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Analysis of (K, r) Incomplete Inspection Policy for Minimizing Inspection Cost subject to a Target AOQ

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출하 품질목표 조건하에 검사비용을 최소화하는(K, r) 부분검사정책의 분석

양문희

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In this paper, we address an optimization problem for minimizing the inspection and rework cost in an inspection-rework system, which forms a network of nodes including a K-stage inspection system, storage areas for items, a source inspection shop, and a re-inspection shop. We assume that (n, 0) acceptance sampling is performed in the source inspection shop and that only 100(1-r)% of items of rejected lots are re-inspected in the re-inspection shop. Since all the nodes are interrelated, in order to formulate our steady-state objective function, we make a steady-state network flow analysis between nodes, and derive both the steady-state amount of flows between nodes and the steady-state fraction defectives by solving a nonlinear balance equation. Finally we provide some fundamental properties and an enumeration procedure for determining the optimal values of (K, r) which both minimizes our objective function and attains a given target average outgoing quality.

Keyword: inspection cost, rework cost, incomplete inspection

1. Introduction

Most rectifying inspection plans for lot-by-lot sampling call for 100% inspection of rejected lots, and rejected lots are handed over to consumers immediately after all the nonconforming items in rejected lots are replaced with conforming items. 100% inspection is restricted to rejected lots and this will in most cases be a small volume of inspection. However, if the burden of inspecting rejected lots placed on the producer is not trivial, producers might avoid 100% inspection of rejected lots and could introduce 100 (1-r)% incomplete inspection for rejected lots instead of 100% screening inspection.

A single acceptance sampling plan, usually designated as (n, c), specifies the sample size n that should be taken and the number of defective units c that cannot be exceeded without the lots's being rejected. In BLU (Back-Light Unit) industries, (n, 0) acceptance sampling plan for inspection is widely used and rejected lots cannot be handed over to consumers immediately. Rejected lots must be 100% inspected later, and if some nonconforming items are found, then they should be reworked and pooled together with recalled items. Hence newly produced items, those reworked items in rejected lots, and those items recalled from consumers are pooled together

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and form lots for inspection in a storage area. Since those items with different fraction defectives would be circulated, even estimating the fraction defectives of items in different storage areas is not easy. Similarly, measuring the effect of (n, 0) plan on the prior and posterior processes throughout the factory is not easy even though theoretical results of (n, 0) plans are simple.

Considering a BLU factory as a network of processes, Yang and Kim (2009) provided a balance nonlinear equation for estimating the fraction defectives in nodes and the sizes of flows between nodes assuming that the system is in steady state. They formulated the cost objective function including the number of items inspected and reworked, and provided some fundamental properties and an enumeration method for determining an optimal value of K which minimizes their objective function. Yang and Kim (2009) also had the same results using a different approach. Those papers assumed, however, that the number of nonconforming items in a lot, which were formed for inspection in storage area, was not a variable but a constant. The "constant" assumption might not be realistic since the fraction defectives of different lots will be different. Assuming that the number of nonconforming items in a lot followed a binomial distribution, Yang (2010) addressed the same problem but had slightly different results.

There have been no papers except Korea, which considered a factory as a network of inspection processes. For reader's reference, we summarize some papers slightly related with K-stage inspection system as below. Raz and Thomas (1982) presented a branch-and-bound method for determining an optimum sequencing inspection plan for a group of inspectors operating at different skill and cost levels. Production and inspection costs for both accepted and rejected items were considered, and dependencies among successive inspections were permitted. Jaraiedi *et al.* (1987) presented a model which could be used to determine the average outgoing quality for a product which had multiple quality characteristics and which was subject to multiple 100% inspections where the inspection was subject to both type I and type II inspection errors. Assuming a fixed sequence of unreliable inspection operations with known costs and inspection error probabilities of two types, Avinadav and Raz (2003) developed a model for selecting the set of inspections that should be activated in order to minimize expected total costs (inspection and penalties), and provided an efficient branch-and-bound algorithm for finding the optimal solution.

Practically speaking, 100% re-inspection for rejected lots would not be likely to be observed in factories and the "100% re-inspection" assumption made by previous papers might not be realistic. In this paper assuming that the number of nonconforming items in a lot follows a binomial distribution, and that 100(1-r)% of rejected lots are inspected, we address an optimization problem and provide some properties and a procedure for determining the optimal values of (K, r) which both minimizes our objective function and satisfies a given target AOQ (Average outgoing quality). In Section 2, for reader's convenience we describe briefly the problem. In Section 3, we make the steady-state flow and cost analysis in order to formulate our cost objective function, and provide an enumeration method for determining the optimal values of (K, r) which minimizes the objective function and satisfies a given target AOQ. Finally, in Section 4, we give our computational results and remarks.

2. Problem Statement

Yang (2007) suggested a K-stage inspection system consisting of K stages, each of which includes an inspection process and a rework process as shown in <Figure 1>. In the first stage, if an item coming off from production lines is classified as conforming, then it is sent to a storage area called as Node 2. Otherwise, it is sent to the first rework process. After reworked, it is sent to the second stage and the same processes are performed. At the last K-th stage, an item classified as conforming is sent to Node 2 but an item classified as nonconforming is reworked and sent immediately to Node 2 without inspection. Assuming that inspectors are perfect in the sense that both type I error and type II error are zeros and using his result, we can express the average fraction defective of items stored at Node 2 as

$$p_K = p_0 p_R^K \tag{1}$$

where p_0= the average fraction defective of items produced from production lines, p_R= the average fraction defective of items reworked. Throughout this paper, we assume that $0 < p_0, p_R < 1$. It follows that $0 < p_K < 1$.

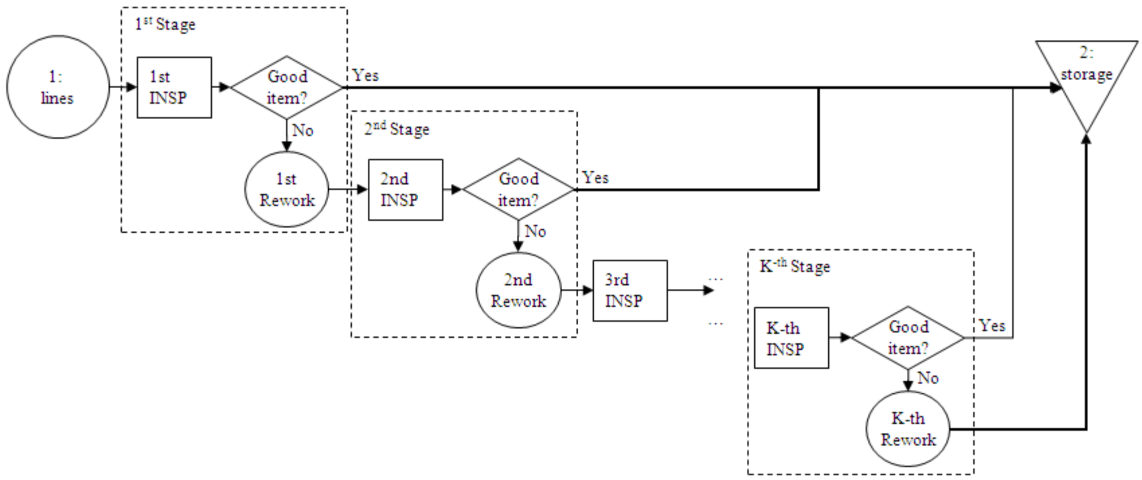

Figure 1. A Conceptual Process Diagram of the K-stage Inspection System

As shown in <Figure 2>, the inspection system consists of the production lines, the K-stage inspection system, the source inspection shop, and finally the re-inspection shop. Items stored in Node 2 in the K-stage inspection system are packed into lots and transferred to Node 3 in the source inspection shop, where they are stored. If demands arrive, the source inspector starts to inspect lots. If all the samples drawn from a lot are judged as conforming by the source inspector, the accepted lot is accumulated and transported just in time to the consumer's production lines. Otherwise, rejected lots are transferred to Node 5 in the re-inspection shop where 100(1-r)% of the items are re-inspected again where $0 \le r \le 1$. If those items inspected in Node 5 are classified as conforming by inspectors, they are transferred to Node 3, the storage area in the source inspection shop. If not, they are sent to Node 6 and reworked in Node 7, located in the re-inspection shop. The nonconforming items returned from Node 9 are also reworked together with the items sent from Node 5.

We may consider various different types of costs. In this paper, we consider only both the cost of inspecting items and the cost of reworking them. The ultimate loss from passing a defective is assumed to be equivalent to the cost of reworking. Now, define NIN (K, r) to be the total number of items inspected at the K-stage inspection system, Node 4, and Node 5, and define NRW (K, r) to be the total number of items reworked at the K-stage inspection system and Node 7 in the long run respectively. Assume that inspection costs per item occurring at different nodes are same and that rework costs per item occurring at different nodes are same too. Define κ to be the ratio of inspection cost per unit to rework cost per unit. Then given K stages and r, the total relevant inspection plus rework cost incurred throughout the factory, denoted by TC (K, r), can be expressed as NRW (K, r)+κNIN (K, r). It may be conjectured that if K increases, the cost incurred at the K-stage inspection system increases while the cost incurred at Node 4, Node 5, and Node 7 decreases. Otherwise, reverse phenomenon will happen. Hence given values of (K, r), it can be expected that there exists an

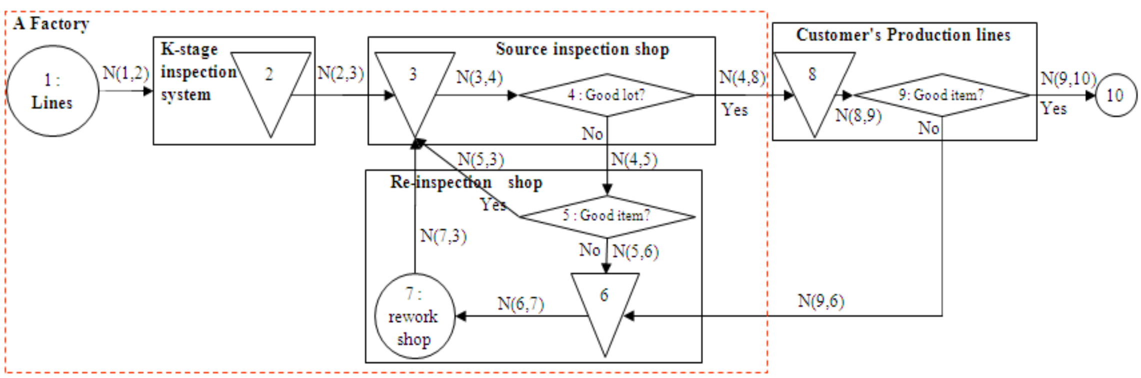

Figure 2. A Conceptual Process Diagram of an Inspection System

optimal value of K minimizing TC (K, r), and our problem can be stated as follows; Find the optimal values of (K, r), denoted by (K^*, r^*), so that we minimize TC (K, r) subject to AOQ (K, r)$\leq p_G$ where AOQ (K, r) is average outgoing quality given (K, r) and p_G is a given target AOQ.

3. Flow and Cost Analysis

In this section, making several reasonable assumptions we will make a steady-state flow analysis and derive a steady-state nonlinear balance equation in order to formulate TC (K, r), and provide a procedure for determining the optimal values of (K, r) as well as fundamental properties.

3.1 Flow Analysis

In order to facilitate our flow analysis, we define the following notations;

$NG(i, j)$ = the steady-state number of conforming items sent from Node i to Node j

$NB(i, j)$ = the steady-state number of nonconforming items sent from Node i to Node j

$N(i, j) = NG(i, j) + NB(i, j)$

$NG(i) = NG(i, i)$ = the steady-state number of conforming items in Node i

$NB(i) = NB(i, i)$ = the steady-state number of nonconforming items in Node i

$N(i) = N(i, i) = NG(i) + NB(i)$

$p(i)$ = the steady-state fraction defective of items in Node i

Consider Node 8. We assume that as soon as the items with the flow size of N (4, 8) are moved to Node 8, they are temporarily stored in Node 8, and are inspected and loaded into consumer's production lines. In other words, we assume that there are actually no stored items in Node 8, and we represent the series of these activities as N (4, 8) = N (8) = N (8, 9). Assume that if items are classified as conforming by the inspectors in Node 9, they are sent to Node 10. Otherwise they are sent to Node 6. Suppose that the number of items required for the consumer's production lines, N (9, 10), is Q items per day. Since the fraction defective of items stored in Node 8 is p (8), in the long run we have

$$N(4, 8) = N(8) = N(8, 9) = \frac{Q}{1-p(8)} \tag{2}$$

$$N(9, 6) = \frac{p(8)Q}{1-p(8)} \tag{3}$$

Consider Node 4. Assume that all the items available in Node 3 are packed into lots and are immediately sent to Node 4 for inspection, i.e., N (3) = N (3, 4). Assume that (i) each lot has exactly N items, which are taken randomly from the population of size N (3), (ii) the probability that a nonconforming item is taken from the population is same and a constant denoted by p (3) or p, that is, if we let X be the number of nonconforming items in a lot, then X follows the binomial distribution, B (N, p), (iii) a lot is accepted only if all the sampled n ($0 < n \leq N$) items per lot are judged as conforming by the source inspectors in Node 4, (iv) rejected lots must be sent to Node 5. Let L (p) be the probability of accepting a lot. Then, from the results of Yang (2010), we have

$$L(p) = (1-p)^n$$
$$p(8) = (1-\lambda)p$$
$$p(5) = (1+\delta)p$$

where $\lambda = n/N$ and $\delta = \lambda L(p)/\{1-L(p)\}$.

Assume that N (3, 4) is very large enough to satisfy that $\lfloor N(3, 4)/N \rfloor N \approx N(3, 4)$ where $\lfloor x \rfloor$ represents the greatest integer less than or equal to x. Since given $N(3, 4)$ the average number of lots accepted by the source inspectors will be $\lfloor N(3, 4)/N \rfloor L(p)$, we have

$$N(4, 8) = \left\lfloor \frac{N(3, 4)}{N} \right\rfloor L(p)N \approx L(p)N(3, 4)$$

$$N(3, 4) \approx \frac{N(4, 8)}{L(p)} = \frac{Q}{L(p)\{1-p(8)\}} \tag{4}$$

$$N(4, 5) = \{1-L(p)\}N(3, 4) \approx \frac{\{1-L(p)\}Q}{L(p)\{1-p(8)\}} \tag{5}$$

Since Q is sufficiently large, and the difference between two values computed by approximation and equal signs respectively is usually within an allowed tolerance, we will use an equal sign instead of an approximation sign from now on. Furthermore we assume that the numbers used in this paper are real.

Consider Node 5. Suppose that 100r% of items transferred from Node 4 to Node 5 are passed to Node 3 without inspection at Node 5 and that only 100(1-r)% of N (4, 5) items are inspected and classified at Node 5. This "partial screening" or "incomplete" inspection strategy has been practically used for controlling AOQ and minimizing the number of items inspected in BLU industries. Then using Equation (5), we have,

$$N(5,\,6) = p(5)(1-r)N(4,\,5) = \frac{(1-r)p\{1-(1-\lambda)L(p)\}Q}{L(p)\{1-(1-\lambda)p\}} \tag{6}$$

$$N(5,\,3) = rN(4,\,5) + \{1-p(5)\}(1-r)N(4,\,5) = \frac{[1-p(1-r)\{1-L(p)\}]-p(1-r)\lambda L(p)}{L(p)\{1-(1-\lambda)p\}}Q \tag{7}$$

$$NG(5,3) = \{1-p(5)\}rN(4,5) + \{1-p(5)\}(1-r)N(4,\,5) = \frac{\{1-L(p)\}Q}{L(p)}$$

$$NB(5,\,3) = p(5)rN(4,\,5) = \frac{rp\{1-(1-\lambda)L(p)\}Q}{L(p)\{1-(1-\lambda)p\}}$$

Consider Node 6. Since all the items corresponding to N (5, 6) and N (9, 6) are nonconforming and reworked in Node 7, using Equation (3) and Equation (6), we have

$$N(6,\,7) = N(5,\,6) + N(9,\,6) = \frac{(1-r)p + r(1-\lambda)pL(p)}{L(p)\{1-(1-\lambda)p\}}Q \tag{8}$$

$N(6,\,7) = N(7) = N(7,\,3)$,

$NG(7,\,3) = (1-p_R)N(6,\,7)$, and

$NB(7,\,3) = p_R N(6,\,7)$

Consider Node 3. Since the steady-state amount of the flow into Node 3 must be equal to the steady-state amount of the flow out of Node 3, we have,

$N(2,\,3) = N(3,\,4) - N(5,\,3) - N(7,\,3) = Q$

$NG(2,\,3) = (1-p_K)Q$ and $NB(2,\,3) = p_K Q$

$$N(3) = N(3,\,4) = \frac{Q}{L(p)\{1-(1-\lambda)p\}} \tag{9}$$

$$NB(3) = NB(2,\,3) + NB(5,\,3) + NB(7,\,3) = \frac{L(p)\{p_K-(1-\lambda)pa_K\} + \{p_R + r(1-p_R)\}p}{L(p)\{1-(1-\lambda)p\}}Q$$

where $a_K = p_K + r(1-p_R)$. Suppose that we form lots in Node 3 after randomly mixing those items transferred separately from Node 5 and Node 2. Now p can be expressed as

$$p = \frac{NB(3)}{N(3)}$$

$$= \{p_K - (1-\lambda)a_K p\}(1-p)^n + \{p_R + r(1-p_R)\}p$$

which can be further reduced to the steady-state nonlinear equation of p,

$$\{p_K - (1-\lambda)a_K p\}(1-p)^n - (1-r)(1-p_R)p = 0 \tag{10}$$

Define p_E to be the steady-state fraction defective of items available in Node 3, which satisfies Equation (10). We will prove that Equation (10) has one and only one root even though it is an (n+1)th-order polynomial equation, and that there exists one and only one steady-state value of N_E corresponding to N (3). Before proving this property, we need the following property. For convenience let $b_K = np_K + (1-r)a_K$ and $\eta = (1-\lambda)r(1-p_R)$.

Property 1 : If $0 < p_R, p_0, \lambda < 1$, then for 0<x<1,

$$g_n(x) = (1-x)^{n-1}\{(1-\lambda)(n+1)a_K x - b_K\} - (1-r)(1-p_R)$$

has the following properties;

(i) when n=1, if $x_1 < 1$, then $g_1(x) < 0$ for $0 < x < x_1$ and $g_1(x) \ge 0$ for $x_1 \le x < 1$, if $x_1 \ge 1$, then $g_1(x) < 0$ for $0 < x < 1$ where $x_1 = \dfrac{(2-\lambda)p_K + \eta + (1-r)(1-p_R)}{2(1-\lambda)a_K}$.

(ii) when $n \ge 2$, $g_n(x) < 0$ for $0 < x < 1$.

Proof : When n=1, $g_1(x) = 2(1-\lambda)a_K(x-x_1)$ where $x_1 > 1$. Since $a_K > 0$, Property1-(i) holds clearly. When $n \ge 2$, the first order derivative of $g_n(x)$ can be derived as follows;

$$g_n{}'(x) = -(1-\lambda)n(n+1)a_K(1-x)^{n-2}(x-x_2)$$

where $x_2 = \dfrac{(n-2\lambda+1)p_K + 2\eta}{(n+1)(1-\lambda)a_K} > 0$. Suppose that $x_2 \ge 1$. Then $g_n{}'(x)$ is positive and $g_n(x)$ is a strictly increasing function of x in (0, 1). Since $g_n(0) = -[\{np_K + (1-\lambda)a_K\} + (1-r)(1-p_R)] < 0$ and $g_n(1) = -(1-r)(1-p_R) < 0$, $g_n(x)$ is negative in (0, 1). Suppose that $x_2 < 1$. Then $g_n{}'(x)$ is positive in $(0,\,x_2)$ and nonpositive in $[x_2, 1)$. Thus, $g_n(x)$ is strictly increasing in $(0, x_2)$ and strictly decreasing in $[x_2, 1)$. Since $g_n(x_2) = -\{(1-x_2)^{n-1}(\lambda p_K + \eta) + (1-r)(1-p_R)\} < 0$, $g_n(x)$ is negative in (0, 1) and Property 1-(ii) holds. □

Property 2 : If $0 < p_R, p_0, \lambda < 1$, then for positive integer n, there exists one and only one value p_E such that

(i) p_E is the solution of the following equation;

$$f(x) = \{p_K - (1-\lambda)a_K x\}(1-x)^n - (1-r)(1-p_R)x = 0, \text{ and}$$

(ii) $N_E = \dfrac{Q}{\{1-(1-\lambda)p_E\}(1-p_E)^n}$.

Proof : When n = 1, the first order derivatives of $f(x)$ can be derived as $f'(x) = 2(1-\lambda)a_K(x-x_1)$. From Property1-(i), if $x_1 \geq 1$, then $f'(x)$ is negative, and $f(x)$ is a strictly decreasing function of x in (0, 1) and since $f(0) > 0$ and $f(1) < 0$, Property 2-(i) holds. From Property 1-(i), if $x_1 < 1$, then $f(x)$ is strictly decreasing in $(0, x_1)$ and increasing in $[x_1, 1)$. Since $f(0) = p_K > 0$ and $f(1) = -(1-p_R)(1-r) < 0$, Property2-(i) holds. When $n \geq 2$, the first order derivative of $f(x)$ can be derived as follows; $f'(x) = (1-x)^{n-1}\{(1-\lambda)(n+1)a_K x - b_K\} - (1-r)(1-p_R) = g_n(x)$. From Property 1-(ii), $f'(x)$ is negative and $f(x)$ is strictly decreasing function of x in (0, 1). Since $f(0) = p_K > 0$ and $f(1) = -(1-p_R)(1-r) < 0$, Property 1-(i) holds. Using Equation (9) and Property 1-(i), Property 2-(ii) holds. □

From Property 2, we can compute the steady-state value of N (i, j) for all i and j, and the steady-state value of the AOQ corresponding to p (8). In addition, it can be observed that p_E and N_E are functions of (K, N, n, p_0, p_R, r). In order to represent explicitly that both N_E and p_E depend upon (K, r), we change those notations to $N_E(K, r)$ and $p_E(K, r)$ respectively for a nonnegative integer K and $0 \leq r \leq 1$. We have the following basic properties.

Property 3 : If $0 < p_R, p_K, \lambda < 1$, then

(i) both $p_E(K, r)$ and $N_E(K, r)$ are strictly decreasing functions of K respectively except if n = 1, $x_1 < 1$, and $x_1 < p_E(K, r)$.

(ii) $\lim_{K \to \infty} p_E(K, r) = 0$,

(iii) $\lim_{K \to \infty} N_E(K, r) = Q$.

Proof : It is enough to prove that if n = 1, $x_1 < 1$, and $x_1 < p_E(K, r)$, then both $p_E(K, r)$ and $N_E(K, r)$ are increasing functions of K respectively, and that if n = 1, $x_1 < 1$, and $p_E(K, r) < x_1$ or if n = 1 and $x_1 \geq 1$ or if $n \geq 2$, then both $p_E(K, r)$ and $N_E(K, r)$ are strictly decreasing functions of K respectively. From Property 2, we have

$$\frac{\partial p_E(K,r)}{\partial K} = -\frac{\{1-(1-\lambda)p_E(K, r)\}\{1-p_E(K, r)\}^n}{g_n(p_E(K, r))}\frac{\partial p_K}{\partial K}$$

$$\frac{\partial N_E(K, r)}{\partial K} = \frac{[(1-\lambda)\{1-p_E(K, r)\} + \{1-(1-\lambda)p_E(K, r)\}n]\,Q}{\{1-(1-\lambda)p_E(K, r)\}^2\{1-p_E(K, r)\}^{n+1}} \times \frac{\partial p_E(K, r)}{\partial K}$$

Since $\partial p_K / \partial K < 0$, from Property 1, Property 3-(i) holds. Since p_K converges to zero as K goes to infinity, Property 3-(ii) holds from Property 2-(i). It follows that Property 3-(iii) holds from Property 2-(ii). □

Note that in exceptional case, theoretically both $p_E(K, r)$ and $N_E(K, r)$ could be increasing functions of K respectively. However, in real situations, n usually exceeds one item and $p_E(K, r)$ is so small that it may not exceed x_1 when $x_1 < 1$. Hence, it can be said practically that both $p_E(K, r)$ and $N_E(K, r)$ are strictly decreasing functions of K respectively. In a similar manner, it may be said practically that both $p_E(K, r)$ and $N_E(K, r)$ are strictly increasing functions of r respectively, as proved in the following property.

Property 4 : If $0 < p_R, p_K, \lambda < 1$, then

(i) both $p_E(K, r)$ and $N_E(K, r)$ are strictly increasing functions of r respectively except if n = 1, $x_1 < 1$, and $x_1 \leq p_E(K, r)$,

(ii) $p_E(K, 1) = \dfrac{p_K}{(1-\lambda)(1+p_K-p_R)}$,

(iii) $N_E(K, 1) = \dfrac{Q}{(1-c_K)\{1-(1-\lambda)^{-1}c_K\}^n}$

where $c_K = p_K(1+p_K-p_R)^{-1}$.

Proof : It is enough to prove that if n = 1, $x_1 < 1$, and $x_1 \leq p_E(K, r)$, then both $p_E(K, r)$ and $N_E(K, r)$ are decreasing functions of r respectively, and that if n = 1, $x_1 < 1$, and $p_E(K, r) < x_1$, or if n = 1 and $x_1 \geq 1$, or if $n \geq 2$, then both $p_E(K, r)$ and $N_E(K, r)$ are strictly increasing functions of r respectively. From Property 2, we have

$$\frac{\partial p_E(K,r)}{\partial r} = -\frac{(1-p_R)p_E(K,r)\left[1-(1-\lambda)\{1-p_E(K,r)\}^n\right]}{g_n(p_E(K,r))}$$

$$\frac{\partial N_E(K,r)}{\partial r} = \frac{\left[(1-\lambda)\{1-p_E(K,\ r)\}+\{1-(1-\lambda)p_E(K,\ r)\}n\right]Q}{\{1-(1-\lambda)p_E(K,\ r)\}^2\{1-p_E(K,\ r)\}^{n+1}} \times \frac{\partial p_E(K,\ r)}{\partial r}$$

Hence, from Property 1, Property 4-(i) holds. If r = 1, from Property 2, Property 4-(ii) and (iii) holds. □

From properties mentioned above, it is not clear whether $p_E(K,\ r)$ is always less than p_K or not. If our inspection system gives $p_E(K,\ r)$ greater than p_K , then there is no reason for the rework shop corresponding to Node 7 to exist. In fact, the following property gives the necessary and sufficient condition for its existence.

Property 5 : $p_E(K,\ r) < p_K$ if and only if $p_R < p_{RUB}(K,\ r)$, $0 < p_R, p_0 < 1$, and $n \geq 2$ where

$$p_{RUB}(K,r) = 1 - \frac{\{1-(1-\lambda)p_K\}(1-p_K)^n}{(1-\lambda)r(1-p_K)^n - (1-r)}.$$

Proof : Since $f(x)$ is a strictly decreasing convex function of x and $f(p_E(K,\ r)) = 0$, $f(p_K)$ is negative if and only if $p_E(K,\ r) < p_K$. Thus solving $f(p_K) < 0$ gives the necessary and sufficient condition. □

3.2 Cost Analysis

Consider the total number of items reworked. Define $NRW_1(K,\ r)$ and $NRW_2(K,\ r)$ to be the number of items reworked at the K-stage inspection system and the number of items reworked at the re-inspection shop respectively. Define NRW (K, r) to be the sum of $NRW_1(K,\ r)$ and $NRW_2(K,\ r)$. Then the following property indicates that NRW (K, r) is invariant irrespective of the values of (K, r). This important result is explicitly the same as that of Yang (2009) even though we make the different assumptions.

Property 6 : NRW (K, r) = $\dfrac{p_0 Q}{1-p_R}$.

Proof : We can express $NRW_1(K,\ r)$ and $NRW_2(K,\ r)$ respectively as

$$NRW_1(K,\ r) = \frac{(1-p_R^K)p_0 Q}{1-p_R} \quad \text{(from Yang (2007))}$$

$$NRW_2(K,\ r) = N(6,\ 7)$$
$$= \frac{(1-r)p_E(K,\ r) + r(1-\lambda)p_E(K,\ r)L(p_E(K,\ r))}{L(p_E(K,\ r))\{1-(1-\lambda)p_E(K,\ r)\}}Q$$
$$= \frac{p_K Q}{1-p_R} \quad \text{(from Equation (8) and Property 2-(i))}$$

Hence NRW (K, r) = $\dfrac{p_0 Q}{1-p_R}$. □

Since NRW (K, r) is constant irrespective of the values of (K, r), we can exclude the rework cost, and our total inspection plus rework cost, TC (K, r), is now redefined as the only inspection costs incurred at three shops; the K-stage inspection system, the source inspection shop, and the re-inspection shop. Utilizing the results of Yang (2007) again, the number of items inspected at the K-stage inspection system, denoted by $N_1(K,\ r)$, can be expressed as

$$N_1(K,\ r) = 0 \quad \text{if } K = 0$$
$$= \left\{1 + \frac{(1-p_R^{K-1})p_0}{1-p_R}\right\}Q \quad \text{if } K \geq 1 \qquad (11)$$

Note that even though $N_1(K,\ r)$ is not a function of r, we insert r for convenience. Assume that the source inspector must examine all of the n samples per lot even though he may happen to find a defective item and reject the lot without inspecting the remaining samples. Then, using Equation (4) and Property 2-(ii), the number of items inspected in Node 4, denoted by $N_2(K,\ r)$, can be expressed as

$$N_2(K,\ r) = n\left\lfloor \frac{N(3,\ 4)}{N} \right\rfloor = \lambda N_E(K) \qquad (12)$$

From Equation (4) and Equation (5), the number of items inspected in Node 5, denoted by $N_3(K,\ r)$, can be expressed as,

$$N_3(K,\ r) = (1-r)N(4,\ 5) \qquad (13)$$
$$= (1-r)\left[1-\{1-p_E(K,\ r)\}^n\right]N_E(K,\ r)$$

Hence, we can express the total relevant inspection cost as TC (K, r)= $\sum_{i=1}^{3} N_i(K,\ r)$. It is not easy to figure out the shape of TC (K, r). However, the following property might be useful to sketch and explain an approximated shape of TC (K, r).

Property 7 : If $0 < p_0,\ p_R < 1$, then we have,
(i) $N_1(K,\ r)$ is a strictly increasing concave function of K.
(ii) $\lim_{K\to\infty} N_1(K,\ r) = \left(1+\frac{p_0}{1-p_R}\right)Q$.
(iii) If $n \geq 2$, $N_2(K,\ r)$ is a strictly decreasing function of K.
(iv) $\lim_{K\to\infty} N_2(K,\ r) = \lambda Q$.
(v) If $n \geq 2$, $N_3(K,\ r)$ is a strictly decreasing function of K.
(vi) $\lim_{K\to\infty} N_3(K,\ r) = 0$.

Proof : For K = 0 and 1, Property 7-(i) is clearly holds since $N_1(0,\ r) = 0$ and $N_1(1,\ r) = Q$ from Equation (11). Since $\ln p_R < 0$, we have,

$$\frac{\partial N_1(K,\ r)}{\partial K} = -\frac{p_0 p_R^{K-1}(\ln p_R)Q}{1-p_R} > 0, \text{ and}$$
$$\frac{\partial^2 N_1(K,\ r)}{\partial K^2} = -\frac{p_0 p_R^{K-1}(\ln p_R)^2 Q}{1-p_R} < 0\ .$$

Hence, $N_1(K,\ r)$ is a strictly increasing concave function of K. Since $\lim_{K\to\infty} p_R^{K-1} = 0$, Property7-(ii) holds. Taking the first derivative of $N_2(K,\ r)$, from Property 3-(i) we have

$$\frac{\partial N_2(K,\ r)}{\partial K} = \lambda \frac{\partial N_E(K,\ r)}{\partial K} < 0$$

if $n \geq 2$. Hence, Property 7-(iii) holds. From Property 3-(iii), Property 7-(iv) holds. Taking the first order derivative of $N_3(K,\ r)$, we have

$$\begin{aligned}\frac{\partial N_3(K,\ r)}{\partial K} &= (1-r)n\{1-p_E(K,\ r)\}^{n-1} \\ &\quad \times N_E(K,\ r)\frac{\partial p_E(K,\ r)}{\partial K} \\ &\quad + (1-r)\left[1-\{1-p_E(K,\ r)\}^n\right]\frac{\partial N_E(K,\ r)}{\partial K} < 0\end{aligned}$$

if $n \geq 2$. Hence, Property 7-(v) holds. From Property 3-(ii), Property 7-(vi) holds. □

3.3 An algorithm for determining $(K^*,\ r^*)$

Let $J_{\max}$ to be $\lfloor r_{inc}^{-1} \rfloor$ and $0 < r_{inc} < 1$. For K = 0, 1, $\cdots$, $K_{\max}$ and J = 0, 1, $\cdots$, $J_{\max}$, define PE (K, J), NE (K, J), TIC (K, J), and AOQ (K, J) to be arrays which correspond to those values of $p_E(K,\ J\times r_{inc})$, $N_E(K,\ J\times r_{inc})$, $TC(K,\ J\times r_{inc})$, and $AOQ(K,\ J\times r_{inc})$ respectively. Since the candidate value of K is limited, an enumeration method for determining (K, r) may work well. Hence we suggest the following procedure; For appropriate values of $(K_{\max},\ r_{inc})$,

Step 1 : Enumeration Phase for computing TIC (K, J) and AOQ (K, J)
  $J_{\max} \leftarrow \lfloor r_{inc}^{-1} \rfloor$
  For J = 1 to $J_{\max}$
   Begin
    $r \leftarrow J \times r_{inc}$
    For K = 0 to $K_{\max}$
    Begin
     Find a solution of the equation in Equation (10) and let PE (K, J) be the solution.
     Compute NE (K, J)
     $= \dfrac{p_K Q}{(1-p_R)\times PE(K,J)}$, TIC(K, J),
     and AOQ (K, J)
    End
   End
Step 2 : Search Phase for determining optimal values of (K, r)
  Find $(K^*,\ r^*)$ which minimizes TIC (K, J) subject to AOQ (K, J) $\leq p_G$.

4. Computational results

Using the same input values of $(Q,\ p_0,\ p_R,\ N,\ n)$ estimated by Yang and Kim (2009) as (4,800 units/day, 16.1%, 5.0%, 240 units, 16 units) and setting r_{inc} = 20%, we obtained the values of TC (K, r), $p_E(K,\ r)$, $N_E(K,\ r)$, and AOQ (K, r) for K = 0, 1, $\cdots$, 6 and J = 0, 1, $\cdots$, 6 as shown in Table 1. Since $n \geq 2$, it can be observed that as K increases, both $p_E(K,\ r)$ and $N_E(K,\ r)$ decrease and converge to zero and 4,800 units respectively as proved in Property 3, and that as r increases, both $p_E(K,\ r)$ and $N_E(K,\ r)$ increase respectively as proved in Property 4. It can be also observed that TC (K, r) is a decreasing function of r except when K = 0, and that if p_G is given as 7,500 PPM, the optimal values of (K, r) are turned out to be (1,40%). If we would obtain more accurate values of r, we could give smaller value of r_{inc}.

When r = 40%, as shown in <Table 2>, $N_1(K, 40\%)$ increases up to 5,613 units. $N_2(K, 40\%)$ and $N_3(K, 40\%)$ decrease up to 320 units and zero respectively as proved in Property 7. Note that $NRW(K, 40\%)$ is computed as 813 units, which is invariant irrespective of (K, r). It can be observed that

Table 1. Computational Results of TC (K, r), $p_E(K, r)$, $N_E(K, r)$, and AOQ (K, r) given that (Q, p_0, p_R, N, n) = (4,800 units/day, 16.1%, 5.0%, 240 units, 16 units) and r_{inc} = 20%

r \ K		0	1	2	3	4	5	6
0.0	TC(K, r)	9443	**5779**	5928	5933	5933	5933	5933
	$N_E(K, r)$	13619	5449	4834	4802	4800	4800	4800
	$p_E(K, r)$	5.9733%	0.7464%	0.0421%	0.0021%	0.0001%	0.0000%	0.0000%
	AOQ(K, r)	5.5750%	0.6966%	0.0393%	0.0020%	0.0001%	0.0000%	0.0000%
0.2	TC(K, r)	8658	**5674**	5921	5933	5933	5933	5933
	$N_E(K, r)$	14703	5472	4835	4802	4800	4800	4800
	$p_E(K, r)$	6.3977%	0.7704%	0.0427%	0.0021%	0.0001%	0.0000%	0.0000%
	AOQ(K, r)	5.9712%	0.7190%	0.0398%	0.0020%	0.0001%	0.0000%	0.0000%
0.4	TC(K, r)	7776	5563	5915	5933	5933	5933	5933
	$N_E(K, r)$	16284	5497	4835	4802	4800	4800	4800
	$p_E(K, r)$	6.9607%	0.7970%	0.0433%	0.0022%	0.0001%	0.0000%	0.0000%
	AOQ(K, r)	6.4966%	0.7439%	0.0404%	0.0020%	0.0001%	0.0000%	0.0000%
0.6	TC(K, r)	6751	5443	5909	5932	5933	5933	5933
	$N_E(K, r)$	18904	5525	4836	4802	4800	4800	4800
	$p_E(K, r)$	7.7771%	0.8269%	0.0440%	0.0022%	0.0001%	0.0000%	0.0000%
	AOQ(K, r)	7.2586%	**0.7718%**	0.0411%	0.0021%	0.0001%	0.0000%	0.0000%
0.8	TC(K, r)	5510	5314	5902	5932	5933	5933	5933
	$N_E(K, r)$	24601	5557	4836	4802	4800	4800	4800
	$p_E(K, r)$	9.2012%	0.8609%	0.0447%	0.0022%	0.0001%	0.0000%	0.0000%
	AOQ(K, r)	8.5878%	0.8035%	0.0417%	0.0021%	0.0001%	0.0000%	0.0000%
1.0	TC(K, r)	5567	5173	5895	5932	5933	5933	5933
	$N_E(K, r)$	83504	5594	4837	4802	4800	4800	4800
	$p_E(K, r)$	15.5266%	0.9003%	0.0454%	0.0023%	0.0001%	0.0000%	0.0000%
	AOQ(K, r)	14.4915%	0.8402%	0.0424%	0.0021%	0.0001%	0.0000%	0.0000%

Table 2. Computational results of p_K, $N_i(K, r)$, $NRW_i(K, r)$, TC (K, r), NRW (K, r), and S (K, r) given that (Q, p_0, p_R, N, n) = (4,800 units/day, 16.1%, 5.0%, 240 units, 16 units), and r = 40%

K	0	1	2	3	4	5	6
p_K	16.1000%	0.8050%	0.0403%	0.0020%	0.0001%	0.0000%	0.0000%
$N_1(K, r)$	0	4800	5573	5611	5613	5613	5613
$N_2(K, r)$	1086	366	322	320	320	320	320
$N_3(K, r)$	6690	396	20	1	0	0	0
TC (K, r)	7776	5563	5915	5933	5933	5933	5933
$NRW_1(K, r)$	0	773	811	813	813	813	813
$NRW_2(K, r)$	813	41	2	0	0	0	0
NRW(K, r)	813	813	813	813	813	813	813
S(K, r)	4,460	264	13	1	0	0	0

there is the greatest reduction of TC (K, 40%) when K changes from zero to one. In detail, the number of items inspected in the K-stage inspection system has increased from zero to only 4,800 units while the number of items inspected in the source inspection shop and the re-inspection shop has decreased from 7,776 units to only 762 units. Hence the number of inspected items reduced throughout the factory becomes as many as 2,213 units in total, including the number of items passed from Node 5 to Node 3 without inspection, denoted by S (K, r) in <Table 2 >. After K ≥ 2, TC (K, 40%) increases a little and converges very rapidly to the value of 5,933 units.

5. Concluding Remarks

In this paper, by introducing the control variable r for rejected lots, we addressed an optimization problem for minimizing our objective function subject to a given AOQ. Since flows between nodes were interrelated, we made a network flow analysis and derived a steady-state nonlinear balance equation for solving the fraction defectives $p_E(K, r)$ and provided a formula for $N_E(K, r)$. Based on the values of $(p_E(K, r), N_E(K, r))$, our objective function was obtained. Since we proved that NRW (K, r) was constant irrespective of K and r, we redefined TC (K, r), and provided an enumeration method for determining the optimal values of (K, r) which both minimized TC (K, r) and attained a given target AOQ.

Our conjectures are that TC (K, r) is a decreasing function of r if K is greater than one and that there may exist some conditions under which the number of items reworked throughout a factory do not change. However, we failed either to prove it or to find the conditions. These conjectures could be further studied. In addition, since one of our assumptions is that inspectors are perfect in the sense that both type I error and type II error are zeros, this assumption may be relaxed and very complicated results could be derived in the future.

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