b)x^{\Delta}(b, \lambda_0) = -S_1^*\eta_1 \\ \frac{\partial}{\partial \lambda}x(b, \lambda_0) = S_2^*\eta_2, & P(b)\frac{\partial}{\partial \lambda}x^{\Delta}(b, \lambda_0) = -S_1^*\eta_2. \end{array}$$

Thus, from (11),

$$\begin{aligned} & \int_{\rho(a)}^{b} x^{*\sigma}(\tau, \lambda)w(\tau)x(\tau, \lambda)\Delta\tau \\ = \; & \frac{\partial}{\partial \lambda}x^{\Delta *}(b, \lambda_0)P(b)x(b, \lambda_0) - \frac{\partial}{\partial \lambda}x^{*}(b, \lambda_0)P(b)x^{\Delta}(b, \lambda_0) \\ = \; & -\eta_2^* S_1 S_2^* \eta + \eta_2^* S_2 S_1^* \eta_1 = 0, \end{aligned}$$

which contradicts the fact that $x(\cdot, \lambda_0)$ is nontrivial. The proof is completed. $\square$

3.2. Relations Between Zeros of $\det X(b, \lambda)$ and $\det X^{\Delta}(b, \lambda)$

Although we are not sure if $\det X(b, \lambda)$ and $\det X^{\Delta}(b, \lambda)$ have common zeros, we can directly make a conclusion below without proof, which is a uniqueness of solutions property.

Theorem 5. *For each* $\xi \in \mathbb{C}^n$ *with* $\xi \neq 0$, $x(b, \lambda)$ *and* $x^{\Delta}(b, \lambda)$ *have no common zeros.*

With a similar argument to that in [2,Theorem 7], we have that between any eigenvalues of one of the problems (1) with (5) and (1) with (6) lies an eigenvalue of the other in the scalar case. That is, by using of Theorem 1, between any zeros of one of $\det X(b, \lambda)$ and $\det X^{\Delta}(b, \lambda)$ lies a zero of the other in the case $n = 1$. This fact can be regarded as a comparison theorem of eigenvalues. But this result isn't true in the case $n \geq 2$. Consider the 2-dimensional vector difference equation:

$$-\Delta(\Delta x(t)) + Q(t)x(t+1) = \lambda x(t+1), \quad t \in \mathbb{I} = \{0, 1, 2, 3\},$$

where Δ is the forward difference operator $\Delta x(t) = x(t+1) - x(t)$, $n = 2$, and $Q(t) = \mathrm{diag}\{1, -t+2\}$. For given $X(0) = 0$ and $\Delta X(0) = \mathrm{I}_2$, the solution is

$$X(2, \lambda) = \mathrm{diag}\{3 - \lambda, 4 - \lambda\},$$

$$X(3,\lambda) = \operatorname{diag}\{\lambda^2 - 6\lambda + 8, \lambda^2 - 7\lambda + 11\},$$

and

$$X(4,\lambda) = \operatorname{diag}\{-\lambda^3 + 9\lambda^2 - 25\lambda + 21, -\lambda^3 + 9\lambda^2 - 24\lambda + 18\}.$$

So

$$\det X(3,\lambda) = \lambda^4 - 13\lambda^3 + 61\lambda^2 - 122\lambda + 88$$

and

$$\det \Delta X(3,\lambda) = \lambda^6 - 16\lambda^5 + 100\lambda^4 - 308\lambda^3 + 483\lambda^2 - 354\lambda + 91.$$

And the zeros of $\det X(3,\lambda)$ are

$$\alpha_1 = 2.0000,\ \alpha_2 = 2.3820,\ \alpha_3 = 4.0000,\ \text{and } \alpha_4 = 4.6180.$$

The zeros of $\det \Delta X(3,\lambda)$ are

$$\beta_1 = 0.5395,\ \beta_2 = 1.1981,\ \beta_3 = 2.5550,\ \beta_4 = 2.7609,$$
$$\beta_5 = 4.2470,\ \text{and } \beta_6 = 4.6996.$$

We find that between the two zeros α_1 and α_2 of $\det X(3,\lambda)$ lies no zero of $\det \Delta X(3,\lambda)$. But, it is interesting that $\det \Delta X(3,\lambda)$ has two zeros β_3 and β_4 in (α_1,α_3) and three zeros β_3, β_4, and β_5 in (α_2,α_4). What's more, both the interval $[\alpha_1,\alpha_3]$ and the interval $[\alpha_2,\alpha_4]$ exactly contain $2+1$ zeros of $\det X(3,\lambda)$. In fact, this conclusion can be made not only for discrete cases but also for any case on time scales. We have the following result.

Theorem 6. *If the interval $[\lambda_1,\lambda_2]$ contains $n+1$ zeros of $\det X(b,\lambda)$ (multiplicity included), $\det X^{\Delta}(b,\lambda)$ has at least one zero in the open interval (λ_1,λ_2).*

To prove Theorem 6, we shall show the following three propositions.

Proposition 1. *The matrices $F(b,\lambda) := X(b,\lambda)+iP(b)X^{\Delta}(b,\lambda)$ and $G(b,\lambda) := X(b,\lambda) - iP(b)X^{\Delta}(b,\lambda)$ are nonsingular for all $\lambda \in \mathbb{R}$.*

Proof. Let $\lambda \in \mathbb{R}$. Using (4) and by Lemma 2, we have

$$\begin{aligned} &W(X,X)(b) = W(X,X)(\rho(a)) \\ &= (P(\rho(a))X^{\Delta}(\rho(a)))^*X(\rho(a)) - X^*(\rho(a))P(\rho(a))X^{\Delta}(\rho(a)) \\ &= R_1R_2^* - R_2R_1^* = 0. \end{aligned}$$

Thus

$$(P(b)X^{\Delta}(b,\lambda))^*X(b,\lambda) = X^*(b,\lambda)P(b)X^{\Delta}(b,\lambda). \tag{14}$$

Clearly, dropping (b,λ)

$$F^*F = X^*X + X^{\Delta *}P^2(b)X^{\Delta} + i(X^*P(b)X^{\Delta} - X^{\Delta *}P(b)X).$$

It follows from (14), we have

$$F^*F = X^*X + X^{\Delta *}P^2(b)X^{\Delta}. \tag{15}$$

Similarly,

$$G^*G = X^*X + X^{\Delta *}P^2(b)X^{\Delta}. \tag{16}$$

By using of Theorem 6, $X^*X + X^{\Delta *}P^2(b)X^{\Delta} > 0$. So, equations (15) and (16) imply that $F(b,\lambda)$ and $G(b,\lambda)$ are nonsingular. The proof is completed. □

Proposition 2. *For all $\lambda \in \mathbb{R}$, the matrix*

$$Q(b,\lambda) := G(b,\lambda)F^{-1}(b,\lambda) \tag{17}$$

is unitary and satisfies the following differential equation:

$$\frac{d}{d\lambda}Q(b,\lambda) = iQ(b,\lambda)\Omega(b,\lambda), \tag{18}$$

where

$$\begin{aligned}\Omega(b,\lambda) \quad = \quad & 2F^{*-1}(b,\lambda)[X^{\Delta *}(b,\lambda)P(b)\frac{\partial}{\partial\lambda}X(b,\lambda) \\ & -X^*(b,\lambda)P(b)\frac{\partial}{\partial\lambda}X^{\Delta}(b,\lambda)]F^{-1}(b,\lambda)\end{aligned}$$

is positive definite.

Proof. Let $\lambda \in \mathbb{R}$. It is clear that $Q(b,\lambda)$ exists by Proposition 1. Furthermore, dropping (b,λ), $Q^*Q = F^{*-1}G^*GF^{-1} = I_n$, because $F^*F = G^*G$, from (15) and (16). Thus $Q(b,\lambda)$ is unitary. Denote $\frac{d}{d\lambda}Q(b,\lambda) =: Q'(b,\lambda)$.

From (17),

$$Q' = [G' - GF^{-1}F']F^{-1} = [G' - QF']F^{-1}.$$

Multiplying from the left by Q^* and using the unitarity of Q, we have

$$Q^*Q' = [Q^*G' - F']F^{-1} = F^{*-1}[G^*G' - F^*F']F^{-1}. \tag{19}$$

However,

$$G^*G' - F^*F' = 2i[X^{\Delta *}(b,\lambda)P(b)\frac{\partial}{\partial\lambda}X(b,\lambda) - X^*(b,\lambda)P(b)\frac{\partial}{\partial\lambda}X^{\Delta}(b,\lambda)],$$

by the definitions of G and F. Therefore, $Q^*Q' = i\Omega$, which means that $Q(b,\lambda)$ satisfies (18). Moreover, by Lemmas 6 and 7, $\Omega(b,\lambda) > 0$. This completes the proof. □

Proposition 3. *The value λ_0 is a zero of $\det X(b,\lambda)$ if and only if -1 is an eigenvalue of $Q(b,\lambda_0)$, and λ_0 is a zero of $\det X^{\Delta}(b,\lambda)$ if and only if $+1$ is an eigenvalue of $Q(b,\lambda_0)$.*

Proof. If there is some λ_0 such that $\det X(b,\lambda_0) = 0$, there will be a $\xi \in \mathbb{C}^n$ with $\xi \neq 0$ such that $X(b,\lambda_0)\xi = 0$, and also $X^{\Delta}(b,\lambda_0)\xi \neq 0$ by Theorem 5. Hence $F(b,\lambda_0)\xi = -G(b,\lambda_0)\xi$. From (17) and with $\eta = F(b,\lambda_0)\xi$,

$$\eta = -G(b,\lambda_0)\xi = -G(b,\lambda)F^{-1}(b,\lambda_0)\eta = -Q(b,\lambda_0)\eta.$$

So $Q(b,\lambda_0)$ must have -1 among its eigenvalues. Conversely, if -1 is an eigenvalue of $Q(b,\lambda_0)$ for some λ_0, so that for some $\eta \neq 0$ we have $\eta = -Q(b,\lambda_0)\eta$. By retracing the above steps, we get that $X(b,\lambda_0)\xi = 0$ for some $\xi \neq 0$. Similarly, λ_0 is a zero of $\det X^{\Delta}(b,\lambda)$ if and only if $+1$ is an eigenvalue of $Q(b,\lambda_0)$. This completes the proof. □

Proof of Theorem 6. By Proposition 2 and Theorem V.6.1 in [6], $Q(b,\lambda)$ have n eigenvalues $\mu_1(\lambda), \mu_2(\lambda), ...\mu_3(\lambda)$, which are continuous in $\lambda \in \mathbb{R}$ and move monotonically and positively around the unit circle as λ increases.

We assume that the open interval (λ_1, λ_2) contains no zeros of $\det X^{\Delta}(b,\lambda)$, if the interval $[\lambda_1, \lambda_2]$ contains $n+1$ zeros of $\det X(b,\lambda)$ (multiplicity included). From Proposition 3,

$$\mu_j(\lambda) \neq 1 \text{ for } 1 \leq j \leq n. \tag{20}$$

So as λ increase in (λ_1, λ_2), the $\mu_j(\lambda)$ will equal -1 at most one time for $1 \leq j \leq n$. The same is true of the closed interval $[\lambda_1, \lambda_2]$, since if $\mu_j(\lambda_1) = -1$, then μ_j must lie in the lower half of the unit circle for $\lambda \in (\lambda_1, \lambda_2)$ from (20). Hence $\mu_i(\lambda) \neq -1$ for $\lambda \in (\lambda_1, \lambda_2]$. Similarly, if $\mu_j(\lambda_2) = -1$, then $\mu_j(\lambda)$ must lie in the upper half part of the unit circle for $\lambda \in (\lambda_1, \lambda_2)$. Hence, $\mu_j(\lambda) \neq -1$ for $\lambda \in [\lambda_1, \lambda_2)$.

In one word, the eigenvalue of $Q(b,\lambda)$ on $[\lambda_1, \lambda_2]$ takes -1 at most n times. It follows that $\det X(b,\lambda)$ has at most n zeros on the closed interval $[\lambda_1, \lambda_2]$ by using of Proposition 3. This contradicts the give condition. The proof is completed. □

Remark 2. Similar results to Proposition 1, Proposition 2(first part), and Proposition 3(second part) can be found in [11, Section 5].

Example. *Consider the following three specific cases:*

$$[\rho(0), 1] \cap \mathbb{T} = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1],$$

$$[\rho(0), 1] \cap \mathbb{T} = [0, \frac{1}{3}] \cup \{\frac{1}{3(N-1)}, \frac{2}{3(N-1)}, \frac{1}{N-1}, \frac{4}{3(N-1)}, \cdots, 1\},$$

and

$$[\rho(0), 1] \cap \mathbb{T} = \{q^k | k \geq 0, k \in \mathbb{Z}\} \cap \{0\},$$

where $N > 3$ *and* $0 < q < 1$.

By Theorem 6, if an interval $[\lambda_1, \lambda_2]$ contains $n+1$ zeros of $\det X(1,\lambda)$ (multiplicity included), $\det X^{\Delta}(1,\lambda)$ has at least one zero in the open interval (λ_1, λ_2). Obviously, the above three cases are not continuous and not discrete. So the existing results for the differential and difference equations are not available now. Thus, our results not only unifies the results in both the continuous and the discrete cases but also contains more complicated time scales.

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