대한수학교육학회지 수학교육학연구 제 21 권 제 1 호
Journal of Educational Research in Mathematics
Vol. 21, No. 1, 33-55. Feb 2011

Calculus Instructors and Students′ Discourseson the Derivative

Park, Jung Eun*

This study explores the characteristics of calculus students' and instructors' discourses on the derivative using a communicational approach to cognition. The data were collected from surveys, classroom observations, and interviews. The results show that the instructors did not explicitly address some aspects of the derivative such as the relationship between the derivative function ($f'(x)$) and the derivative at a point ($f'(a)$), and $f'(x)$ as a function, and that students incorrectly described or used these aspects for problem solving. It is also found that both implicitness in the instructors' discourse, and students'incorrect descriptions were closely related to their use of the word, ″derivative″ without specifying it as ″the derivative function″ or ″the derivative at a point.″ Comparison between instructors' and students' discourses suggests that explicit discussion about the derivative including exact use of terms will help students see the relationship that $f'(a)$is a number, a point-specific value of $f'(x)$ that is a function, and overcome their mixed and incorrect notion ″the derivative″ such as the tangent line at a point.

I. Introduction

Recently, there has been an increasing interest in collegiate mathematics education, especially teaching and learning calculus (e.g., Carlson, Oehrtman, & Thompson, 2008; Speer, Smith, & Horvath, 2010). Of calculus concepts, the derivative is known as a difficult concept for students to understand because it involves various concepts such as ratio, limit, and function (e.g., Thompson, 1994) and it can be represented in multiple ways (e.g., Zandieh, 2000).

This study explored and compared university calculus instructors' and students' discourse on the derivative with the lens of a communicational approach to cognition (Sfard, 2008). Specifically, it examined how they describe a) the concept of derivative, b) the relationships between a function ($f(x)$), the derivative function ($f'(x)$), and the derivative at a point ($f'(a)$), and c) $f'(x)$ as a function. Motivation of this study came from my teaching experience in South Korea and the United States (US). The most prominent difference between the two countries was use of the word, ″derivative.″ Observing US instructors' and students' discussions using the word without specifying it as ″the derivative function,″ or ″the derivative at a point″ made me wonder if they meant the same object. The corresponding terms in Korean are "미분계수, Mi-Bun-Gye-Su" for the derivative at a point (translated as a differential coefficient), and "도함수, Do-Ham-Su" for the derivative function (translated as a path function). These Korean terms cannot be represented by one word, ″derivative,″ since they do not share a common word. Research in Korea has

* Michigan State University, parkju20@msu.edu

pointed out inconsistency in these terms may hamper student understanding of the relationship between the two concepts (한대희, 1998). However, little empirical research has been done in use of mathematical terms regarding students' learning of advanced concepts such as the derivative. In this study, I explore main features of instructors' and students' discourses on the derivative including how they use terms, "derivative," "the derivative at a point," and "the derivative function."

This study contributes to the field of mathematics education in the following ways. First, by exploring the use of words, this study may reveal the role of the mathematical terms in students' learning. Research showed the importance of word use in students' thinking about numbers and fractions (e.g., Fuson & Kwon, 1992; Sfard, 2008). This current study may expand our understanding of the role of mathematical terms in learning an advanced concept, the derivative. Second, this study may provide a better understanding of how the derivative is taught in classrooms. Though there have been studies about calculus textbooks (Wu, 1997; Hurley, Koehn, & Ganter, 1999), research have pointed out a lack of studies addressing what is happening in calculus classrooms (Speer et al., 2010). Exploring classroom discourse may reveal various aspects of derivative lessons such as how instructors address key topics, and whether and how students participate in classroom. Third, by exploring instructors' and students' discourse on the derivative, this study may provide explanations of discrepancies between the two types of discourses. Such explanations may lead to suggestions for instructors to help students overcome their difficulties with, and incorrect notions of the derivative. If instructors' vague use of words affects students' thinking of the derivative, instructors should consider specifying exact meaning of the terms while communicating with students.

II. Theoretical Background

1. Previous Research on Students' Thinking about the Derivative

Existing studies about students' thinking about the derivative can be divided into their thinking about a) the related concepts such as ratio, limit, and function, and b) various representations of the derivative: symbolic, graphical, physical, and algebraic.

A. Students' Thinking on the Related Mathematics Concepts

The definition of derivative function, $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, consists of ratio, limit, and function. Studies have shown that students' thinking of these concepts is related to their thinking of the derivative. Studies have found that a) students have a strong procedure-based concept of the ratio, which hampers appreciating the limit process of obtaining instantaneous rate of change (IRC) from the sequence of average rate of changes (ARC) (Hauger, 1998; Orton, 1983), and b) an operational understanding of ARC in relation to the change in the function helps students understand calculus concepts and theorems (Thomson, 1994). Regarding the concept of limit, studies have showed that students' thinking of the derivative is related to their difficulty appreciating a tangent line as a limit of secant lines; they stated that the secant lines approaching to a tangent line could never get to the tangent line, or the limit of the secant lines was zero or a point or did not exist (Tall, 1986; Orton, 1983). About the concept of function, students' thinking of its co-varying nature and the derivative function as a function, and their tendency to find an algebraic representation of a function are found to be related to their thinking of the derivative. Thompson (1994) reported students' image that a function has two sides separated by an equal sign, a short expression on the left and a long expression on the right (p. 268), can lead students to consider a function as one thing changing rather than as a relation of co-varying quantities. Monk (1994) defined two types of understandings: "pointwise understanding" as the value of the function at a point, and "across-time understanding" as a dynamic quantity on an interval. Monk (1994) found that "pointwise understanding" is prerequisite to "across-time understanding," and students have a fragmented concept of the derivative function rather than realizing it as a dynamic quantity. A study also found students' difficulty recognizing the independent variable of the derivative function (Carlson et al., 2008).

B. Students' Thinking on the Representations of the Derivative

The derivative can be represented algebraically, symbolically, graphically, and physically. Studies have found that students prefer algebraic representations yet show many errors when applying the formulas (Asiala, Cottrill, Dubinsky, Schwingendorf, 1997; Hirst, 2002). Studies also found that students recognized the symbol, dy/dx, as a signal of differentiating without distinguishing independent and dependent variables properly (Santos & Thomas, 2003), and did not see the equivalence between the two symbols $\frac{d^2y}{dx^2}$ and $\frac{d(\frac{dy}{dx})}{dx}$ (Thomas, 2002). Studies reported that students assumed that the graphs of a function and its derivative had resemblance (e.g., Nemirovsky & Rubin, 1992), and could not provide correct units or interpret the derivative as the rate of change (Bezuidenhout, 1998).

Studies also have shown that students have different levels of thinking of different representations of the derivative, and thus have difficulty explaining their relationships (Hahkioniemi, 2005; Santos & Thomas, 2002; Zandieh, 2000). Bingolbali & Monaghan (2008) found that students in different disciplines developed different images of the derivative; mechanical engineering students and mathematics students tended to interpret the derivative as the rate of change, and the slope of a tangent line, respectively.

2. Limitations of the Existing Studies

Existing studies, which have revealed various aspects of students' thinking of the derivative, have limitations regarding topic, method, and data. First, studies did not explore students' thinking about the derivative in relation to the function. For example, students' difficulty interpreting velocity or acceleration, which was classified as their *misconception* of the derivative in a study (Bezuidenhout, 1998), can be addressed in terms of how they explain the first and second derivatives in relation to the original function, which is a position. Given that the definition of the derivative function and the derivative at a point are derived from a function (Weir, Hass, & Giordano, 2006), how students described the derivative in terms of the function and how they used the descriptions in problem solving would be important to explore.

Second, calculus instructors and students tend to use the word, "derivative," without specifying it as "the derivative function," or "the derivative at a point." However, how instructors use these terms may be different from how their students use these terms, and this ambiguous word use may be related to students' thinking on the derivative. For example, a student' difficulty finding differentiability of a piecewise function (Ferrini-Mundy & Graham, 1994) may come from her confusion between the derivative function that has two equations on both sides and the derivative at a point that may be the same on both sides. However, how calculus students or instructors use these terms has not been investigated in a systemic way.

Third, in terms of data, existing research did not explore how the derivative is taught in calculus classrooms. Importance of investigating how teachers teach mathematical concepts in their classes has been emphasized by many researchers (e.g., Newton, 2008; Speer, Smith, & Horvath, 2010). Calculus students' opportunity to learn the derivative in their classrooms can be explored by analyzing instructors' use of curriculum materials or their classroom discourses.

Regarding the limitations mentioned above, this study expands the previous work by including information about how the derivative is taught and learned in classrooms with a new perspective, a communicational approach to cognition. In this study, instructors' and students' discourse on the derivative in classrooms and interview settings are explored, focusing on their explanations of the derivative function and the derivative at a point, the relationships between a function, the derivative function, and the derivative at a point, and the derivative as a function.

3. Theoretical Frameworks for this Research

As discussed in the previous section, this study explores the calculus instructors' and students' discourse on the derivative using a communicational approach to cognition (Sfard, 2008). This approach assumes that thinking is an "individualized version of interpersonal communication" (Sfard, 2008, p. xvii), and mathematics is a form of discourse with four features: *Word use, Visual mediator, Routine*, and *Endorsed narrative*. *Words* in mathematics signify mathematical objects, and what a word refers to may differ across speakers. For example, an instructor's utterance, "the derivative is increasing here," can be understood as "the derivative at a point is increasing here" by students who consider the tangent line as "the derivative at a point" (Park, 2008). *Visual mediators* refer to visual objects used as a means of communication such as notations, drawings, gestures, or concrete objects. The discourse on the derivative can be mediated by the definition, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, the graph of a tangent line, or the rate of change of a moving object. *Narrative* is utterances which can be endorsed as true or rejected as false, and *endorsed narratives* are ones accepted as true by speakers (Sfard, 2008). Students' endorsed narratives do not always coincide with those of mathematicians. For example, teachers can endorse a narrative, "the derivative function is also a function," whereas students may endorse an incorrect statement, "the tangent line is the derivative function" based on a narrative, "the derivative is a linear function represented by a tangent line" (Park, 2008). Routines are repetitive patterns in discourse defined as metarule that determines situations where a performance is appropriate to start ("applicability condition") and finish ("closure"), and a course of action in the performance (Sfard, 2008, p. 208). If an instructor solves problems that involve finding $f'(a)$ using the definition $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ by graphing $f(x)$, computing $f'(a)$ algebraically, drawing a tangent line to the graph of $f(x)$, and confirming the answer by comparing its slope to $f'(a)$, this pattern can be considered as a course of action. The problems can be considered as an applicability condition and the last step as a closure. These four main characteristics of mathematical discourses provided me a lens to look through how instructors and students addressed or used the concepts of the derivative at a point and the derivative of a function, and their relationships during the class and interviews.

III. Design of Study

The purpose of this study is to explore three calculus instructors and their students' discourse on the derivative in class and interview settings with the following research questions:

1. How do instructors introduce or define the derivative at a point and the derivative function?
2. To what extent do instructors address the relationships between a function, the derivative at a point, and the derivative function?
3. To what extent do instructors address the derivative of a function as another function?
4. How do students describe the derivative at a point and the derivative of a function?
5. How do students describe or use the relationships between a function, the derivative at a point, and the derivative function?
6. How do students use the derivative function as another function?

To this end, I used a mixed design of classroom observations, student surveys, and interviews with instructors and students. I videotaped derivative lessons from three calculus classes in a large public university in the Midwest of the US. I chose three instructors who had different backgrounds and teaching styles. One instructor, Ian, was an adjunct faculty with Ph. D in mathematics and taught one section of Calculus I in a residential college at the university with the textbook, *University Calculus* (Hass, Weir, & Thomas, 2008). Two instructors, Tyler and Alan, were doctoral students with masters' degrees in mathematics and taught *Calculus I* in the mathematics department, with the textbook, *Thomas' Calculus* (Weir, et al., 2006)[1]. These two books are written by the same author team and include two chapters of the derivative with the same titles but slightly different sections.

After the derivative unit was over, the survey including problems involving the derivative, most of which are from Epstein's (2006) Calculus Concept Inventory, was taken by the students in the classroom. To further investigate their solution processes, I purposefully recruited 12 students, whose answers were close to the answers chosen by most students in their class and heterogeneous in terms of their achievement on the survey, and interviewed them individually for about an hour. I also interviewed the instructors about how they would explain survey problems to their students. Interviews were videotaped and transcribed focusing on what is said (e.g., verbal expressions including pauses) and what is done (e.g., drawings and gestures).

Transcripts were coded based on four topics: the derivative function, the derivative at a point, the relationships between them, and the original function (Table III-1). Analysis of instructors' discourse explored how and to what extent they addressed each topic, focusing on the key features of the discourse: word use, visual mediator, routine, and endorsed narratives. Cases in which they stated and used a relationship were classified as *explicit* whereas cases in which they used the relationship without stating it was classified as *implicit*. Analysis of students' discourse focuses on

1) The names of the instructors are pseudonyms.

how they described and used each topic in problem solving situations. My coding scheme (Table III-1) was used as an analytical framework for the cases in which the instructors and students addressed the relationships between a function, the derivative function, and the derivative at a point, and the derivative as a function. The four features of the mathematical discourse, which described in the previous section, provided me a lens to look through how they addressed these topics in each of these instances.

IV. Findings

This section addresses research questions about each of instructors' and students' explanations of a) the definitions of the derivative function, *f '(x)*, and the is a function or *f '(x₀)* is a specific value of *f '(x)* at a point; thus, it is a number. Alan first defined the rate of change at x_0, $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ and then the derivative of a function, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. Although he did not give a definition or notation of $f'(x_0)$, or mention that the instantaneous rate of change is the same as the derivative, he explicitly addressed the difference between $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ as a number and *f '(x)* as a function. Ian first defined the derivative at a point, $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ and then the derivative function using the same representation by stating that it "is a function of *a*." Although he defined these two concepts using the same notation, he did not mention that the derivative at a point is a specific value of the derivative of a function. Later, he changed *a* to *x* without addressing that x is the independent variable of *f '(x)*. In summary, although there are some differences across the instructors, they in general did not explicitly addressed the relationships

<Table III-1> *Cases Belonging to Each Coding Category*

Categories		Cases
Relationship between *f(x)* and *f '(x)*		Graphing or describing *f(x)* using the sign of *f '(x)* on an interval
		Proving the differentiation rules
		Interpreting the chain rule as a product of rates of change
		Finding anti-derivative algebraically or from graphs
		Finding the concavity of *f(x)* using whether *f '(x)* increases or decreases
Relationship between *f(x)* and *f '(a)*		Graphing *f '(x)* using values of the derivative at points on the domain
		Determining the differentiability of *f(x)* using *f '(a)*
		Specifying *f '(a)* = 0 when *f(x)* has an extreme value at *x* = *a*
Relation-ship between *f '(x)* and *f '(a)*	*f '(a)* as a value of *f '(x)*	Mentioning that *f '(a)* is a value of *f '(x)* at *x* = *a*
		Difference between non-differentiability of function and at a point
		Interpreting a point on the graph of *f '(x)* as the slope of a tangent line
	Transition from *f '(a)* to *f '(x)*	Mentioning that several values of the derivative at points form the derivative of a function
	Transition from *f '(x)* to *f '(a)*	Finding the equation of *f '(x)* and then substituting a number to evaluate *f '(a)*
f'(x) as a function		Mentioning that *f '(x)* is a function as a part of its definition.
		Mentioning that *f '(x)* is a function that one could graph.
		Mentioning that *f '(x)* is as a function that has its own derivative, *f ''(x)*

between $f'(a)$ and $f'(x)$, which is that $f'(a)$ is a number, a value of $f'(x)$ that are a function, and the values of the derivative at points on the domain forms the function, $f'(x)$.

B. Relationships between $f(x)$, $f'(x)$, and $f'(a)$

The relationship between $f'(a)$ and $f'(x)$ was also implicitly addressed in later lessons. This relationship was mostly identified when the instructors a) found $f'(a)$ by substituting $x = a$ in $f'(x)$, b) determined the differentiability of a function, and c) described the behavior of f(x) based on the sign of the derivative. Although all instructors mentioned and used that $f'(a)$ can be calculated by substituting a value in $f'(x)$, they did not state that $f'(a)$ is a value of $f'(x)$ at $x = a$. Similarly, they determined the differentiability of a function, $f(x)$, based on the existence of $f'(a)$ without stating the relationship between $f'(a)$ and $f'(x)$ or distinguishing the differentiability of a function on an interval or at a point. The implicit approach was prominently found when the instructors described the behavior of $f(x)$ on an interval based on the sign of the derivative. Instructors used the sign of the point-specific value, $f''(a)$ instead of $f'(x)$ on an interval and used the word, "derivative" ambiguously without specifying it as "the derivative function" or "the derivative at a point." In most cases, they used one arm gesture indicating the tangent line at a single point instead of several gestures for multiple tangent lines on the interval, and stated that the slope was positive or negative on the interval. These word use and gestures tended to make unclear whether "derivative" referred to "derivative function" or "the derivative at a point" because instructors used the word as a point-specific object, $f'(a)$, which is the representative of the derivative function, $f'(x)$ on an interval.

In contrast, the instructors explicitly addressed the relationships between $f(a)$ and $f(x)$, which $f'(a)$ is the slope of $f(x)$ at a point, and used it in various situations such as graphing $f'(x)$ and computing the extreme values of $f(x)$. They explicitly related the characteristics of $f(x)$ such as "beats, valleys, humps" or "turn-around" to the derivative being zero at the point. However, when most of them used this relationship, the derivative being zero at a point where $f(x)$ has a horizontal tangent line to graph $f'(x)$, they again made the relationship between $f'(x)$ and $f'(a)$ implicit. All of them plotted x intercepts of $f'(x)$ where $f(x)$ has a horizontal tangent line without addressing that why they "match[ed] up" these points. None of them explicitly mentioned that $f'(a)$ is a specific value of $f'(x)$, for example, the derivative at several points form a function, $f'(x)$. The instructors also explicitly addressed the relationship between $f''(x)$ and $f(x)$ when deriving and using differentiation rules and describing the behavior of $f(x)$ based on the sign of $f'(x)$. However, in the latter case, their use of $f'(a)$ instead of $f'(x)$ to determine the sign of the derivative on an interval again made the relationship between $f'(x)$ and $f'(a)$ implicit.

C. The Derivative Function as a Function

The instructors' implicit discussion on $f'(x)$ and $f'(a)$ seems to connected to their discussion on $f'(x)$ as a function. Each instructor stated that $f'(x)$ is a function during the derivative unit with different topics; Tyler mentioned it once when he graphed $f'(x)$ by saying it is a function that one can "graph", Alan mentioned it once when he read aloud the definition of $f'(x)$ from the textbook, and Ian mentioned it twice while defining $f'(x)$ and $f''(x)$ as the derivative of $f'(x)$ by saying that "it is a function" which "deserves its own derivative". Aside from those cases, they used

$f'(x)$ as a function without stating it. For example, all instructors applied theorems about a function (e.g., Intermediate Value Theorem) to the derivative function without explicitly stating that the derivative function is a function that satisfies the conditions of the theorems.

The instructors' ambiguous use of the word "derivative" without specifying it as the derivative of a function, $f'(x)$, or the derivative at a point, $f'(a)$, also seems to related to their implicit discussion of $f'(x)$ as a function. Especially, the instructors' explanations about the behavior of $f(x)$ based on the sign of $f'(a)$ were mainly based on the slope of the tangent line at a single point rather than the change in slopes over an interval. Combined with the instructors' ambiguous use of the word, "derivative," their explanation emphasized the concept of "the derivative" as "the derivative at a point" instead of "the derivative function," in relation to the behavior of the original function over an interval. This emphasis on "the derivative at a point" and ambiguous word use may restrict the concept of "the derivative" only as a point-specific value rather than as a dynamic quantity that changes it value on an interval (Monk, 1994). On the other hand, since the relationship between $f(x)$ and $f'(x)$ was mostly addressed when the instructors explained and used differentiation rules, their explanation may develop two disconnected realization of the derivative; the derivative at a point as the slope of tangent line on the graph of $f(x)$ and the derivative of a function as the result of applying differentiation rules.

In addition, given that a function is a dynamic object that changes as its independent variable (IV) changes, identifying the IV of the derivative of a function can be considered as one aspect of conceiving of the derivative as a function. The importance of recognizing the IV is also argued in existing research (Carlson et al., 2008). However, none of the instructors explicitly stated that $f(x)$ and $f'(x)$ have the same IV. The discussion of the IV was limited to the procedure of differentiating implicit functions with multiple IVs.

2. Students' Discourse

A. Definitions of $f'(x)$ and $f'(a)$

Students' concepts of the derivative were investigated while they were answering the question, "What is the derivative?" during interviews. Of the 12 interviewees, seven chose the derivative function, $f'(x)$, as their concept of the derivative, three chose the derivative at a point, $f'(a)$, and two said that the same explanation can be applied to both $f'(x)$ on an interval and $f'(a)$ at a point. The slope is the most common interpretation of the derivative, and differentiation rules and the rate of change follows. Ten students used the same explanation for $f'(x)$ and $f'(a)$, two could not explain $f'(a)$, and one interpreted $f'(x)$ and $f'(a)$ differently. Two students did not provide any explanation about the derivative at a point because "the derivative" was only defined on an interval not at a point in their descriptions. Their descriptions imply that they consider "the derivative" only as a function on an interval, not a number, a point-specific value.

Most students, nine out of 12, correctly explained the relationship between $f'(x)$ and $f'(a)$ by interpreting $f'(a)$ as a point-specific value of $f'(x)$, for example, "slope at a point" versus "slopes at all the points," or "velocity at a point" versus "velocity over time." Two of the students who gave an algebraic explanation that $f'(a)$ can be found by "plug[ging]" $x = a$ in $f'(x)$. One student showed a confusion between $f'(a)$ and $f(a)$ and incorrectly explained that $f'(a)$ could be also obtained from $f(x)$ as well as correct explanation about substituting $x = a$ in $f'(x)$.

Based on their descriptions of the derivative, students explained what $f'(x)$ and $f'(a)$ inform about $f(x)$ by repeating their descriptions of $f'(x)$ and $f'(a)$, and only four students explained that $f'(x)$ and $f'(a)$ describe the behavior of $f(x)$ based on the signs of the $f'(x)$ and $f'(a)$: for example, "If $f'(x)$ on an interval is negative/positive, $f(x)$ decreases/increases on the interval," or "If $f'(a)$ is negative/positive, $f(x)$ decreases /increases at the point." Regarding the incorrect descriptions, one student consistently explained $f'(x)$ as an "extension or contraction of" (stretching or shrinking of the graph of) $f(x)$.

B. Relationships between $f(x)$, $f'(x)$, and $f'(a)$

During the second part of the interview, the students explained how they used the relationship between f(x), $f'(a)$, and $f'(x)$ while taking the survey. Regarding the relationship between $f(x)$ and $f'(x)$ most students correctly used the rules to differentiate $f(x)$, and described the behavior of f(x) in terms of sign of $f'(x)$. However, one student, Neal who used and explained the relationship correctly in a graphical situation, could not apply it to $f'(x)$ and $f''(x)$. Only a few students mentioned and used the concavity of a function in relation to the behavior of its derivative (e.g., increasing or decreasing). Some students also described the relationship incorrectly, and tended to use these descriptions inconsistently. For example, two students, who mentioned that graphs of $f(x)$ and $f'(x)$ go in the same direction, but they applied this relationship differently when $f(x)$ was given and when $f'(x)$ was given. One student, who stated that graphs of all derivative functions are linear or a piecewise linear in one problem, drew a concave up curve for the derivative function in another problem. The most common incorrect description of the relationship between $f(x)$ and $f'(x)$ was that $f'(x)$ increases/ decreases when $f(x)$ increases /decreases. Although most of students did not use this relationship to justify their answers, only two of them accepted that the statement was wrong and corrected it. Even after corrections, the statement was also identified several times later in the interview.

Regarding the relationship between $f(x)$ and $f'(a)$, most students interpreted $f'(a)$ as the slope of $f(x)$ at $x = a$, and used this interpretation to explain that the derivative is zero at a point where $f(x)$ changes its direction and to find the extreme values of $f(x)$. However, when asked to interpret $f'(a)$ in the given problem context, only a few students could distinguish $f'(a)$ from $f(a)$ and recognize the units correctly. About half of the students, who explained the relationship between $f(x)$ and $f'(a)$ correctly, confused $f'(a)$ as the rate of change of $f(x)$ with the change in $f(x)$ between two consecutive x values. This answer is mathematically acceptable to some extent because $f'(a)$ can be used as the estimate for the change. However, the quantities and units of $f'(a)$ are different from this change, which were not appreciated by most students. In graphical situations, students' incorrect interpretations of the relationship between $f(x)$ and $f'(a)$ seem closely related to the tangent line. Three students interpreted the tangent line at a point as the derivative at a point. Two of them integrated the equation of the tangent line to find $f(x)$. These incorrect interpretations shows their inability to conceive of $f'(a)$ as a value.

All 12 students used the relationship between $f'(x)$ and $f'(a)$ correctly identifying $f'(a)$ as a value of $f'(x)$ at $x = a$ when the equation or graph of $f'(x)$ was given. They also correctly recognized this relationship when $f(x)$ has a local extreme value. However, most of them did not apply the substitution correctly beyond the simple computation. For example, in survey problem

9, which asks them to use the relationship that $f'(a)$ is a value of $f'(x)$ to make the slope at $x = a$ positive, most students, who specified the derivative as slope, answered incorrectly or changed their choices several times. This result shows that their ability to perform a simple calculation did not always show their thinking about $f'(a)$ as a specific value of $f'(x)$.

C. The Derivative Function as a Function

Many students did not correctly explain $f'(x)$ as a function and what $f'(x)$ represents in relation to the function, f(x). Five students incorrectly stated that $f(x)$ increases/decreases if and only if $f'(x)$ increases/decreases, and thus their graphs go in the same direction. Three of them described "derivative as a tangent line," which can link to their concept of $f'(x)$ involving the tangent line because the graph $f(x)$ and its tangent line at a point go in the same direction locally. This description shows their mixed notion of the derivative as a point-specific value and as a function on an interval. Students' difficulty in appreciating $f'(x)$ as a function was also found when they described the IV of $f'(x)$. Most did not correctly identify the IV; for example, a student stated that the IV of $f'(x)$ is time regardless of what the original function $f(x)$ represents, and another student mentioned that the IV of $f'(x)$ was the rate of change of the IV of $f(x)$.

Student discourse analysis suggests that many students do not conceive of a) $f'(a)$ as a number, a value of $f'(x)$ and b) $f'(x)$ as a function consisting of the derivative as several points. Their difficulty realizing these aspects of the derivative seems to be related to their instructors' discourse since classroom discourse analysis showed that the instructors did not explicitly address the relationship between $f'(x)$ and $f'(a)$, and the derivative function as a function. The next section compares what extent instructors addressed various aspects of the derivative and how students described or used those aspects in problem solving situations.

V. Discussion

The analyses of instructors' and students' discourse suggest some relationships between students' descriptions and uses of various aspects of the derivative in problem solving and how their instructors addressed those aspects in the classroom. This section compares instructors' and students' discourse in a table format to find out the relationship between the nature of classroom discourse (implicit or explicit) and students' performance on survey. Since not all aspects of the derivative in Table III-1 were identified in students' discourses, the tables in this section only include the aspects that were identified in at least three students' discourse. All instructors addressed each of these aspects entirely explicitly or implicitly. The first table shows the aspects that instructors addressed explicitly and students' descriptions or uses on the aspects. The second table includes the same information but about the aspects that instructors addressed implicitly. The first column in Table V-1 shows the aspect that instructors explicitly addressed. The second and third columns show how many students described or used each aspect, and their descriptions or uses, respectively. The suggestions for instructions based on this comparison follow (Table V-3).

1. Comparison between Explicit Instructors' Discourse and Students' Performance

This section addresses the mathematical aspects of the derivative addressed explicitly by instructors, and students' descriptions or uses of the aspects (see Table V-1). The number of students for each aspect does not always add up to 12 because not all the students described or used the aspect, and some of their descriptions or uses of the aspect overlapped.

Note. Students' mathematically incorrect descriptions and uses are italicized.

Instructors explicitly addressed what *f '(a)* and *f '(x)* represent in relation to *f(x)* by stating them as the slope of *f(x)* or describing the behavior of *f(x)* based on their signs. They addressed the relationship between *f(x)* and *f '(x)* with differentiation rules, and the relationship between *f(x)* and *f '(a)* by addressing the property of the derivative function at a point where *f(x)* has extreme values. How *f '(x)* and *f '(a)* are related was explicitly addressed with the substitution method.

Most students correctly described and used these explicit aspects in problem solving, which suggests the students perform better on problems involving the aspects that the instructors addressed explicitly. However, three students incorrectly stated and used, "the derivative as a tangent line," and five students stated or used "*f '(x)* increases/decreases where *f(x)* increases/decreases" or "*f(x)* and *f '(x)* go in the samedirection." Three of the five students used the first statement to support the second because f(x) and its tangent line go in the same direction. These incorrect notions seem related to students' concepts of the derivative as a point-specific value but as a function on an interval simultaneously, and the instructors' implicit discussion on these two aspects and their ambiguous use of the word, "derivative" without specifying its referent. These will be addressed in the following section.

2. Comparison between Implicit Instructors' Discourse and Students' Performance

This section addresses the aspects of the derivative, which were addressed implicitly by three instructors in the classroom, and students' descriptions and uses of the aspects (Table V-2).

Note. Students' mathematically incorrect descrip-

<Table V-1> *Instructors' Explicit Discussions and Students' Descriptions or Uses of Mathematical Aspects of the Derivative with Frequencies*

Mathematical Aspects	Number of Students	Students' Descriptions or Uses
Defining *f '(a)* as Slope of Tangent line	7	*f '(a)* is the same as slope of *f(x)* at $x = a$
	8	Using the slopes of *f(x)* to graph *f '(x)*
	3	*Tangent line is the derivative at a point*
Describing *f(x)* using *f '(a)*	5	The sign of *f '(a)* indicates the behavior of *f(x)* at $x = a$
Using *f '(a)* = *0* at Local Extremes of *f(x)*	9	*f '(a)* = *0* when *f(x)* has an extreme at $x = a$
Discussing Differentiation Rules	12	*f '(x)* can be obtained from *f(x)* by applying the differentiation rule
Describing *f(x)* Based on the sign of *f '(x)*	11	*f '(x)* is negative/positive where *f(x)* decreases/increases
	11	*f '(x)* is the same as the slope of *f(x)*
	5	*f '(x)* *increases/decreases where f(x) increases/decreases*
Substituting $x = a$ in *f '(x)* to Find *f '(a)*	12	*f '(a)* can be found by substituting a in *f '(x)*
	12	*f '(a)* is a *y* value on graph of *f '(x)* at $x = a$

Note. Students' mathematically incorrect descriptions and uses are italicized tions and uses are italicized.

As shown in Table V-2, many students incorrectly described or used the mathematical aspects of the derivative, which are implicitly addressed by their instructors. Comparison between instructors' and students' discourses in those aspects not only shows that students' descriptions and uses are not always mathematically correct while instructors' were correct, but also implies that there are systemic differences between their discourses. I organized these differences around three themes: a) word use; b) explanations of the derivative; and c) the derivative as a function.

A. Word Use

The most prominent difference between instructors' and students' discourse was found in their use of the word, "derivative." Although both used the word without specifying it as "the derivative function" or "the derivative at a point," they used the word differently. Throughout the derivative unit, instructors used the word "derivative" to refer to "the derivative function" in most cases; when they referred to "the derivative at a point," they specified the point. This way of using the word, "derivative," is consistent with the way the terms, "function" and "function at a point," are used. In contrast, students sometimes inconsistently used the word, "derivative." They used it to refer to "the derivative function," or "the derivative at a point," but sometimes changed its meaning frequently within a single interview question to support their incorrect concept such as the derivative as a tangent line or a constant function. This way of using the word, "derivative," appeared to be related to the way students described the derivative as a point-specific value and as a function on an interval, which is addressed in the next section.

B. Point-wise and Across-time Explanations of the Derivative

Monk (1994) defined two types of understandings:

< Table V-2 > Instructors' Implicit Discussions and Students' Descriptions or Uses of Mathematical Aspects of the Derivative with Frequencies

Mathematical Aspects	Number of Students	Students' Descriptions or Uses
Using one Word "derivative" for "derivative function" and "derivative at a point"	4	"Derivative" at a point is a linear or constant function
Interpreting $f'(a)$ as Rate of Change	3	$f'(a)$ is the rate of change of $f(x)$ at $x = a$
	5	The unit of $f'(a)$ is (unit of $f(x)$) / (units of x)
	4	*$f'(a)$ is change in $f(x)$ between $x = a$ and $a + 1$*
	6	*$f'(a)$ and $f(a)$ has the same unit*
Interpreting a Point on Graph of $f'(x)$ as Slope of Tangent Line	7	*$f'(x)$ determines the behavior of $f(x)$*
	7	*The slope of $f'(x)$ is the derivative at a point*
	3	*The derivative is tangent line at the point*
	3	*$f'(a)$ can be gained by substituting a value in the equation of the tangent line.*
	5	*$f'(x)$ is a linear or constant function*
	3	*The graph of $f'(x)$ is (piecewise) linear*
Identifying Independent Variable of $f'(x)$	5	*The independent variables of a function and its derivative function are different.*

Note. Students' mathematically incorrect descriptions and uses are italicized "pointwise understanding" as the value of a function at a point, and "across-time understanding" of a function as a dynamic quantity on an interval. The results of this current study show that students' explanations of the derivative included both types of understanding but not always in a mathematically accepted way. One of the most prominent incorrect notions of "the derivative" was the tangent line at a point. This notion may come from students' descriptions of the derivative at a point, most of which include the tangent line, and the derivative function, which has variable x or is defined on an interval. The use of the word, "derivative" without specifying it as the derivative function or the derivative at a point also seems to play a part in forming this incorrect notion by allowing students to consider "derivative" as one object "pointwise" and "across-time," simultaneously. Similarly, some students considered the derivative at a point as a constant function instead of a number. As shown in Table V-2, the instructors implicitly addressed what $f'(x)$ and $f'(a)$ represents in relation to $f(x)$ they graphed $f'(x)$ using the slopes of tangent lines of $f(x)$ without stating that the slopes are the derivative at several points, and $f'(a)$ is a specific case of $f'(x)$.

C. Derivative Function as Function

Regarding the aspect of the derivative function as a function, some students incorrectly described the derivative as a linear function represented by a tangent line or as a constant function of the derivative at a point. Some of them also stated that the derivative graphs are linear or piecewise linear. Such descriptions show that students have a concept of the derivative as a function but do not correctly conceive of what the derivative represents in relation to the original function. While introducing or using this aspect of the derivative, instructors did not explicitly address what the derivative at a point, $f'(a)$, and the derivative function, $f'(x)$, represent in terms of the original function. As shown in Table V-2, none of them explicitly explained $f'(a)$ as the rate of change of $f(x)$ while solving problems about the related rates, or $f'(x)$ as the slope of $f(x)$ while graphing $f'(x)$.

Additionally, the property of the derivative function as a function can be addressed discussing what the independent variable (IV) that the derivative function defends on. Most students did not correctly identify the IV of $f'(x)$ correctly or recognize that $f(x)$ and $f'(x)$ have the same IV. None of the instructors explicitly addressed these two aspects of the IV of $f'(x)$ the discussion was limited to the procedure of differentiating implicit functions with multiple variables.

The comparison between the nature of the instructors' discourse (implicit and explicit), and students' descriptions and uses of the derivative suggests the positive relationship between explicitness of classroom discussion and students' performance. This comparison also points out the similarities and differences in main feature of their discourses. For example, although both instructors and students use the word, "derivative" without specifying its referent, instructors use the word in a way that it is coherent with the way that the word, "function" is used. However, some students used the word, "derivative" to signify the tangent line, which shows their mixed and incorrect notion of the derivative as a point-specific object and as a function simultaneously. The mathematical aspects that the students have difficulty with suggest that instructors need to address these aspects explicitly in the classroom. Students may benefit from their instructors' explicit discussion which makes how $f'(x)$ and $f'(x)$ are different but related, and what property of the original function *f(x)* these concepts represent. Details suggestions for instruction will be addressed in the following section.

3. Suggestions for Instruction

The following table shows the mathematical aspects that the instructors implicitly discussed in their classrooms and suggestions for explicit discussions on these aspects.

The first mathematical aspect of the derivative in Table V-3, the relationship between $f'(x)$ and $f'(a)$, can be explicitly addressed by explaining that $f'(a)$

<Table V-3> *Suggestions on Cases for Mathematical Aspects of the Derivative*

Aspects	Case	Suggestion
Relationship between $f'(x)$ and $f'(a)$	Making transition of definitions from $f'(a)$ to $f'(x)$	Remind students that a is used as a point and x is used as the variable, and emphasize $f'(a)$ is a number that is a value of $f'(x)$, which is a function.
	Graphing $f'(x$ using several values of $f'(a)$	Remind students that they can graph $f'(x)$ by plotting the derivative at several points. Emphasize that each value $f'(a)$ is a specific value of $f'(x)$, and thus values of the derivative at several points form $f'(x)$.
	Determining differentiability of *f(x)* using the tangent line at a point	Distinguish between the differentiability of a function at a point and on an interval by clarifying where the function is defined, continuous, and differentiable. Justify why $f'(a)$ can be used to determine the differentiability of a function by stating that $f'(a)$ is a value of $f'(x)$
	Deciding sign of $f'(x)$ on interval using sign of $f'(a)$	Explain why the sign of $f'(x)$ stays the same on an interval between two consecutive critical points. Emphasize that $f'(a)$ is one value of $f'(x)$ on the interval.
	Proving differentiation rules using a tangent line	Draw several tangent lines, or use the arm gestures at several points on interval and state that the explanation at one point works at any point on the domain
$f'(x)$ as a function	Defining $f''(x)$ and $f'''(x)$	State that $f'(x)$ has its own derivative by talking about differentiation as an operation on functions. For example, compare the process of deriving $f''(x)$ from $f'(x)$ with the process of deriving $f'(x)$ from *f(x)*
	Applying the theorems for a function to $f'(x)$	Check explicitly that $f'(x)$ is a function that satisfies the conditions of the theorem. Remind the students how to apply the theorem to a function.
	Identifying the independent variable of the derivative function	Identify the variable that the derivative function depends on and stating that the derivative function has the independent variable because it is a function.
	Defining $f'(x)$ as the slope of a tangent line	Distinguish between a tangent line and its slope by stating that $f'(x)$ is a function representing the slope not the tangent line itself. Display the value of the slope as y value of $f'(x)$ using dynamic software.
	Specifying the units of $f'(x)$	Emphasize that $f'(x)$ represents a different quantity than *f(x)* with different units by graphing *f(x)* and $f'(x)$ on separate transparent sheets. To compare the relationship between their behaviors, overlay the transparent sheets. Use the limit definition of $f'(x)$ to addressed units of $f'(x)$

is a specific value of $f'(x)$. Reminding students that the function at a point, $f(a)$, is a value of the function, $f(x)$ would help students see the equivalence between the relationship between $f(x)$ and $f(a)$, and the relationship between $f'(x)$ and $f'(a)$. The equivalence can be addressed by reviewing the notations of the function at a point, $f(a)$, and the function, $f(x)$, and use of the letters a as a constant and x as a variable. Similarly, while graphing $f'(x)$, reminding students about the graphing process of $f(x)$ based on several values of the function would help them recognize the equivalence in the processes of graphing $f(x)$ and $f'(x)$. This equivalency would help students realize that $f'(x)$ consists of the values of the derivative at several points. When the instructors use $f'(a)$ as the representative of $f'(x)$ on an interval in explanations of differentiability and sign diagrams, it would be crucial to state that $f'(a)$ is a specific case of $f'(x)$ in order to address why the method at a specific point works at any point on the interval. When they prove the differentiation rules using the tangent line at a point on an interval, it is important to show that the proof is not point-specific. They can show that the same explanation works on the whole interval by showing the behavior of the tangent line at several points with multiple drawings or gestures.

The second mathematical aspect of the derivative, the derivative function as a function, can also be explicitly addressed by reminding students about the property of a function. Defining $f''(x)$ or $f'''(x)$, and applying theorems for the function to $f'(x)$ are also good places to include instruction that $f'(x)$ is a function on which they can define its own derivative or apply the theorem for functions. Drawing parallels between the treatments of $f(x)$ and $f'(x)$ would help students see both of them as functions. The derivative as a function also can be addressed by emphasizing its dynamics as the independent variable changes and identifying its independent variable not only in the calculation process but also in context of the derivative function as a function. Regarding students' incorrect interpretations about what $f'(x)$ represents in terms of $f(x)$, the instructors may emphasize that each value of $f'(x)$ represents the slope of the tangent line, not the tangent line itself. They can show the process of plotting several points for $f'(x)$ based on the slopes of tangent lines of $f(x)$ at several points with dynamic geometry software. The different units of $f(x)$ and $f'(x)$ can be addressed by graphing $f'(x)$ and $f(x)$ on separate planes and explaining the y axes represent different quantities: a function value, and the rate of change of the function. They also can address the units of $f'(x)$ algebraically with its limit definition. To compare the sign of $f'(x)$ and the behavior of $f(x)$ on the same interval, instructors can graph them on two transparent sheets and overlay them.

The suggestions above, including the correct use of the terms and explicit discussions on the relationships between concepts may not be directly related to the positive impact on students' achievement on the derivative. However, these explicit discussions, which remind students about the properties of a function that they have learned previously, would help students see the equivalence in the concepts of a function and the derivative, and thus help them understand the relationship between $f'(x)$ and $f'(a)$, and $f'(x)$ as a function. These discussions would provide students a better learning opportunity and help them overcome their incorrect notions about the concepts and relationships, and expedite development of their mathematical discourse on the derivative closer to that of mathematicians.

VI. Conclusion

This study contribute to the current field of mathematics education by pointing out a) the importance of word use in relation to students' thinking of the derivative, b) the need for instructors' explicit discussion of certain aspects of the derivative including the exact use of the mathematical terms. The comparison between instructors' and students' discourses on the derivative also provided various suggestions that may contribute to explicit classroom discourses (Table V-3), and thus may lead to improvement in students' discourse on the derivative. However, there are other important topics to address in future studies to deepen our understanding of learning and teaching of the derivative.

Students' thinking on the derivative may also stem from other factors than their classroom exposure on the derivative. It could originate in their previous knowledge about functions. Existing studies shows that the transition from point-wise understanding of the function to the across-time understanding is difficult for student to achieve, and students have difficulties in understanding function notations (Carlson, 1998; Monk, 1994). Regarding these results, it is possible that the students in the calculus class might not yet achieve the across-time understanding of the function, or could not appreciate the relationship and difference between two notations, $f(a)$ and $f(x)$, nor the use of the letter a for a constant and x for a variable.

Since studies have addressed curriculum materials as a key factor of students thinking of mathematical concepts (e.g., Fuson, Stigler, & Bartsch, 1988; Senk & Thompson, 2003, p. x), textbooks may also affect students' thinking on the derivative. It could be explored how textbooks introduce the derivative and use the terms, "derivative," "the derivative function," and "the derivative at a point" in their description, and what extent they addressed the relationships between the concepts. These analyses may provide a good comparison to classroom discourses and reveal possible reasons for students' incorrect descriptions and uses of the derivative.

The instructors' discourse, in which some mathematical aspects of the derivative were implicitly addressed, can also be further investigated in terms of instructors' beliefs about their students' thinking of a function and the derivative. The implicitness of discussions does not seem to come from a lack of the instructors' content knowledge on these aspects because they correctly answered and explained the survey problems involving these aspects during the interview. However, they seemed to assume that some of these aspects are obvious to their students. For example, they said the problems about the relationship between $f'(x)$ and $f'(a)$ were "too simple" to use or only good for "review about a function." Although instructors' beliefs are beyond the scope of this study, they are important to address to explain the nature of their classroom discourse.

The results of this study show that both implicitness in the instructors' discourse, and students' incorrect descriptions and uses were closely related to their ambiguous use of the word, "derivative." As mentioned earlier, in some languages such as Korean, using a word "derivative" for both "the derivative function" and "the derivative at a point" is not possible because the terms for these two concepts are not consistent. How instructors and students explain these concepts in other languages, such as Korean, would provide more thorough information about the role of word use in learning the derivative. This exploration may reveal the affordances and constraints in consistency versus the difference in terms for the related but different mathematical concepts, "the derivative function" and "the derivative at a point."

One of the results of this study showing that students have incorrect notions of the derivative is not new to the field of mathematics education, and their instructors' explanations may not be the only source of their learning. However, looking at students' thinking of the derivative through their discourse with the lens of a communicational approach, revealed how their ambiguous use of word can contribute to forming incorrect notions of the derivative. Using the same lens to analyze the instructors' discourse made it possible to show how instructors and students talk about the same mathematical objects differently. Although there are many possible factors that affect students' learning of the derivative and instructors' teaching practice as listed above, it is important for instructors to acknowledge what aspects of the derivative that students have difficulties with, what kind of incorrect notions of the derivative that they have, and how they form those incorrect notions. With these acknowledgements, instructors could explicitly address important mathematical aspects of the derivative in a way that would help students appreciate coherence between their previous knowledge about functions and their new knowledge about the derivative, and thus prepare them to see the derivative as an operation on functions in more advanced mathematics courses such as differential equations and analysis.

Bibliography

한대희(1998). 미분법 단원에서의 용어의 문제. 대한 수학교육학회논문집 8 (2), 429-507

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. (1997). The development of students' graphical understanding of the derivative, *Journal of Mathematical Behavior*, 16 (4), 399-430

Bingolbali, E., & Monaghan, J. (2008). Concept image revisited, *Educational studies in Mathematics, 68*, 19-35.

Bezuidenhout, J. (1998). First-year university students' understanding of rate of change. International Journal of Mathematics Education in Science and Technology 29(3) 389-399.

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education III. Issues in Mathematics Education*, 7, 115-162.

Carlson, M. P., Oehrtman, M. C., & Thompson, P. W. (2008). Foundational Reasoning Ability that Promote Coherence in Students' Function Understanding. In M. P. Carlson & C. Rasmussen (Eds.), Making the connection: Research and practice in *undergraduate mathematics* (pp. 150-171). Washington, DC: Mathematical Association of America.

Epstein, J. (2006). "The Calculus Concept Inventory," presentation at the 2006 Conference on Research in Undergraduate Mathematics Education, Piscataway, N.J., abstract online at http://cresmet.asu.edu/crume2006/Abstracts.html#Epstein

Ferrini-Mundy, J., & Graham, K. (1994). Research in calculus learning: Understanding of limits, derivatives, and integrals. Research Issues in Undergraduate Mathematics learning; Preliminary Analyses and Results. MAA notes 33, 31-45.

Fuson, K., & Kown, Y., (1992). Korean Children's Understanding Multidigit Addition and Subtraction, *Child development*, 63(2), 491-506.

Fuson, K., Stigler, J., & Bartsch, K . (1988). Grade placement of addition and subtraction topics in Japan, mainland China, the Soviet Union, Taiwan, and the United States. Journal for Research in Mathematics Education, 19, 449-456.

Hahkioniemi, M. (2005). Is there a limit in the derivative? - Exploring students'understanding of the limit of the difference quotient. Retrieved January 20, 2009 from http://cerme4.crm.es/Papers%20definitius/14/Haikionemi.pdf.

Hass, J., Weir, M. D., & Thomas, G. B. (2008) *University Calculus: Elements with Early Transcendentals* Pearson Addison Wesley.

Hauger, G. (1998). *High school and college students' knowledge of rate of change*. Unpublished doctorial dissertation, Michigan State University, Michigan.

Hirst, K. (2002). Classifying students' Mistakes in Calculus. *Proceedings of the 2nd* International conference on the Teaching of Mathematics, Crete, Greece July 2002.

Hurley, J. F., Koehn, U., and Ganter, S. L. (1999) Effects of Calculus Reform: Local and National. *American Mathematical Monthly*, 106(9), 800-811.

Miura, I., Okamoto, Y., Vlahovic-Stetic, V., Kim, C., & Han., J. (1999) Language supports for children's understanding of numerical fractions: cross-national comparison, Journal of Experimental Child Psychology, 74, 356-265

Nemirovsky, R. & Rudin, A. (1992). Students' Tendency to assume resemblances between a function and its Derivatives. *TERC Working Paper*, 2-92, Cambridge, Massachusetts.

Newton, J. (2008). *Discourse analysis as a tool to investigate the relationship between written and enacted curricula: the case of fraction multiplication in a middle school standards-based curriculum*. Unpublished doctorial dissertation, Michigan State University, Michigan.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, 14, 235-250

Park, J. (2008). Students' *understanding of the derivative*. Unpublished doctoral practicum, Michigan State University, East Lansing.

Santos, A. Thomas, M. (2003). Representational ability and understanding of derivative. Proceedings of the 2003 Joint Meeting of PME and PMENA, 2, 325-332.

Senk, S. L., & Thompson, D. R. (2003). Standards-based school mathematics curricula: What are they? What do students learn? Mahwah, NJ: Lawrence Erlbaum

Sfard A. (2008). *Thinking as communication*. Cambridge University Press

Speer, N. M., Smith, J. P., Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior*, 19, 99-114

Tall, D. (1986). Construction the concept image of a tangent; *Proceeding of the eleventh* conference of the International Group for the Psychology of Mathematics Education, Montreal, 3, 69-75.

Thomas, M.O. J. (2002). *Versatile Thinking in Mathematics* In Tall, D. & Thomas, M (Eds.) Intelligence, Learning and Understanding (179-204). Flaxtom, Queensland, Australia: Post Pressed.

Thompson, P. (1994). Images of rate and operational understanding of the fundamental theorem of calculus, *Educational studies in mathematics* 26 (2/3), 229-274.

Weir, M. D., Hass, J., & Giordano, R.F. (2006) *Thomas' Calculus: Early Transcendentals*, Pearson Addison Wesley.

Wu, H. (1997). The mathematics education reform: Why you should be concerned and what you can do. American Mathematical Monthly, 103, 954 - 962. J. F. Hurley, U. Koehn, and S. L. Ganter. Effects of calculus reform: Local and national. American Mathematical Monthly, 106(9):800-811.

Zandieh, M. (2000) A theoretical Framework for analyzing students understanding of the concept of derivative. CBMS *Issues in Mathematics Education*, 8, 103-127.

미적분학 강사와 학생의 미분에 관한 담화

박 정 은 (미시간주립대학교)

미적분학 강사와 학생의 미분에 관한 담화의 특징을 인식에 관한 의사소통적 접근을 통해 조사하였다. 이 연구의 자료는 설문, 수업 관찰, 그리고 인터뷰를 통해 수집되었다. 연구의 결과는 강사들이 도함수와 미분계수의 관계, 함수로써의 도함수 등 미분의 성질들을 명백히 서술함 없이 사용한다는 것과 학생들의 문제 풀이에 있어 이런 성질들을 부정확하게 서술하고 사용한다는 것을 보여준다. 미분에 관한 교사들의 암묵적인 담화와 학생들의 부정확한 설명은 그들이 용어, "미분"을 "미분계수" 혹은 "도함수"로 구분하지 않고 사용한다는 사실과 밀접한 관련이 있는 것으로 밝혀졌다. 강사와 학생의 담화 비교는 분명한 용어 사용을 포함한 미분의 수학적인 성질에 대한 명백히 설명이 학생들이 도함수의 한 값으로의 미분계수를 이해하는 것과 접선과 같은 도함수의 부정확한 개념을 극복하는 데 도움을 줄 수 있음을 암시한다.

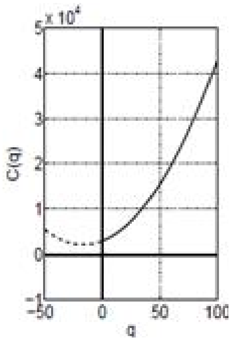
* Key Words : Derivative (미분), Derivative at a point (미분계수), Derivative function (도함수), Communicational Approach to Cognition (인식에 관한 상호 의사소통의 접근방식)

논문접수: 2010. 12. 28.
논문수정: 2011. 1. 21.
심사완료: 2011. 2. 11.

Appendix: Survey Questions

Please solve the following problems and show your work.

1. $C(q)$ is the total cost (in dollars) required to set up a new rope factory and produce q miles of the rope. If the cost satisfies the equation $C(q)=3000+100q+3q^2$, and the graph is given as follows.



(a) Find the value of $C(2)$

(b) What are the units of 2 in (a)?

(c) What are the units of $C(2)$?

(d) What is the meaning of $C(2)$ in the problem context?

(e) Find the value of $C(2)$.

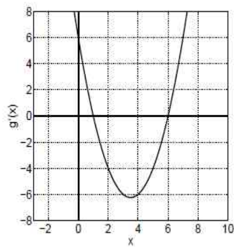
(f) What are the units for 2 in (e)?

(g) What are the units of $C'(2)$?

(h) What is the meaning of $C'(2)$ in the problem context?

2. The derivative of a function f, is given as $f'(x) = x^2 - 7x + 6$. What is the value of $f'(2)$?

3. The graph of the derivative, $g'(x)$ of function g is given as follows. What is the value of $g'(2)$?



a) -4

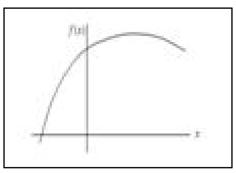
b) -2

c) 0

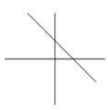
d) 2

e) 4

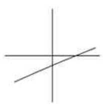
4. Below is the graph of a function $f(x)$, which choice a) to e) could be a graph of the derivative, $f'(x)$?



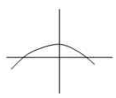
a)



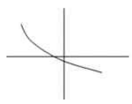
b)



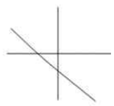
c)



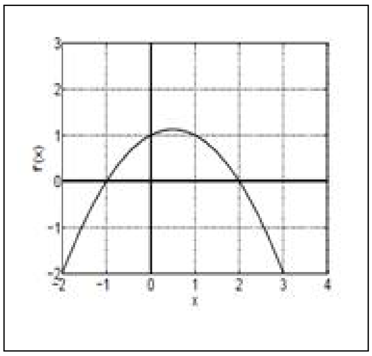
d)



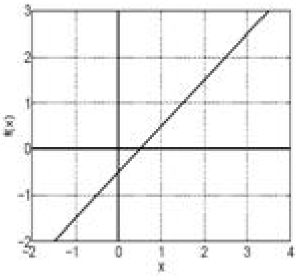
e)



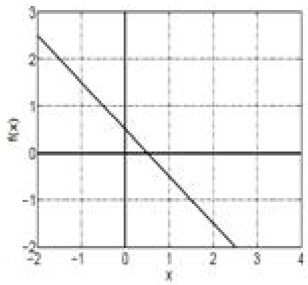
5. Below is the graph of the derivative $f'(x)$ of a function $f(x)$. Which choice a) to e) could be a graph of the function $f(x)$?



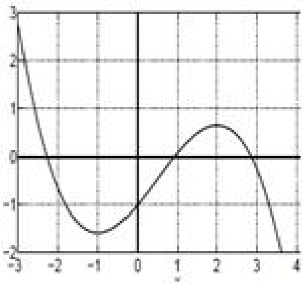
a)



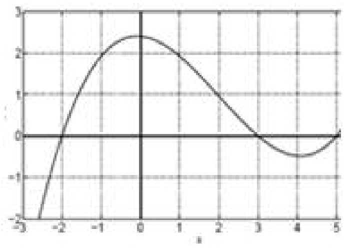
b)



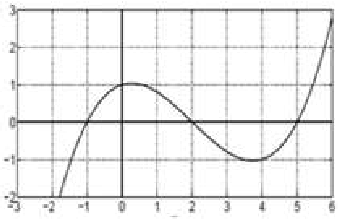
c)



d)



e)



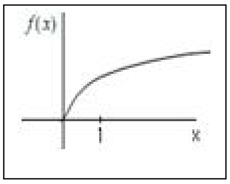
f) None of these

6. If a function is always positive, then what must be true about its derivative function?
   a) The derivative function is always positive.
   b) The derivative function is never negative.
   c) The derivative function is increasing.
   d) The derivative function is decreasing.
   e) You can't conclude anything about the derivative function.

7. The derivative of a function *f(x)* is negative on the interval *x=2 to x =3*. What is true for the function *f(x)*?
   a) The function *f(x)* is positive on this interval.
   b) The function *f(x)* is negative on this interval.
   c) The maximum value of the function *f(x)* over the interval occurs at *x=2*.
   d) The maximum value of the function *f(x)* over the interval occurs at *x=3*.
   e) We cannot tell any of the above.

8. Consider the graph below. The tangent line to this graph of *f(x)* at *x = 1* is given by $y = \frac{1}{2}x + \frac{1}{2}$. Which

of the following statements is true and why?



a) $\frac{1}{2}x+\frac{1}{2}=f(x)$ b) $\frac{1}{2}x+\frac{1}{2}\leq f(x)$ c) $\frac{1}{2}x+\frac{1}{2}\geq f(x)$

d) $\frac{1}{2}x=\frac{1}{2}f(x)$ e) None of these

9. The derivative of a function, *f*, is $f'(x)=ax^2+b$. What is required of the values of a and b so that the slope of the tangent line to the function f will be positive at $x = 0$.

a) a and b must both be positive numbers.

b) a must be positive, while b can be any real number.

c) a can be any real number, while b must be positive.

d) a and b can be any real numbers.

e) None of these

Why?