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THE LOCATION FOR EIGENVALUES OF COMPLEX MATRICES BY A NUMERICAL METHOD†

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Abstract. In this paper, we adopt a numerical method to establish the smallest set to contain all Geršgorin discs of a given complex matrix and its some similar matrices. With the smallest set, a new estimation for all eigenvalues of the matrix is obtained.

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1. Introduction

The well-known Geršgorin theorem mainly reads as follows (see [1-3]): For any given complex matrix $A = (a_{ij}) \in C^{n\times n}$, if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are its eigenvalues, then they can be included by Gerŝgorin set

$$\Gamma(A) = \bigcup_{i=1}^{n} \Gamma_i(A),$$

where $\Gamma_i(A) = \{z \in C : |z - a_{ii}| \le r_i(A)\}$, $r_i(A) = \sum_{j\neq i} |a_{ij}|$, $i \in \{1, 2, \cdots, n\}$.

However, just as literature [3] said, if there is a non-singular $n \times n$ matrix X such that $X^{-1}AX = B = (b_{ij}) \in C^{n\times n}$ (it means that A and B have the same eigenvalues), then a new Geršgorin set

$$\Gamma(B) = \bigcup_{i=1}^{n} \Gamma_i(B)$$

also contains $\lambda_1, \lambda_2, \cdots, \lambda_n$, where $\Gamma_i(B) = \{z \in C : |z - b_{ii}| \le r_i(B)\}$, $r_i(B) = \sum_{j\neq i} |b_{ij}|$, $i \in \{1, 2, \cdots, n\}$. We note that, in general, $a_{ii} \neq b_{ii}$, $r_i(A) \neq r_i(B)$, and then $\Gamma(A) \neq \Gamma(B)$. Thus a problem comes, that is how to find a

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smaller Geršgorin set between $\Gamma(A)$ and $\Gamma(B)$ to contain all eigenvalues of A such that the location for eigenvalues is more accurate. Furthermore, we may be find finite or countable $n \times n$ non-singular matrices $X_1, X_2, \cdots, X_k$ such that $X_1^{-1}AX_1 = B_1, X_2^{-1}AX_2 = B_2, \cdots, X_k^{-1}AX_k = B_k$. Thus we can obtain finite or countable Geršgorin sets $\Gamma(B_1), \Gamma(B_2), \cdots, \Gamma(B_k)$ and $\Gamma(A)$ to contain all eigenvalues of A. In this case, it requires us to find out the smallest Geršgorin set from $K+1$ Geršgorin sets to contain all eigenvalues of A such that this location is best one.

From 2004 to 2006, the similar problems had been proposed in some literatures continuously. Such as literature [3], the author used intersection of Geršgorin sets to determine the smallest set to locate the eigenvalues of a given matrix and some special cases were studied in detail. However just as literature [3] said, in generally, such set is difficult to determine. The difficulty lies in finding out an explicit and calculable numerical formula to express this set. In this paper, we will explore a new way to solve the problem by an effective numerical formula.

2. Main results

Lemma 2.1. Let$A = (a_{ij}) \in C^{n\times n}$, then there must be a minimal disc in complex plane containing all Geršgorin discs of A.

Proof. By the Geršgorin theorem, we know that all Geršgorin discs of A are

$$\Gamma_i(A) = \{z \in C : |z - a_{ii}| \le r_i(A)\},$$

where $r_i(A) = \sum_{j\neq i} |a_{ij}|$, $i \in \{1, 2, \cdots, n\}$. If we treat every $\Gamma_i(A)(i = 1, 2, \cdots, n)$ as a particle or rigid body, according to the central principle, the center of all particles or rigid bodies is $\frac{1}{n}\sum_{i=1}^{n} a_{ii} = \frac{trA}{n}$. To seek the smallest disc, we establish the following model:

$$\min\left|z - \frac{trA}{n}\right| \qquad s.t. \qquad |z_i - a_{ii}| \le r_i(A), \quad i = (1, 2, \cdots, n).$$

Since

$$\begin{aligned}\left|z_i - \frac{trA}{n}\right| &= \left|z_i - a_{ii} + a_{ii} - \frac{trA}{n}\right| \\ &\le |z_i - a_{ii}| + \left|a_{ii} - \frac{trA}{n}\right| \\ &\le r_i(A) + \left|a_{ii} - \frac{trA}{n}\right|,\end{aligned}$$

the solution to the above model is that

$$\left|z - \frac{trA}{n}\right| = \max_{i\in\{1,2,\cdots n\}}\left|z_i - \frac{trA}{n}\right| \le \max_{i\in\{1,2,\cdots n\}}\left[r_i(A) + \left|a_{ii} - \frac{trA}{n}\right|\right].$$

That is, all Geršgorin discs of A must belong to the smallest disc with radius $\max_{i\in\{1,2,\cdots n\}}\left[r_i(A) + \left|a_{ii} - \frac{trA}{n}\right|\right]$ and center at $\frac{trA}{n}$. Thus, the proof is complete.

For convenience, we denotes the smallest disc by $\Omega(A)$, then

$$\Omega(A)=\left\{z\in C:\left|z-\frac{trA}{n}\right|\leq\max_{i\in\{1,2,\cdots n\}}\left[r_i(A)+\left|a_{ii}-\frac{trA}{n}\right|\right]\right\}.$$

Theorem 2.1. Let $A\in C^{n\times n}$, then all eigenvalues of A can be contained by $\Omega(A)$.

Proof. By Geršgorin theorem and Lemma 2.1, we can draw the conclusion easily, so here we omit the proof.

Lemma 2.2. Let $A\in C^{n\times n}$. If $B_1,B_2,\cdots,B_k$ are similar to A, then we can derive a minimal disc in complex plane which can contain all Geršgorin discs of at least one matrix among $B_1,B_2,\cdots,B_k$ and A.

Proof. Since $B_1,B_2,\cdots,B_k$ are similar to A, we have $trB_i=trA(i=1,2,\cdots,k$. It shows that all discs $\Omega(B_i)(i=1,2,\cdots,k$ and $\Omega(A)$ are concentric discs whose centers at $\frac{trA}{n}$. So, it only needs us to find the disc with the smallest radius from $k+1$ concentric discs.

If we use $\Omega_{min}(A)$ to denote disc with the smallest radius, then we have

$$\Omega_{min}(A)=\left\{z\in C:\left|z-\frac{trA}{n}\right|\right\}$$

$$\leq\min_{j\in\{1,2,\cdots k\}}\left\{\max_{i\in\{1,2,\cdots,n\}}\left[r_i(A)+\left|a_{ii}-\frac{trA}{n}\right|\right],\max_{i\in\{1.2.\cdots,n\}}\left[r_i(B_j)+\left|b_{ii}-\frac{trA}{n}\right|\right]\right\}.$$

It shows that the radius and centre of the smallest disc can be determined by the entries of $B_1,B_2,\cdots,B_k$ and A. Therefore the proof is complete.

Theorem 2.2. Let $A\in C^{n\times n}$ and $\lambda_1,\lambda_2,\cdots,\lambda_n$ be n eigenvalues of A. If $B_1,B_2,\cdots,B_k$ are similar to A, then $\lambda_1,\lambda_2,\cdots,\lambda_n$ can be contained by $\Omega_{min}(A)$.

Proof. By Geršgorin theorem and Lemma 2.2, we can draw the conclusion easily, so here we also omit the proof.

3. Numerical examples

In this section, we provide two numerical examples to assess our theoretical results.

Example 3.1. Let

$$A=\begin{bmatrix}-1&1&0\\-4&3&0\\1&0&2\end{bmatrix}.$$

We can obtain

$$\Omega(A)=\left\{z\in C:\left|z-\frac{trA}{n}\right|\leq\max_{i\in\{1,2,\cdots,n\}}\left[r_i(A)+\left|a_{ii}-\frac{trA}{n}\right|\right]\right\}$$

$$=\left\{z\in C:\left|z-\frac{4}{3}\right|\leq\max_{i\in\{1,2,\cdots,n\}}\left[\frac{10}{3},\frac{17}{3},\frac{5}{3}\right]=\frac{17}{3}\right\}.$$

By Theorem 2.1, we know all eigenvalues of A should be contained in $\Omega(A)$, i.e.

$$\left|\lambda - \frac{4}{3}\right| \leq \frac{17}{3}.$$

By direct calculations, the eigenvalues of A are

$$\lambda_1 = 2,\ \lambda_2 = \lambda_3 = 1.$$

Thus the Theorem 2.1 is valid.

Example 3.2. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

If we let

$$P = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

then

$$P^{-1}AP = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = B$$

We can obtain

$$\begin{aligned} \Omega(A) &= \left\{ z \in C : \left| z - \frac{trA}{n} \right| \leq \max_{i \in \{1,2,\cdots,n\}} \left[r_i(A) + \left| a_{ii} - \frac{trA}{n} \right| \right] \right\} \\ &= \left\{ z \in C : \left| z - \frac{1}{3} \right| \leq \max_{i \in \{1,2,\cdots,n\}} \left[\frac{4}{3}, \frac{8}{3}, \frac{4}{3} \right] = \frac{8}{3} \right\}. \end{aligned}$$

and

$$\begin{aligned} \Omega(B) &= \left\{ z \in C : \left| z - \frac{trB}{n} \right| \leq \max_{i \in \{1,2,\cdots,n\}} \left[r_i(B) + \left| b_{ii} - \frac{trB}{n} \right| \right] \right\} \\ &= \left\{ z \in C : \left| z - \frac{1}{3} \right| \leq \max_{i \in \{1,2,\cdots,n\}} \left[\frac{4}{3}, \frac{2}{3}, \frac{2}{3} \right] = \frac{4}{3} \right\}. \end{aligned}$$

Obviously,

$$\Omega_{min}(A) = \Omega(B) = \left\{ z \in C : \left| z - \frac{1}{3} \right| \leq \frac{4}{3} \right\}.$$

By Theorem 2.2, we know all eigenvalues of A should be contained in $\Omega_{min}(A)$, i.e.,

$$\left|\lambda - \frac{1}{3}\right| \leq \frac{4}{3}.$$

By direct calculations, the eigenvalues of A are

$$\lambda_1 = -1,\ \lambda_2 = \lambda_3 = 1.$$

Therefore, Theorem 2.2 is valid.

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