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EXISTENCE OF GLOBAL SOLUTIONS FOR A PREY-PREDATOR MODEL WITH NON-MONOTONIC FUNCTIONAL RESPONSE AND CROSS-DIFFUSION†

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Abstract. In this paper, using the energy estimates and the bootstrap arguments, the global existence of classical solutions for a prey-predator model with non-monotonic functional response and cross-diffusion where the prey and predator both have linear density restriction is proved when the space dimension $n < 10$.

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1. Introduction

In this paper, we are interested in the following the prey-predator model with non-monotonic functional response and cross-diffusion where the prey and predator both have linear density restriction

$$\begin{aligned}
&u_t = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + [a - ku - \frac{bv}{1 + mu + \beta u^2}]u, \quad \text{in } \Omega \times [0, \infty),\\
&v_t = \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] + [c - ev + \frac{du}{1 + mu + \beta u^2}]v, \quad \text{in } \Omega \times [0, \infty),\\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times [0, \infty),\\
&u(x, 0) = u_0(x) \geq 0, \ \ v(x, 0) = v_0(x) \geq 0, \quad \text{in } \Omega,
\end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^{\mathbb{N}}(\mathbb{N} \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector of the boundary$\partial\Omega$. α_{ij} are given nonnegative constants for $i, j = 1, 2$. And $d_1, d_2, a, \frac{1}{k}, b$ and d are positive constants which stand for the random diffusion rates of the two species, prey intrinsic growth rate, carrying capacity, capturing rate and conversion rate respectively. e is a

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nonnegative constant and c is a constant which may change sign with $c < 0$ if $e = 0$. The $-ev^2$ represents the self-limitation for the predator. u_0 and v_0 are nonnegative functions. In system (1.1), u and v represent the population densities of the prey and predator species, respectively, α_{11} and α_{22} are self-diffusion rates, and α_{12} and α_{21} are cross-diffusion rates. The interaction term, $\frac{u}{1+mu+\beta u^2}$ is of the Holling type IV functional response, the constants $\beta > 0$ and m are assumed satisfy $m > -2\sqrt{\beta}$ so that functional response $\frac{u}{1+mu+\beta u^2}$ remains nonnegative for $u(x,t) \geq 0$ in $\overline{\Omega}$, more explanations for the response functions of this type can be found in [1]. When $\beta, m = 0$, the system (1.1) reduces to the well-known Lotka-Volterra prey-predator SKT model which has been investigated by [2]. Note that $m = 0$, $\frac{u}{1+mu+\beta u^2}$ is reduced to the function used in [3]. If $e = m = 0$, the corresponding ODE system (1.1) has been discussed by many authors; see [3, 4, 5]and the references therein. In the case of $\beta = 0$ and $m > 0$, the interaction term $\frac{u}{1+mu}$ is known as Holling type II or Michaelis-Menten functional response which was proposed by Michaelis-Menten and Holling in studying enzymatic reactions and predator-prey models, for more explanations for response functions of this type, refer to[6].

The corresponding weakly coupled reaction-diffusion system (1.1) has received a lot of attention, see[7, 8], the biologically more interesting case $e > 0$ was studied in [8]. But up to now, the corresponding researches chiefly concentrate on non-existence and the existence of the positive steady-state solution, Hopf bifurcation can occur, and existence of non-constant positive steady-state solutions of the weakly-coupled reaction-diffusion system (1.1). To the best of our knowledge, especially the case $e > 0$, when $\alpha_{12} > 0$ or $\alpha_{21} > 0$ is positive, (1.1) is a strongly-coupled reaction-diffusion system which occurs frequently is biological and it is very difficult to analyze, there are very few results for the (1.1), there is only one work of Chen *et. al.*[5] available, in which she obtained a number of existence and non-existence results concerning non-constant steady states (patterns) of (1.1) when $e = m = 0$.

Local existence (in time) of solutions to (1.1) was established by Amann in a series of important papers [9, 10, 11]. Referring to the result stated in Theorem 8 in [12]. However, very few results are know for global existence of solutions to (1.1), in particular, the global existence of classical solutions for (1.1) is open and interesting question to understand the problem in the high-dimensional space. The main purpose of this paper is to understand the global existence of classical solutions of (1.1) for higher $n(n < 10)$. We remark that while there have been many results on global solutions to Lotka-Volterra competition systems with cross-diffusion, such as [13, 14, 15, 16], predator-prey systems with cross-diffusion seem to be far less studied. Scaling the parameters we may assume that $e = k = 1$. Thus we will concentrate on the system (namely, the system (1.1) for $\alpha_{12} = 0$)

$$
\begin{aligned}
&u_t = \Delta[(d_1 + \alpha_{11} u)u] + [a - u - \frac{bv}{1 + mu + \beta u^2}]u, \quad \text{in } \Omega \times [0, \infty),\\
&v_t = \Delta[(d_2 + \alpha_{21} u + \alpha_{22} v)v] + [c - v + \frac{du}{1 + mu + \beta u^2}]v, \quad \text{in } \Omega \times [0, \infty),\\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times [0, \infty),\\
&u(x, 0) = u_0(x) \geq 0, \;\; v(x, 0) = v_0(x) \geq 0, \quad \text{in } \Omega.
\end{aligned}
\tag{1.2}
$$

Theorem 1.1. *Let $\alpha_{22} > 0$ and assume that $u_0 \geq 0, v_0 \geq 0$ satisfy zero Neumann boundary condition and belong to $C^{2+\lambda}(\overline{\Omega})$ for some $0 < \lambda < 1$. Then (1.2)possesses a unique non-negative solution $u, v \in C^{2+\lambda, 1+\frac{\lambda}{2}}(\overline{\Omega} \times [0, \infty))$ if $\alpha_{11} > 0$ and $n < 10$.*

The paper is organized as follows. In section 2, we present some known results which are useful in later section. In section 3, we establish L^r-estimates of the solution v of (1.2) and we give a proof of Theorem 1.1.

2. Preliminaries

We list here some notation.

$Q_T = \Omega \times [0, T)$,

$\|u\|_{L^{p,q}(Q_T)} = \Big(\int_0^T (\int_\Omega |u(x, t)|^p dx)^{\frac{q}{p}} dt \Big)^{1/q}, \;\; L^p(Q_T) := L^{p,p}(Q_T)$,

$\|u\|_{W_p^{2,1}(Q_T)} := \|u\|_{L^p(Q_T)} + \|u_t\|_{L^p(Q_T)} + \|\nabla u\|_{L^p(Q_T)} + \|\nabla^2 u\|_{L^p(Q_T)}$,

$\|u\|_{V_2(Q_T)} := \sup_{0 \leq t \leq T} \|u(., t)\|_{L^2(\Omega)} + \|\nabla u(x, t)\|_{L^2(Q_T)}$,

where T be the maximal existence time for the solution (u, v) of (1.2). In order to establish L^r-estimates for solutions of (1.2), we need the following preliminary results.

Lemma 2.1. *Let u, v be a solution of (1.2) in $[0, T)$. Then $0 \leq u \leq m$ and $v \geq 0$ in Q_T, where $m = \max\{a, \|u_0\|_{L^\infty(\Omega)}\}$.*

Proof. The first equation in (1.2) is expressed as

$$
u_t = (d_1 + 2\alpha_{11} u)\triangle u + 2\alpha_{11} \nabla u \cdot \nabla u + u[a - u - \frac{bv}{1 + mu + \beta u^2}], \tag{2.1}
$$

and the second equation is written as

$$
\begin{aligned}
v_t = &(d_2 + \alpha_{21} u + 2\alpha_{22} v)\triangle v + 2(\alpha_{21} \nabla u + \alpha_{22} \nabla v)\nabla v\\
&+ v[c - v + \frac{du}{1 + mu + \beta u^2}].
\end{aligned}
\tag{2.2}
$$

Then application of the maximum principle for (2.1)and (2.2)yields the nonnegative of u and v. Applying the maximum principle to (2.1)again one can also show the boundedness of u. $\square$

Lemma 2.2. *There exists a positive $C_1(T)$ such that*

$$\sup_{0\le t\le T}\|u(.,t)\|_{L^1(\Omega)} < C_1(T), \qquad \sup_{0\le t\le T}\|v(.,t)\|_{L^1(\Omega)} < C_1(T),$$
$$\|u\|_{L^2(Q_T)} < C_1(T), \qquad \|v\|_{L^2(Q_T)} < C_1(T).$$

Proof. Integrating the first equation in (1.2) over the domain Ω, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u dx &= \int_\Omega u[a-u-\frac{bv}{1+mu+\beta u^2}]dx\\ &\le a\int_\Omega u dx - \int_\Omega u^2 dx\\ &\le a\int_\Omega u dx - \frac{1}{|\Omega|}\left(\int_\Omega u dx\right)^2,\end{aligned}$$

where we used Hölder's inequality. Then we have $\|u(.,t)\|_{L^1(\Omega)} \le M_1'$, where $M_1' = \max\{\|u_0\|_{L^1(\Omega)}, a|\Omega|\}$. Furthermore,

$$\sup_{0\le t\le T}\|u(.,t)\|_{L^1(\Omega)} < C_1(T).$$

Since

$$\frac{d}{dt}\int_\Omega u\, dx \le a\int_\Omega u\, dx - \int_\Omega u^2 dx. \tag{2.3}$$

Integrating (2.3) from 0 to T, we have

$$\|u\|^2_{L^2(Q_T)} \le M_1'|Q_T| + \|u_0\|_{L^1(\Omega)}.$$

Therefore,

$$\|u\|_{L^2(Q_T)} < C_1(T).$$

Now, integrating the first, second equation in the system (1.2) over the domain Ω, and after a linear combination, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega (du+bv)\mathrm{dx} &= -\int_\Omega (du^2+bv^2)\mathrm{dx} + \int_\Omega (\mathrm{adu}+\mathrm{bcv})\mathrm{dx}\\ &\le -\frac{1}{2}\min\{\frac{1}{d},\frac{1}{b}\}\left[\int_\Omega (du+bv)\mathrm{dx}\right]^2 + \max\{a,c\}\int_\Omega (du+bv)\mathrm{dx}.\end{aligned}$$

Therefore, $\|v(.,t)\|_{L^1(\Omega)} \le M_2'$, where M_2' depending only on u_0, v_0 and the coefficients of (1.2), then

$$\sup_{0\le t\le T}\|v(.,t)\|_{L^1(\Omega)} < C_1(T).$$

Since $\frac{d}{dt}\int_\Omega (du+bv)\mathrm{dx} = \int_\Omega (\mathrm{adu} - \mathrm{du}^2 - \mathrm{bv}^2 + \mathrm{bcv})\mathrm{dx}$, then integrating the equation from 0 to T, we have

$$b\int_{Q_T} v^2 dxdt \le ad\int_0^T M_1' dt + bc\int_0^T M_2' dt + d\|u_0\|_{L^1(\Omega)} + b\|v_0\|_{L^1(\Omega)},$$

which implies

$$\|v\|_{L^2(Q_T)} < C_1(T).$$

□

Lemma 2.3. *Let $w_1 = (d_1 + \alpha_{11} u)u$, and u be a solution of the problem*

$$u_t = \Delta[(d_1 + \alpha_{11} u)u] + [a - u - \frac{bv}{1 + mu + \beta u^2}]u, \quad (x,t) \in \Omega \times (0,T),$$

$$\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega \times (0,T),$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in \Omega,$$

where $d_1, \alpha_{11}, a, b, m, \beta$ are positive constants, $0 \leq v \in L^2(Q_T)$, $u_0 \in W_2^2(\Omega) \cap L^\infty(\Omega)$. Then there exists a constant $C_2(T)$, depending on $\|u_0\|_{W_2^1(\Omega)}$ and $\|u_0\|_{L^\infty(\Omega)}$ such that

$$\|w_1\|_{W_2^{2,1}(Q_T)} \leq C_2(T).$$

Furthermore, $\nabla w_1 \in V_2(Q_T)$.

Proof. The proof of Lemma 2.5 is similar to [15] Lemma 2.2, we omit it. □

Lemma 2.4. *Let $q > 1, \widetilde{q} = 2 + \frac{4q}{n(q+1)}$, $\widetilde{\beta}$ in $(0,1)$ and let $C_T > 0$ be any number which may depend on T. Then there is a constant M_1 depending on $q, n, \Omega, \widetilde{\beta}$ and C_T such that for any g in $C([0,T), W_2^1(\Omega))$ with $(\int_\Omega |g(.,t)|^{\widetilde{\beta}} dx)^{\frac{1}{\widetilde{\beta}}} \leq C_T$ for all $t \in [0,T]$, we have the following inequality*

$$\|g\|_{L^{\widetilde{q}}(Q_T)} \leq M_1 \left\{ 1 + \left(\sup_{0 \leq t \leq T} \|g(.,t)\|_{L^{2q/q+1}(\Omega)} \right)^{4q/n(q+1)\widetilde{q}} \|\nabla g\|_{L^2(Q_T)}^{2/\widetilde{q}} \right\}.$$

Proof. The proof may be found in [15] Lemma 2.3 and Lemma 2.4. □

Lemma 2.5. *There exists a constant $C_3(T)$ such that*

$$\|\nabla u\|_{L^4(Q_T)} \leq C_3(T).$$

Proof. The proof of Lemma 2.5 is similar to [2] Lemma 3.1, we omit it. □

3. Proof of the Theorem 1.1

In this section, we present a proof of our main result Theorem 1.1. It consists of three steps that are devoted to obtain $L^r(Q_T), L^\infty(Q_T)$ and $C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{Q}_T)$-estimates, respectively, for the solution $(u(x,t), v(x,t))$ to the system (1.2), and in its conclusion these estimates are combined and applied to Theorem 8 in [12] to derive the global existence.

step1. L^r-estimates

Lemma 3.1. *Let $r > 2$ and $p_r = \frac{2r}{r-2}$ be two positive numbers. Assume that $\alpha_{22} > 0$ and assume also that there is a constant $M_{r,T} > 0$ depending only on r, T, Ω and the coefficients of (1.2) such that $\|\nabla u\|_{L^r(Q_T)} \leq M_{r,T}$. Then there exists positive constants $C_7(T)$ and $C_T$such that*

$$\|v\|_{V_2(Q_T)} \leq C_7(T),$$

and

$$\|v\|_{L^r(Q_T)} \le C_T, \, if \quad r < \frac{4(n+1)}{(n-2)_+},$$

where $a_+ = \max\{a, 0\}$.

Proof. For any constant $q > 1$, multiplying the second equation of (1.2) by qv^{q-1} and using the integration by parts, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega v^q dx &= q\int_\Omega v^{q-1}\nabla\cdot[(d_2+\alpha_{21}u+2\alpha_{22}v)\nabla v+\alpha_{21}v\nabla u]dx \\
&\quad + q\int_\Omega v^q(c-v+\frac{du}{1+mu+\beta u^2})dx \\
&= -q(q-1)\int_\Omega v^{q-2}(d_2+\alpha_{21}u+2\alpha_{22}v)|\nabla v|^2dx - \alpha_{21}(q-1)\int_\Omega \nabla(v^q)\cdot\nabla u dx \\
&\quad + q\int_\Omega v^q(c-v+\frac{du}{1+mu+\beta u^2})dx \\
&\le -q(q-1)d_2\int_\Omega v^{q-2}|\nabla v|^2dx - 2\alpha_{22}q(q-1)\int_\Omega v^{q-1}|\nabla v|^2dx \\
&\quad - \alpha_{21}(q-1)\int_\Omega \nabla(v^q)\cdot\nabla u dx + q\int_\Omega v^q(c-v+\frac{du}{1+mu+\beta u^2})dx \\
&= -\frac{4(q-1)d_2}{q}\int_\Omega |\nabla(v^{\frac{q}{2}})|^2dx - \frac{8\alpha_{22}q(q-1)}{(q+1)^2}\int_\Omega |\nabla(v^{\frac{q+1}{2}})|^2dx \\
&\quad - \alpha_{21}(q-1)\int_\Omega \nabla(v^q)\cdot\nabla u dx + q\int_\Omega v^q(c-v+\frac{du}{1+mu+\beta u^2})dx.
\end{aligned}$$

Integrating (3.1) from 0 to t, we have

$$\begin{aligned}
&\int_\Omega v^q(x,t)dx + \frac{4(q-1)d_2}{q}\int_{Q_t}|\nabla(v^{\frac{q}{2}})|^2dxdt + \frac{8\alpha_{22}q(q-1)}{(q+1)^2}\int_{Q_t}|\nabla(v^{\frac{q+1}{2}})|^2dxdt \\
\le &\int_\Omega v^q(x,0)dx - \alpha_{21}(q-1)\int_{Q_t}\nabla(v^q)\cdot\nabla u dxdt + q\int_{Q_t} v^q(c-v+\frac{du}{1+mu+\beta u^2})dxdt.
\end{aligned} \tag{3.1}$$

By Hölder's inequality, we have

$$\begin{aligned}
&q\int_{Q_t} v^q(c-v+\frac{du}{1+mu+\beta u^2})dxdt \\
&\le -q\|v\|^{q+1}_{L^{q+1}(Q_t)} + (\frac{d}{m}+c)q\|v\|^q_{L^q(Q_t)} \\
&\le -q\|v\|^{q+1}_{L^{q+1}(Q_t)} + \frac{q(cm+d)}{m}\left[|Q_T|^{\frac{1}{q}-\frac{1}{q+1}}\|v\|_{L^{q+1}(Q_t)}\right]^q \\
&= -q\|v\|^{q+1}_{L^{q+1}(Q_t)} + \frac{q(cm+d)}{m}|Q_T|^{\frac{1}{q+1}}\|v\|^q_{L^{q+1}(Q_t)} \\
&\le -q\|v\|^{q+1}_{L^{q+1}(Q_t)} + \frac{q(cm+d)}{m}\left[\frac{|Q_T|^{\frac{q}{q+1}}}{\varepsilon^q} + \varepsilon\|v\|^{q+1}_{L^{q+1}(Q_t)}\right] \\
&\le C_4,
\end{aligned}$$

where $\varepsilon = \frac{m}{cm+d}$, $C_4 = \frac{q(cm+d)^{q+1}}{m^{q+1}}|Q_T|^{\frac{q}{q+1}}$.

On the other hand, since that $\frac{1}{r}+\frac{1}{2}+\frac{1}{p_r}=1$ and ∇u is in $L^r(Q_T)$, using the Hölder's inequality, we have

$$\begin{aligned}\left|-\int_{Q_t}\nabla(v^q)\cdot\nabla u dxdt\right| &= \frac{2q}{q+1}\left|\int_{Q_t} v^{\frac{q-1}{2}}\cdot\nabla(v^{\frac{(q+1)}{2}})\cdot\nabla u dxdt\right| \\ &\le \frac{2q}{q+1}\|v^{\frac{q-1}{2}}\|_{L^{p_r}(Q_t)}\cdot\|\nabla(v^{\frac{q+1}{2}})\|_{L^2(Q_t)}\cdot\|\nabla u\|_{L^r(Q_t)} \\ &\le \frac{2q}{q+1}\|v\|^{\frac{q-1}{2}}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}\cdot\|\nabla(v^{\frac{q+1}{2}})\|_{L^2(Q_t)}\cdot\|\nabla u\|_{L^r(Q_t)} \\ &\le \frac{2q}{q+1}M_{r,T}\|v\|^{\frac{q-1}{2}}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}\cdot\|\nabla(v^{\frac{q+1}{2}})\|_{L^2(Q_t)}.\end{aligned}$$

Therefore, (3.1) yields

$$\begin{aligned}&\int_\Omega v^q(x,t)dx+\frac{4(q-1)d_2}{q}\int_{Q_t}|\nabla(v^{\frac{q}{2}})|^2dxdt+\frac{8\alpha_{22}q(q-1)}{(q+1)^2}\int_{Q_t}|\nabla(v^{\frac{q+1}{2}})|^2dxdt \\ &\le C_5+C_6\|v\|^{\frac{q-1}{2}}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}\cdot\|\nabla(v^{\frac{q+1}{2}})\|_{L^2(Q_t)} \\ &\le C_5+\frac{C_6}{4\varepsilon}\|v\|^{q-1}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}+C_6\varepsilon\|\nabla(v^{\frac{q+1}{2}})\|^2_{L^2(Q_t)},\end{aligned}$$

where $C_5>0$ depending on q,T,Ω coefficients of (1.2) and initial datal v_0. For any $\varepsilon>0$, from above expression and by choosing a sufficiently small ε, such that $C_6\varepsilon<\frac{8\alpha_{22}q(q-1)}{(q+1)^2}$, we have

$$\begin{aligned}&\|v(.,t)\|^q_{L^q(\Omega)}+\|\nabla(v^{q/2})\|^2_{L^2(Q_t)}+\|\nabla(v^{(q+1)/2})\|^2_{L^2(Q_t)} \\ &\le C(r,q,T)\left(1+\|v\|^{q-1}_{L^{\frac{p_r(q-1)}{2}}(Q_t)}\right).\end{aligned}\tag{3.2}$$

Set $\widetilde{v}=v^{\frac{(q+1)}{2}}$, and

$$\begin{aligned}E &\equiv \sup_{0\le t\le T}\int_\Omega v^q(x,t)dx+\int_{Q_T}|\nabla(v^{(q+1)/2})|^2dxdt \\ &= \sup_{0\le t\le T}\int_\Omega \widetilde{v}^{2q/q+1}dx+\int_{Q_T}|\nabla\widetilde{v}|^2dxdt.\end{aligned}$$

Let $r_0=4, p_0=\frac{2r_0}{r_0-2}$. By Lemma 2.5, we see that ∇u is in $L^{r_0}(Q_T)$. So, from(3.2), we have

$$E+\|\nabla(v^{\frac{q}{2}})\|^2_{L^2(Q_T)}\le C(r_0,q,T)\left(1+\|\widetilde{v}\|^{\frac{2(q-1)}{q+1}}_{L^{\frac{p_0(q-1)}{q+1}}(Q_T)}\right).\tag{3.3}$$

For any $q>1$, if

$$(np_0-2n-4)q\le 2n+np_0,\tag{3.4}$$

then, $\frac{p_0(q-1)}{q+1} \leq \widetilde{q} = 2 + \frac{4q}{n(q+1)}$. By Hölder′s inequality

$$\|\widetilde{v}\|_{L^{\frac{p_0(q-1)}{q+1}}(Q_T)} \leq C_8(q,T)\|\widetilde{v}\|_{L^{\widetilde{q}}(Q_T)}, \tag{3.5}$$

where $C_8(q,T) = |Q_T|^{\frac{q+1}{p_0(q-1)} - \frac{1}{\widetilde{q}}}$. Setting $\widetilde{\beta} = 2/(q+1) \in (0,1)$, by Lemma 2.2 we have

$$\|\widetilde{v}(.,t)\|_{L^{\widetilde{\beta}}(\Omega)} = \|v(.,t)\|_{L^1(\Omega)}^{\frac{1}{\widetilde{\beta}}} \leq (C_1(T))^{\frac{1}{\widetilde{\beta}}}, \forall t \in [0,T). \tag{3.6}$$

Therefore, by (3.6), Lemma 2.4 and the definition of E, the expression (3.5) yields

$$\|\widetilde{v}\|_{L^{p_0(q-1)/q+1}_{(Q_T)}} \leq C_8(q,T)\|\widetilde{v}\|_{L^{\widetilde{q}}(Q_T)} \leq C_8(q,T)M_1\left\{1 + E^{2/n\widetilde{q}}E^{\frac{1}{\widetilde{q}}}\right\}. \tag{3.7}$$

Then, by(3.3) and (3.7), we have

$$E \leq C_9(q,T)(1 + E^{\mu}E^{\nu}) \tag{3.8}$$

with

$$\mu = \frac{4(q-1)}{n\widetilde{q}(q+1)}, \quad \nu = \frac{2(q-1)}{\widetilde{q}(q+1)}.$$

Since

$$\mu + \nu = \frac{2(q-1)}{\widetilde{q}(q+1)}\left[\frac{2}{n} + 1\right] < \frac{1}{\widetilde{q}}\left[\frac{4q}{n(q+2)} + 2\right] = 1,$$

it follows from (3.8) that there exists a positive constant C_{10} such that $E \leq C_{10}$. By (3.17) and (3.8) we get $\widetilde{v} \in L^{\widetilde{q}}(Q_T)$ which in turn implies that $v \in L^r(Q_T)$ with $r = \frac{\widetilde{q}(q+1)}{2}$ for any q satisfying (3.4). Now, Looking at (3.4), if $n \leq 2$, we have

$$np_0 - 2n - 4 = 2(n-2) \leq 0,$$

then (3.4) holds for all q. So for $n \leq 2$, $v \in L^r(Q_T)$ for all $r > 1$. If $n > 2$, then (3.4) is equivalent to

$$1 < q < q_0 \doteq \frac{2n + np_0}{(np_0 - 2n - 4)} = \frac{3n}{n-2}.$$

By

$$\frac{\widetilde{q}(q+1)}{2} = q + 1 + \frac{2q}{n} \leq \overline{r}_1 \doteq q_0 + 1 + \frac{2q_0}{n} = \frac{4(n+1)}{n-2}.$$

We have that v is in $L^r(Q_T)$ for all $1 < r \leq \overline{r}_1$. Since (3.7) holds true for $q = 2$. So E is bounded for $q = 2$. It follows from (3.3) and (3.7), we see that $\|v\|_{V_2(Q_T)}$ is bounded for any n. Namely, there exist positive constants C_T such that $\|v\|_{L^r(Q_T)} \leq C_T$ for $r < \frac{4(n+1)}{(n-2)_+}$, this completes the proof of Lemma 3.1. □

step2. $L^\infty(Q_T)$-estimates

Lemma 3.2. *Let $\alpha_{11} > 0$ and suppose that there are $r_1 > \max\{\frac{n+2}{2}, 3\}$ and a positive constant $C_{r_1,T}$ such that*

$$\|v\|_{L^{r_1}(Q_T)} \le C_{r_1,T}.$$

Then, there exists a positive M_2 such that

$$\|v\|_{L^r(Q_T)} \le M_2 \quad \textit{for any } \ r > 1.$$

Proof. First, the equation for u can be written in the divergence form as

$$u_t = \nabla \cdot [(d_1 + 2\alpha_{11}u)\nabla u] + u[a - u - \frac{bv}{1 + mu + \beta u^2}], \tag{3.9}$$

where $d_1 + 2\alpha_{11}u$ is bounded in $\overline{Q}_T$ by Lemma 2.1 and $u[a - u - \frac{bv}{1+mu+\beta u^2}]$ is in L^{r_1} with $r_1 > \frac{n+2}{2}$. Application of the Hölder continuity result see [17, Theorem 10.1, p.204]to (3.9) yields

$$u \in C^{\beta, \frac{\beta}{2}}(\overline{Q}_T) \ \text{ with some } \ \beta > 0. \tag{3.10}$$

Moreover, we have

$$w_{1t} = (d_1 + 2\alpha_{11}u)\triangle w_1 + f_1, \tag{3.11}$$

where $w_1 = (d_1 + \alpha_{11}u)u$, $f_1 = (d_1 + 2\alpha_{11}u)u[a - u - \frac{bv}{1+mu+\beta u^2}]$. Since u is bounded and by the assumption of this Lemma, we see that f_1 is in $L^{r_1}(Q_T)$. From (3.11), Lemma 2.1 and Lemma 3.1, applying Theorem 9.1 [17, p.341-342] and its remark[17, p.351], we have

$$w_1 \in W^{2,1}_{r_1}(Q_T)$$

this implies $\nabla u = \frac{1}{d_1+2\alpha_{11}u}\nabla w_1$ in $L^{r_1}(Q_T)$. Now, following the proof of Lemma 3.1 with r_1 instead of r_0 and $p_1 = \frac{2r_1}{r_1-2}$ instead of p_0, we see that either v is in $L^r(Q_T)$ for any $r > 1$ or else v is in $L^{r_2}(Q_T)$ with

$$r_2 \doteq \frac{(n+1)r_1}{n+2-r_1}.$$

The later case happens if and only if

$$n + 2 - r_1 > 0.$$

If v is in $L^{r_2}(Q_T)$, we see that f_1 is in $L^{r_2}(Q_T)$. Therefore, applying Theorem 9.1 [17, p.341-342] and its remark [17, p.351] again, we have ∇u in $L^{r_2}(Q_T)$. Then we go back and do the same argument again. Keep doing likes this we will get a sequence of numbers

$$r_{k+1} \doteq \frac{(n+1)r_k}{n+2-r_k}. \tag{3.12}$$

We stop and get the conclusion that v is in $L^r(Q_T)$ for any $r > 1$ when

$$n + 2 - r_k \le 0. \tag{3.13}$$

Since $r_1 > 3$, it is not hard to verify by inducting that $r_k > 3, k = 1, 2, \cdots$. Then, we have

$$\frac{r_{k+1}}{r_k} = \frac{n+1}{n+2-r_k} \geq \frac{n+1}{n-1} > 1.$$

Therefore, (3.13) holds for some k. We stop at this k and conclude that

$$\|v\|_{L^r(Q_T)} \leq M_2 \quad \text{for any} \quad r > 1.$$

$\square$

step3. $C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{Q}_T)$-estimates

Similar to the proof of Theorem1.2 of [2], *we can obtain* $C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{Q}_T)$-*estimates of the solution of (1.2). So the proof of Theorem 1.1 is completed.*

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