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TWO-DIMENSIONAL MUTI-PARAMETERIZED SCHWARZ ALTERNATING METHOD†

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Abstract. The convergence rate of a numerical procedure based on Schwarz Alternating Method(SAM) for solving elliptic boundary value problems depends on the selection of the interface conditions applied on the interior boundaries of the overlapping subdomains. It has been observed that the *Robin condition* (mixed interface condition), controlled by a parameter, can optimize SAM's convergence rate. In [7], one had formulated the *multi-parameterized SAM* and determined the optimal values of the multi-parameters to produce the best convergence rate for one-dimensional elliptic boundary value problems. However it was not successful for two-dimensional problem. In this paper, we present a new method which utilizes the one-dimensional result to get the optimal convergence rate for the two-dimensional problem.

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1. Introduction

Schwarz-type alternating methods have become some of the most important approaches in domain decomposition techniques for solution of the boundary value problems (BVP's). These methods are based on a decomposition of the BVP domain into overlapping subdomains. The original BVP is reduced to a set of *smaller* BVP's on a number of subdomains with appropriate *interface conditions* on the interior boundaries of the overlapping areas, whose solutions are coupled through some iterative scheme to produce an approximation of the solution of the original BVP. It is known [1], [6] that under certain conditions the sequence of the solutions of the subproblems converges to the solution of the original problem.

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One of the objectives of this research is to study a class of Schwarz alternating methods (SAM's) whose interface conditions are parameterized and estimate the values of the parameters involved that speed up the convergence of these methods for a class of BVP's. In the context of elliptic BVP's the most commonly used *interface conditions* are of Dirichlet type. For this class of numerical SAM, there are several studies about the convergence, which include [9], [11], [14], [15], [12], [2], [10], [16]. The effect of parameterized mixed interface conditions has been considered by a number of researchers [3], [13], [5], [17]. Among them, Tang proposed a generalized Schwarz splitting [17]. The main part of his approach to the solution of a BVP is to use the mixed boundary condition, known as *Robin condition*,

$$B(u) = \omega u + (1-\omega)\frac{\partial u}{\partial n} \tag{1}$$

on the artificial boundaries. In [5], a multi-parameter SAM is formulated in which the mixed boundary conditions

$$B_i(u) = \omega_i u + (1-\omega_i)\frac{\partial u}{\partial n} \tag{2}$$

are controlled by a distinct parameter ω_i for the i-th overlapping area. Fourier analysis is applied to determine the values of ω_i parameters that make the convergence factor of SAM be zero.

In [7], one formulated a multi-parameter SAM at the matrix level where the parameters α_i are used to impose mixed interface conditions. The relation between the parameters α_i and ω_i is given by

$$\alpha_i = \frac{1-\omega_i}{1-\omega_i+\omega_i h} \tag{3}$$

(Refer to [7]), where h is the grid size. One determined analytically the optimal values of α_i's for one-dimensional(1-dim) boundary value problems, which minimize the spectral radius of the block Jacobi iteration matrix associated with the SAM matrix.

For the case of two-dimensional(2-dim) boundary value problems, however, the result was not satisfactory. Although reducing a 2-dim problem into several 1-dim problems by using tensor product of matrices, one could not determine analytically the optimal values of α_i's for the 2-dim problem. One obtained some result for the case of two subdomains ($k = 2$) only. In this paper, however, we use a distinct multi-parameter $\alpha_{j,i}$ for each j-th grid point of the i-th interfaces of the subdomains to get the best convergence for the 2-dim problem, while we used only one parameter α_i for the i-th interfaces of the subdomains in the previous paper [7].

In the following section, we summarize the result of the multi-parameterized SAM on one-dimensional problem, which has been presented in [7]. In section 3, we formulate the multi-parameterized SAM on two-dimensional problem where we impose distinct parameters on each grid point on the interfaces of the

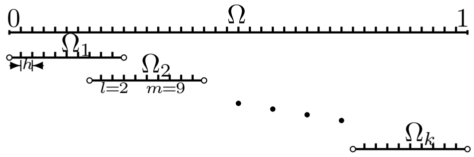

FIGURE 1. An example of the k-way decomposition of the domain of one dimensional boundary value problem (4).

subdomains. We show that the two-dimensional case can be reduced to the one-dimensional ones and obtain the optimal values of the multi-parameters which minimize the spectral radius of the block Jacobi iteration matrix associated with the SAM matrix of two-dimensional problem.

2. Multi-Parameterized Schwarz Splitting for 1-dim problem

We consider the two-point boundary value problem:

$$\begin{aligned} -u''(t) + q\,u(t) &= f(t),\ t \in (0,1) \\ u(0) = a_0,\ u(1) &= a_1, \end{aligned} \tag{4}$$

with $q \geq 0$ being a constant. We will formulate a numerical instance of SAM based on a k-way decomposition (i.e. the number of subdomains is k) of the problem domain. An example of k-way decomposition is depicted in Figure 1.

Let $T_j(x, y, z)$ be a $j \times j$ tridiagonal matrix such that

$$T_j(x,y,z) = \begin{bmatrix} x & \text{-1} & 0 & \cdots & 0 \\ \text{-1} & y & \text{-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \text{-1} & y & \text{-1} \\ 0 & \cdots & 0 & \text{-1} & z \end{bmatrix} \tag{5}$$

and let

$$T_j(x) \equiv T_j(x,x,x). \tag{6}$$

If we discretize the problem (4) by a second order central divided difference discretization scheme with a uniform grid of mesh size $h = \frac{1}{n+1}$, we obtain a linear system

$$A\,x = f \tag{7}$$

where $A = T_n(\beta)$ with $\beta = 2 + qh^2$.

If we consider 3-way ($k = 3$) decomposition, then $Ax = f$ has three overlapping diagonal blocks as follows.

$$\begin{bmatrix} T_{m\text{-}l} & \text{-}F & 0 & 0 & 0 \\ \text{-}E & T_l & \text{-}F & 0 & 0 \\ 0 & \text{-}E & T_{m\text{-}2l} & \text{-}F & 0 \\ 0 & 0 & \text{-}E & T_l & \text{-}F \\ 0 & 0 & 0 & \text{-}E & T_{m\text{-}l} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} \tag{8}$$

where $T_j = T_j(\beta,\beta,\beta)$ in (5) and m and l are the numbers of nodes in each subdomain and the overlapping regions, respectively, such that $l < \frac{m-1}{2}$. In (8), the matrix E have zero elements everywhere except for a 1 at the rightmost top position and the matrix F have zero elements everywhere except for a 1 at the leftmost bottom position. So the matrices E and F have compatible sizes with the following forms.

$$E = \begin{bmatrix} 0 \cdots 0 & 1 \\ 0 \cdots 0 & 0 \\ \vdots \quad \vdots & \vdots \\ 0 \cdots 0 & 0 \end{bmatrix}, \; F = \begin{bmatrix} 0 & 0 \cdots 0 \\ \vdots & \vdots \quad \vdots \\ 0 & 0 \cdots 0 \\ 1 & 0 \cdots 0 \end{bmatrix} \tag{9}$$

The numerical version of SAM [16] for the problem (4) is equivalent to a *block Gauss-Seidel iteration procedure* for a new linear system, called *Schwarz Enhanced Matrix Equation* ,

$$\tilde{A}\tilde{x} = \tilde{f} \tag{10}$$

where

$$\tilde{A} = \tilde{A}(\beta) = \begin{bmatrix} T_{m\text{-}l} & \text{-}F & 0 & 0 & 0 & 0 & 0 \\ \text{-}E & T_l & 0 & \text{-}F & 0 & 0 & 0 \\ \text{-}E & 0 & T_l & \text{-}F & 0 & 0 & 0 \\ 0 & 0 & \text{-}E & T_{m\text{-}2l} & \text{-}F & 0 & 0 \\ 0 & 0 & 0 & \text{-}E & T_l & 0 & \text{-}F \\ 0 & 0 & 0 & \text{-}E & 0 & T_l & \text{-}F \\ 0 & 0 & 0 & 0 & 0 & \text{-}E & T_{m\text{-}l} \end{bmatrix}, \; \tilde{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_2' \\ x_3 \\ x_4 \\ x_4' \\ x_5 \end{bmatrix}, \; \tilde{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_2 \\ f_3 \\ f_4 \\ f_4 \\ f_5 \end{bmatrix}. \tag{11}$$

$\tilde{A}(\beta)$ means that $\tilde{A}$ is a function of β. Note that the solution x of (7) is obtained from the solution $\tilde{x}$ of (10), vice versa. In [17], it is shown that a good choice of the splitting of T_l's can significantly improve the convergence of SAM. Applying for some splittings of T_l's into $\tilde{A}$ in (11), we have a new equation

$$A'\tilde{x} = \tilde{f} \tag{12}$$

with

$$A' = A'(\beta,\alpha_1,\alpha_2) = \begin{bmatrix} T_{m\text{-}l} & \text{-}F & 0 & 0 & 0 & 0 & 0 \\ \text{-}E & B_1 & C_1 & \text{-}F & 0 & 0 & 0 \\ \text{-}E & C_1' & B_1' & \text{-}F & 0 & 0 & 0 \\ 0 & 0 & \text{-}E & T_{m\text{-}2l} & \text{-}F & 0 & 0 \\ 0 & 0 & 0 & \text{-}E & B_2 & C_2 & \text{-}F \\ 0 & 0 & 0 & \text{-}E & C_2' & B_2' & \text{-}F \\ 0 & 0 & 0 & 0 & 0 & \text{-}E & T_{m\text{-}l} \end{bmatrix} \tag{13}$$

where $B_i, C_i',\ i = 1,2$ are some matrices such that $(B_i - C_i')$ is non-singular and

$$T_l = B_i + C_i = B_i' + C_i',\ i = 1,2. \tag{14}$$

Note that two linear system (10) and (12) are equivalent in the sense that they have the same solutions. If C_i' and C_i are chosen such that they are the $l \times l$ matrices with all zero entries except for an α_i in the positions $(1,1)$ and (l,l), respectively, as follows,

$$C_i' = \begin{bmatrix} \alpha_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad C_i = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \alpha_i \end{bmatrix}, \tag{15}$$

the resulting matrix A' is given as follows

$$A' = A'(\beta, \alpha_1, \alpha_2) = \begin{bmatrix} T_m(\beta,\beta,\beta\text{-}\alpha_1) & \text{-}F_1{}' & 0 \\ \text{-}E_1{}' & T_m(\beta\text{-}\alpha_1,\beta,\beta\text{-}\alpha_2) & \text{-}F_2{}' \\ 0 & \text{-}E_2{}' & T_m(\beta\text{-}\alpha_2,\beta,\beta) \end{bmatrix} \tag{16}$$

where $T_m(x, y, z)$'s are $m \times m$ matrices defined in (5) and $E_i{}'$'s are the $m \times m$ matrices with zero elements everywhere except that

$$\begin{aligned} (1, m-l \quad)\text{-th entry} &= \ \ 1, \\ (1, m-l{+}1)\text{-th entry} &= -\alpha_i \end{aligned}$$

and $F_i{}'$'s are the $m \times m$ matrices with zero elements everywhere except that

$$\begin{aligned} (m, l{+}1)\text{-th entry} &= \ \ 1, \\ (m, l \quad)\text{-th entry} &= -\alpha_i. \end{aligned}$$

If the number of subdomains k is more than 3, the matrix A' is a block $k \times k$ matrix of the form

$$A' = A'(\beta, \mathbf{a}) = \begin{bmatrix} G_1 & \text{-}F_1{}' & 0 & 0 & \cdots & 0 \\ \text{-}E_1{}' & G_2 & \text{-}F_2{}' & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \text{-}E'_{k\text{-}2} & G_{k\text{-}1} & \text{-}F'_{k\text{-}1} \\ 0 & \cdots & 0 & 0 & \text{-}E'_{k\text{-}1} & G_k \end{bmatrix} \tag{17}$$

where $\mathbf{a} = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_k)$ with $\alpha_0 = \alpha_k = 0$ and G_i's are defined as

$$G_i = T_m(\beta - \alpha_{i\text{-}1}, \beta, \beta - \alpha_i), \ i = 1, 2, \cdots, k. \tag{18}$$

We call the matrix A' as *Multi-Parameterized Enhanced Matrix.* If we define

$$M{=}M(\beta, \mathbf{a}){=}\begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & G_k \end{bmatrix} \text{ and } N{=}N(\beta, \mathbf{a}){=}\begin{bmatrix} 0 & F_1{}' & 0 & 0 & \cdots & 0 \\ E_1{}' & 0 & F_2{}' & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & E'_{k\text{-}2} & 0 & F'_{k\text{-}1} \\ 0 & \cdots & 0 & 0 & E'_{k\text{-}1} & 0 \end{bmatrix} \tag{19}$$

with $\mathbf{a} = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_k)$, then we can write the multi-parameterized enhanced matrix A' as

$$A' = M - N \tag{20}$$

which is called *a Multi-Parameterized Schwarz Splitting (MPSS)* .

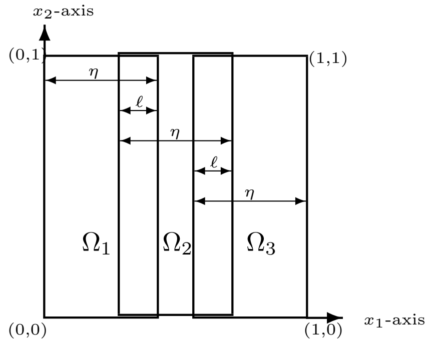

FIGURE 2. A 3-way splitting of the unit square Ω.

The convergence behavior of MPSS depends on the spectral radius of the following block Jacobi matrix

$$J = M^{-1}N = \begin{bmatrix} 0 & G_1^{-1}\tilde{F}_1 & 0 & 0 & \cdots & 0 \\ G_2^{-1}\tilde{E}_1 & 0 & G_2^{-1}\tilde{F}_2 & 0 & \cdots & 0 \\ 0 & G_3^{-1}\tilde{E}_2 & 0 & G_3^{-1}\tilde{F}_3 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & G_{k-1}^{-1}\tilde{E}_{k-2} & 0 & G_{k-1}^{-1}\tilde{F}_{k-1} \\ 0 & \cdots & 0 & 0 & G_k^{-1}\tilde{E}_{k-1} & 0 \end{bmatrix}. \tag{21}$$

Note that J is a function of the parameters α_i's, which correspond to the parameters ω_i's in the mixed interface condition (2), respectively. The convergence rate of SAM can be optimized by controlling these parameters α_i's. In [7], one determined the optimal values of the multi-parameter α_i's that make the spectral radius of the block Jacobi matrix J in (21) to be zero. The result of [7] is presented in the following theorem.

Theorem 1. *Let $\theta = \cosh^{-1}(\frac{\beta}{2})$ with $\beta = 2 + qh^2$ and let $p \in \{1, 2, \cdots, k-1\}$ and let*

$$\Theta(x) \equiv \begin{cases} \sinh(x\,\theta) & , \ \theta > 0 \\ x & , \ \theta = 0. \end{cases}$$

If the values α_i, $i = 0, 1, \cdots, k$, are given by

$$\begin{aligned} \alpha_0 &= \alpha_k = 0 \\ \alpha_i &= \tfrac{\Theta(m-l) - \alpha_{i-1}\Theta(m-l-1)}{\Theta(m-l+1) - \alpha_{i-1}\Theta(m-l)}, \quad i = 1, 2, \cdots, p, \\ \alpha_i &= \tfrac{\Theta(m-l) - \alpha_{i+1}\Theta(m-l-1)}{\Theta(m-l+1) - \alpha_{i+1}\Theta(m-l)}, \quad i = p+1, \cdots, k-1, \end{aligned}$$

then the spectral radius of the block Jacobi matrix J in (21) is zero.

3. Multi-Parameterized Schwarz Splitting for 2-dim Problem

Consider the two-dimensional boundary value problem

$$\begin{aligned} -\nabla^2 u(x) + q\, u(x) = f(x), \quad x \in \Omega, \\ u(x) = g(x), \quad x \in \Gamma \end{aligned} \tag{22}$$

where Γ is the boundary of $\Omega \equiv (0,1) \times (0,1)$ and $q \geq 0$ is a constant. We formulate a SAM based on a k-way splitting of the domain Ω, i.e., we decompose our domain into k overlapping subdomains Ω_i along the x_1-axis and make a strip-type decomposition of the rectangular domain Ω (for instance, see Figure 2). Next we apply the interface conditions on the two interior boundaries between subdomains Ω_i and Ω_{i+1}. Let ℓ be the length of the overlap in x_1-direction and η be the length of each subdomain in the same direction. Figure 2 depicts such a 3-way splitting of the unit square Ω.

To begin our analysis we use a 5-point finite difference discretization scheme with uniform grid of mesh size $h = \frac{1}{n+1}$ on both x_1- and x_2-axes and discretize the BVP in (22) to obtain a linear system of the form

$$Bx = f. \tag{23}$$

The natural ordering of the nodes is adopted starting from the origin and going in the x_2-direction first so that the resulting matrix A can be partitioned into block matrices corresponding to the subdomains, respectively. Using tensor product notation $\otimes$ (See [4], and [8] in which tensor products in connection with BVP's were introduced.), the matrix B in (23) can be written as

$$B = T_n(\beta) \otimes I_n + I_n \otimes T_n(2) \tag{24}$$

where $T_j(x)$ is defined in (6) and $\beta = 2 + qh^2$.

Define $l+1 = \frac{\ell}{h}$ and $m+1 = \frac{\eta}{h}$ such that $n = mk - l(k-1)$ and $l < \frac{m-1}{2}$. The numerical version of SAM for the problem (22) is equivalent to a *block Gauss-Seidel iteration procedure* for a new linear system, called the *Schwarz Enhanced Matrix Equation* ,

$$\tilde{B}\tilde{x} = \tilde{f} \tag{25}$$

with

$$\tilde{B} = \tilde{A}(\beta) \otimes I_n + I_{km} \otimes T_n(2) \tag{26}$$

where I_{km} is the $km \times km$ identity matrix and $\tilde{A}(\beta)$ is the $k \times k$ block matrix as that defined in (11), which is the case of $k = 3$. Note that each diagonal block in $\tilde{A}(\beta)$ is $m \times m$ matrix. .

Let X_n be the $n \times n$ orthogonal matrix whose columns are the eigenvectors of the matrix $T_n(2)$. Since the eigenvalues of the matrix $T_n(2)$ are known to be $\gamma_i = 2 + 2\cos(\frac{i\pi}{n+1}),\ i = 1, 2, \cdots, n$, we can write

$$X_n^T\, T_n(2)\, X_n = D_n \equiv diag(\gamma_1, \gamma_2, \cdots, \gamma_n). \tag{27}$$

Let $X = I_{km} \otimes X_n$, then its inverse is given by $X^{-1} = I_{km} \otimes X_n^T$, so we have

$$\begin{aligned} X^{-1}\tilde{B}X &= (I_{km} \otimes X_n^T)(\tilde{A}(\beta) \otimes I_n)(I_{km} \otimes X_n) \\ &\quad +(I_{km} \otimes X_n^T)(I_{km} \otimes T_n(2))(I_{km} \otimes X_n) \\ &= (I_{km}\, \tilde{A}(\beta)\, I_{km}) \otimes (X_n^T\, I_n\, X_n) + I_{km} \otimes (X_n^T\, T_n(2)\, X_n) \\ &= \tilde{A}(\beta) \otimes I_n + I_{km} \otimes D_n. \end{aligned}$$

If P is the permutation matrix that maps

$$\text{row } (i-1)n + j \quad \text{into} \quad \text{row } (j-1)km + i$$

for $i = 1, 2, \cdots, km$ and for $j = 1, 2, \cdots, n$, then we have

$$\begin{aligned} \hat{B} \equiv P^{-1}X^{-1}\tilde{B}XP &= P^{-1}(\tilde{A}(\beta) \otimes I_n)P + P^{-1}(I_{km} \otimes D_n)P \\ &= I_n \otimes \tilde{A}(\beta) + D_n \otimes I_{km} \\ &= diag(\tilde{A}(\beta+\gamma_1), \tilde{A}(\beta+\gamma_2), \cdots, \tilde{A}(\beta+\gamma_n)) \\ &= diag(\tilde{A}(\zeta_1), \tilde{A}(\zeta_2), \cdots, \tilde{A}(\zeta_n)) \end{aligned} \tag{28}$$

where

$$\zeta_j = \beta + \gamma_j, \ j = 1, 2, \cdots, n. \tag{29}$$

Note that the solution $\tilde{x}$ of linear system (25) is obtained by $\tilde{x} = XP\,\hat{x}$ if we solve the linear system

$$\hat{B}\hat{x} = \hat{f} \tag{30}$$

where $\hat{f} = P^{-1}X^{-1}\tilde{f}$ with $\tilde{f}$ in (25).

From (28) and (30), we know that the two-dimensional problem (25) is reduced to n number of one dimensional problems

$$\tilde{A}(\zeta_j) = \hat{f}_j, \ j = 1, 2, \cdots, n,$$

where $\hat{f}_j$ is the corresponding sub-vector of $\hat{f}$. Based on (28), the *Multi-Parameterized Schwarz Enhanced Matrix* for $\hat{B}$ in (28) is defined as

$$B' = diag(\ A'(\zeta_1, \mathbf{a}), A'(\zeta_2, \mathbf{a}), \cdots, A'(\zeta_n, \mathbf{a})\ ), \tag{31}$$

where $A'(x, \mathbf{a})$ is defined in (17). If we let

$$\begin{aligned} M &= diag(\ M(\zeta_1, \mathbf{a}), M(\zeta_2, \mathbf{a}), \cdots, M(\zeta_n, \mathbf{a})\ ), \\ N &= diag(\ N(\zeta_1, \mathbf{a}), N(\zeta_2, \mathbf{a}), \cdots, N(\zeta_n, \mathbf{a})\ ) \end{aligned} \tag{32}$$

where $M(x, \mathbf{a})$ and $N(x, \mathbf{a})$ are defined in (19), then we can write the multi-parameterized enhanced matrix B' in (31) as

$$B' = M - N \tag{33}$$

which is called a *Multi-Parameterized Schwarz Splitting (MPSS)* . The convergence behavior of MPSS depends on the spectral radius of the following block Jacobi matrix

$$J = M^{-1}N = diag(\ L_1(\mathbf{a}), L_2(\mathbf{a}), \cdots, L_n(\mathbf{a})\ ) \tag{34}$$

where

$$L_j(\mathbf{a}) = M(\zeta_j, \mathbf{a})^{-1}N(\zeta_j, \mathbf{a}), \ j = 1, 2, \cdots, n.$$

In [7], one failed to determine a parameter vector $\mathbf{a}$ such that the spectral radius of the block Jacobi matrix J in (34) is zero because it is not possible to find such a parameter vector $\mathbf{a}$ that makes all of the spectral radii of the diagonal blocks $L_j(\mathbf{a})$'s in (34) zero simultaneously.

Now we adopt distinct parameter vector $\mathbf{a}_j$ for each diagonal block as follows

$$J = M^{-1}N = diag(\ L_1(\mathbf{a}_1), L_2(\mathbf{a}_2), \cdots, L_n(\mathbf{a}_n)\ ) \tag{35}$$

where

$$\mathbf{a}_j = (\alpha_{j,0}, \alpha_{j,1}, \alpha_{j,2}, \cdots, \alpha_{j,k}),\ j = 1, 2, \cdots, n.$$

Note these double indices (j, i) in multi-parameter $\alpha_{j,i}$ are related to the idea that one adopts variable parameter $\omega_i(x, y)$ instead of constant parameter ω_i in (2) along the i-th interface, i.e., we have

$$B_i(u) = \omega_i(x, y)u + (1 - \omega_i(x, y))\frac{\partial u}{\partial n} \tag{36}$$

as the mixed interface condition on the i-th interface boundary.

Now, using these double-index multi-parameter $\alpha_{j,i}$'s, we have the following theorem for two-dimensional multi-parameterized Schwarz splitting $B' = M - N$ in (33).

Theorem 2. *For $j = 1, 2, \cdots, n$, let $\theta_j = \cosh^{-1}(\frac{\zeta_j}{2})$ with ζ_j in (29) and let $p \in \{1, 2, \cdots, k-1\}$ and let*

$$\Theta_j(x) \equiv \begin{cases} \sinh(x\,\theta_j)\ , & \theta > 0 \\ x & , \theta = 0. \end{cases}$$

If the values $\alpha_{j,i},\ j = 1, 2, \cdots, n,\ i = 0, 1, \cdots, k$ are given by

$$\begin{aligned} \alpha_{j,0} &= \alpha_{j,k} = 0 \\ \alpha_{j,i} &= \tfrac{\Theta_j(m\text{-}l) - \alpha_{j,i\text{-}1}\Theta_j(m\text{-}l-1)}{\Theta_j(m\text{-}l+1) - \alpha_{j,i\text{-}1}\Theta_j(m\text{-}l)}, \quad i = 1, 2, \cdots, p, \\ \alpha_{j,i} &= \tfrac{\Theta_j(m\text{-}l) - \alpha_{j,i+1}\Theta_j(m\text{-}l\text{-}1)}{\Theta_j(m\text{-}l+1) - \alpha_{j,i+1}\Theta_j(m\text{-}l)}, \quad i = p+1, \cdots, k-1, \end{aligned}$$

then the spectral radius of the block Jacobi matrix J in (35) is zero.

4. Numerical Experiments

In this section, we present a numerical experiment to prove the result of the previous section. we will compare the results of *Multi-Parameterized SAM (MP-SAM)* with those of the *Classical SAM* . Consider the following model problem

$$\begin{gathered} -\nabla^2 u(x, y) = 0,\ \ (x, y) \in \Omega = (0, 1) \times (0, 1), \\ u(x, y) = f(x, y),\ \ (x, y) \in \Gamma, \end{gathered} \tag{37}$$

where Γ is the boundary of Ω, with solution

$$u(x, y) = \sin(2\pi x)\cos(2\pi y).$$

In all the experiments, the vector with all its components 0.0 was used as initial guess of the solution vector. The *relative residual* r_p is computed as the

TABLE 1. The classical *SAM* is applied to the BVP (37).

m	l	*relative residual* r_k		
		$k=3$	$k=4$	$k=8$
10	1	1.2553E-01	1.0218E-01	7.6206E-02
10	4	9.6040E-03	1.0148E-02	2.2181E-02
20	1	1.6078E-01	1.3224E-01	7.9805E-02
20	9	5.2315E-03	7.3636E-03	1.9280E-02

TABLE 2. The *MPSAM* is applied to the BVP (37).

m	l	*relative residual* r_k		
		$k=3$	$k=4$	$k=8$
10	1	5.7060E-16	8.8938E-16	1.0943E-15
10	4	5.1131E-16	5.6130E-16	1.1373E-15
20	1	1.0603E-15	1.3460E-15	1.7441E-15
20	9	9.3843E-16	9.9606E-16	1.4703E-15

ratio of ℓ_2-norms of the residuals of the corresponding system of equations after p iterations, i.e.,

$$r_p = \frac{||Bx^{(p)} - f||_2}{||Bx^{(0)} - f||_2}.$$

Table 1 shows the relative residuals of *SAM* computed after k iterations for $k = 3, 4$ and 8 subdomains, $m = 10$ and 20 local grids and minimum ($l = 1$) and half ($l = [m\text{-}1]/2$) overlaps. The results indicate slow convergence. Table 2 shows the performance of *MPSAM* under the same conditions. It shows the optimal convergence. Indeed, the relative residuals by *MPSAM* are less than 1.7441×10^{-15} after k iterations for the case of k subdomains.

The convergence rate is very sensitive to the computed optimal value of parameter $\alpha_{j,i}$'s and the symmetric choice of them (i.e. Take $p = [k/2]$ in Theorem (2)) reduces the error propagation when we compute the optimal value of parameters $\alpha_{j,i}$'s.

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