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The Linear Discrepancy of a Fuzzy Poset

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Abstract

In 2001, the notion of a fuzzy poset defined on a set X via a triplet (L, G, I) of functions with domain $X \times X$ and range $[0, 1]$ satisfying a special condition $L + G + I = 1$ is introduced by J. Negger and Hee Sik Kim, where L is the 'less than' function, G is the 'greater than' function, and I is the 'incomparable to' function. Using this approach, we are able to define a special class of fuzzy posets, and define the 'skeleton' of a fuzzy poset in view of major relation. In this sense, we define the linear discrepancy of a fuzzy poset of size n as the minimum value of all maximum of $I(x, y)|f(x) - f(y)|$ for $f \in \mathcal{F}$ and $x, y \in X$ with $I(x, y) > \frac{1}{2}$, where $\mathcal{F}$ is the set of all injective order-preserving maps from the fuzzy poset to the set of positive integers. We first show that the definition is well-defined. Then, it is shown that the optimality appears at the same injective order-preserving maps in both cases of a fuzzy poset and its skeleton if the linear discrepancy of a skeleton of a fuzzy poset is 1.

Key Words: Partially ordered set, Fuzzy poset, Linear discrepancy.

1. Introduction

Almost all relationships can be understood by a partial order properties, namely, reflexivity, anti-symmetry and transitivity, in which a practical meaning of "x is less than y" is not always true in the real world. In order to agree that the statement is true, it can be accompanied with a condition such as "a certain fraction of the time". Indeed, "x may be greater than y" during a certain period, while it is even possible that x and y are incomparable in a part of the time as well.

In 2001, J. Neggers and Hee Sik Kim introduced three functions L, G, I from $X \times X$ to $[0.1]$, and they constructed a sort of posets [4]. These three functions $L(x, y)$, $G(x, y)$ and $I(x, y)$ for $(x, y) \in X \times X$ represent three fractions of time when "x is less than y", "x is greater that y", and "x is incomparable to y", respectively. Hence, it is natural to suppose that

$$L(x, y) + G(x, y) + I(x, y) = 1. \tag{1}$$

If "x is less than y" is to be an acceptable statement, then we would expect the statement to hold during more than half of the time, i.e., $L(x, y) > \frac{1}{2}$. Similarly, "x is greater than y" would require $G(x, y) > \frac{1}{2}$, and "x and y are incomparable" $I(x, y) > \frac{1}{2}$.

In certain situations, the relationship may be asymmetric. For example, suppose that a relation is defined as a person's ability in a company X. A member x of X may feel that he is more able than a member y of X during the most of time., i.e., $L(x, y) < \frac{1}{2}$ and $G(x, y) > \frac{1}{2}$. However, y may also feel that he is more able than x during the most of time, i.e., $L(y, x) < \frac{1}{2}$ and $G(y, x) > \frac{1}{2}$. Hence, $L(x, y) > \frac{1}{2}$ may not be true though $G(y, x) > \frac{1}{2}$. If we accept this majority rule along with the condition that we shall always be able to decide which of these conditions holds, then we suppose that

$$\max\{L(x, y), G(x, y), I(x, y)\} > \frac{1}{2}. \tag{2}$$

Let $P = (X, \leq_P)$ be a poset with a ground set X and a relation $\leq_P$. Then, by the anti-asymmetry, $x \leq_P y$ and $y \leq_P x$ imply $x = y$. In other word, there are no distinct elements x and $y \in X$ such that $x \leq_P y$ and $y \leq_P x$, i.e., for distinct x and $y \in X$, either $x \leq_P y$ or $y \leq_P x$. This can be interpreted, in terms of the fuzziness and the majority rule, that either $L(x, y) > \frac{1}{2}$ or $L(y, x) > \frac{1}{2}$ for distinct x and $y \in X$. In order to emphasize this condition, we consider the condition

$$L(x, y) + L(y, x) \leq 1. \tag{3}$$

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Remark 1.1. For distinct x and y in a set X, suppose that $L(x, y)$ is is to mean the fraction of occurrences that $x \leq_P y$ for a relation $\leq_P$ on X, and satisfies (3). If both $x \leq_P y$ and $y \leq_P x$ hold during more than a half of the time, i.e., $L(x, y) > \frac{1}{2}$ and $L(y, x) > \frac{1}{2}$. Then $L(x, y)+L(y, x) > 1$. However, this is a contradiction to (3). Hence, there are no distinct such x and y in X, i.e., $x = y$. This means that the antisymmetry holds.

Similarly, if $I(x, y) > \frac{1}{2}$ is to reflect that x and y are mostly incomparable, then it is a reasonable requirement to expect some symmetry as in the condition that

$$I(x, y) > \frac{1}{2} \text{ implies } I(y, x) > \frac{1}{2}. \qquad (4)$$

Now, we introduce some propositions as follows.

Proposition 1.2. (J. Negger, and Hee sik Kim [4])
Let X be a set, and let three functions L, G and I from $X \times X$ to $[0, 1]$ satisfy (1), (2), (3) and (4). Then the followings hold.

(1) If $L(x, y) > \frac{1}{2}$, then $G(y, x) > \frac{1}{2}$.

(2) If $G(x, y) > \frac{1}{2}$, then $L(y, x) > \frac{1}{2}$.

To introduce transitivity into the structure, we suppose the following property:

$$\text{if } L(x, y) > \frac{1}{2} \text{ and } L(y, z) > \frac{1}{2}, \text{ then } L(x, z) > \frac{1}{2}. \qquad (5)$$

Remark 1.3. For distinct x and y in a set X, suppose that $L(x, y)$ is is to mean the fraction of occurrences that $x \leq_P y$ for a relation $\leq_P$ on X, and satisfies (5). For x, y and z be distinct elements in X, suppose that both $x \leq_P y$ and $y \leq_p z$ hold during more than a half of the time, i.e., $L(x, y) > \frac{1}{2}$ and $L(y, z) > \frac{1}{2}$. Then, from (5), we have $L(x, z) > \frac{1}{2}$. This can be interpreted as $x \leq_P z$ during more than a half of the time. Therefore, (5) represents the transitivity.

From Proposition 1.2, we obtain the following proposition.

Proposition 1.4. (J. Negger, and Hee sik Kim [4])
Let X be a set, and let three functions L, G and I from $X \times X$ to $[0, 1]$ satisfy (1), (2), (3), (4) and (5). Then the followings hold.

(1) $G(x, y) + G(y, x) \leq 1$.

(2) If $G(x, y) > \frac{1}{2}$ and $G(y, z) > \frac{1}{2}$, then $G(x, z) > \frac{1}{2}$.

Remark 1.5. Let $P = (X, \leq_P)$ be a poset. Then we can define three functions L, G and I from $X \times X$ to $[0, 1]$ as follows.

$$L(x, y) = \begin{cases} 1, & \text{if } x \leq_P y, \\ 0, & \text{otherwise,} \end{cases}$$

$$G(x, y) = \begin{cases} 1, & \text{if } y \leq_P x, \\ 0, & \text{otherwise,} \end{cases}$$

$$I(x, y) = \begin{cases} 1, & \text{if } x \| y, \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is clear that L, G and I satisfy (1), (2), (3), (4) and (5). We can redefine the poset $P = (X, \leq_P)$ with respect to three fuzzy relations L, G and I.

From (1), (2), (3), (4) and (5), we can induce the antisymmetry and the transitivity. Using all three fuzzy relations L, G and I, we can induce a poset $F = (X, L, G, I)$ as follows.

Definition 1.6. Let X be a set, and L, G and I maps from $X \times X$ to $[0, 1]$. Then we define a *majority rule fuzzy poset* induced by fuzzy relations L, G and I, simply *F-poset*, as $F = (X, L, G, I)$ with the following properties: for distinct x, y, $z \in X$,

(1) $L(x, y) + G(x, y) + I(x, y) = 1$,

(2) $\max\{L(x, y), G(x, y), I(x, y)\} > \frac{1}{2}$,

(3) $L(x, y) + L(y, x) \leq 1$,

(4) $I(y, x) > \frac{1}{2}$ if $I(x, y) > \frac{1}{2}$,

(5) $L(x, z) > \frac{1}{2}$ if $L(x, y) > \frac{1}{2}$ and $L(y, z) > \frac{1}{2}$,

(6) $L(x, x) = \frac{1}{2}$, $G(x, x) = \frac{1}{2}$, and $I(x, x) = 0$.

Remark 1.7. For a given $P = (X, \leq_P)$, every pair $(x, x) \in X \times X$ always holds the reflexivity during the evolution of P along the time. Hence, it is impossible that x is incomparable to x. We define $I(x, x) = 0$ in Definition 1.6. Moreover, in order to satisfy the condition $L(x, x) + G(x, x) + I(x, x) = 1$, we also define $L(x, x) = \frac{1}{2}$ and $G(x, x) = \frac{1}{2}$ in Definition 1.6.

Remark 1.8. Definition 1.6 is introduced by J. Negger and Hee Sik Kim[4] in 2001. (6) in Definition 1.6 can not be found in the original definition of J.Negger and Hee Sik Kim. For the reflexivity, we include (6) in the definition.

In an F-poset F, when L, G, or I is greater that $\frac{1}{2}$, the corresponding relation can be emphasized so that we can obtain a new object. The following definition explains the new object.

Definition 1.9. Let $F = (X, L, G, I)$ be an F-poset with a ground set X, functions L, G, I from $X \times X$ to $[0, 1]$. Define a new function $L_{sk} : X \times X \to [0, 1]$ as

$$L_{sk}(x, y) = \begin{cases} 1, & \text{if } L(x, y) > \frac{1}{2}, \\ \frac{1}{2}, & \text{if } L(x, y) = \frac{1}{2} \\ 0, & \text{if } L(x, y) < \frac{1}{2} \end{cases}$$

L_{sk} is called a 'skeleton less than' function on an F-poset F. Similarly, we can define a 'skeleton greater than' function G_{sk} and a 'skeleton incomparable to' function I_{sk} on F. In this time, $sk(F) = (X, L_{sk}, G_{sk}, I_{sk})$ is a called a skeleton of an F-poset F.

Obviously, we note that $sk(F)$ is also an F-poset.

Proposition 1.10. Let $F = (X, L, G, I)$ be an F-poset. Then, $sk(F) = (X, \leq_{sk(F)})$ is a poset where $\leq_{sk(F)}$ is a relation defined as

(1) $x \leq_{sk(F)} y$ if and only if $L_{sk}(x, y) = 1$ for distinct x, $y \in X$,

(2) $x \leq_{sk(F)} x$ for $x \in X$.

Proof. Let $sk(F) = (X, \leq_{sk(F)})$ where $\leq_{sk(F)}$ is a relation defined as

(1) $x \leq_{sk(F)} y$ if and only if $L_{sk}(x, y) = 1$ for distinct x, $y \in X$,

(2) $x \leq_{sk(F)} x$ for $x \in X$.

Clearly, $sk(F)$ satisfies the reflexivity.

For distinct x, $y \in X$, suppose that $x \leq_{sk(F)} y$ and $y \leq_{sk(F)} x$, i.e., $L_{sk}(x, y) = 1$ and $L_{sk}(y, x) = 1$. Then we have $L(x, y) > \frac{1}{2}$ and $L(y, x) > \frac{1}{2}$ so that we have $G(y, x) > \frac{1}{2}$ and $G(x, y) > \frac{1}{2}$ by Proposition 1. This implies that $L(x, y)+G(x, y)+I(x, y) > \frac{1}{2}+\frac{1}{2}+I(x, y) > 1$, which is a contradiction. Hence $x = y$, i.e., $sk(F)$ satisfies the anti-symmetry.

For distinct x, y, $z \in X$, suppose that $x \leq_{sk(F)} y$ and $y \leq_{sk(F)} z$, i.e., $L_{sk}(x, y) = 1$ and $L_{sk}(y, z) = 1$. Then we have $L(x, y) > \frac{1}{2}$ and $L(y, z) > \frac{1}{2}$ so that we have $L(x, z) > \frac{1}{2}$ since F is an F-poset. This implies that $L_{sk}(x, z) = 1$, i.e., $x \leq_{sk(F)} z$. Hence, $sk(F)$ satisfies the transitivity. Therefore, $sk(F)$ is a poset with a relation $\leq_{sk(F)}$ defined as $x \leq_{sk(F)} y$ if $L_{sk}(x, y) = 1$ for distinct $x, y \in X$. □

Example 1.11. Let $X = \{a, b, c, d\}$. Supposet that a F-poset $F = (X, L, G, I)$ is defined as follows: From the

	(a,a)	(a,b)	(a,c)	(a,d)	(b,a)	(b,b)	(b,c)	(b,d)
L	0.5	0.2	1	0.1	0.2	0.5	0.1	1
G	0.5	0.2	0	0.1	0.1	0.5	0.1	0
I	0	0.6	0	0.8	0.7	0	0.8	0
	(c,a)	(c,b)	(c,c)	(c,d)	(d,a)	(d,b)	(d,c)	(d,d)
L	0	0.1	0.5	0.2	0.1	0	0.1	0.5
G	1	0	0.5	0.2	0.1	1	0.2	0.5
I	0	0.9	0	0.6	0.8	0	0.7	0

definition of F, we can obtain three skeleton functions L_{sk}, G_{sk}, and I_{sk} as follows. Then, we can obtain that the relation $\leq_{sk(F)}= \{(a, a), (a, c), (b, b), (b, d), (c, c), (d, d)\}$, illustrated by a Hesse diagram as in Figure 1, i.e., $sk(F)$ is a disjoint sum of two 2-element chains, i.e., $\mathbf{2} + \mathbf{2}$.

	(a,a)	(a,b)	(a,c)	(a,d)	(b,a)	(b,b)	(b,c)	(b,d)
L_{sk}	0.5	0	1	0	0	0.5	0	1
G_{sk}	0.5	0	0	0	0	0.5	0	0
I_{sk}	0	1	0	1	1	0	1	0
	(c,a)	(c,b)	(c,c)	(c,d)	(d,a)	(d,b)	(d,c)	(d,d)
L_{sk}	0	0	0.5	0	0	0	0	0.5
G_{sk}	1	0	0.5	0	0	1	0	0.5
I_{sk}	0	1	0	1	1	0	1	0

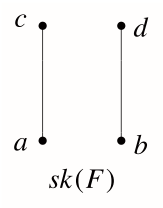

Figure 1: $\mathbf{2} + \mathbf{2}$

2. The Linear Discrepancy of an F-Poset

For positive integers a and b with $a \leq b$, $[a, b]$ denotes $\{a,\ a + 1,\ \ldots, b\}$. Especially, $[b]$ denotes $[1, b]$. The *chain* of order n, denoted by $\mathbf{n} = (X, \leq_{\mathbf{n}})$, is a poset such that $|X| = n$ and $x \leq_{\mathbf{n}} y$ or $y \leq_{\mathbf{n}} x$ for all $x, y \in X$. For a given poset $P = (X, \leq_P)$, an injective map $f : X \rightarrow \mathbb{Z}$ is called an *isotone* if it preserves the order-relation of P, i.e., $f(x) \leq f(y)$ if $x \leq_P y$ in P. When the image of an isotone f of $P = (X, \leq_P)$ with $n = |X|$ is $[n]$, we call such f a *labeling* of P. The *tightness* of an isotone f on P, written as $T_f(P)$, is the maximum difference between the values of f of incomparable pairs in P. We define $T_f(\mathbf{n})$ = 0 for a chain $\mathbf{n}$. The *linear discrepancy* of $P = (X, \leq_P)$, written as $ld(P)$, is the minimum tightness over all isotones on P, i.e.,

$$ld(P) = \min_{f \in \mathcal{F}} T_f(P)$$
$$= \min_{f \in \mathcal{F}} \max\{|f(x) - f(y)| : x \| y \text{ and } x, y \in X\}$$

where $\mathcal{F}$ is the set of all isotones of P. An isotone f on P is called *optimal* if $T_f(P) = ld(P)$. For an optimal isotone f on P, if $f(y) - f(x) = ld(P)$ for some $(x, y) \in P$, then (x, y) is called an *ld-pair* of P.

For a given F-poset $F = (X, L, G, I)$, an injective map $f : X \rightarrow \mathbb{Z}$ is called an *fuzzy isotone* on F if $f(x) \leq f(y)$ whenever $L(x, y) > \frac{1}{2}$. When the image of an isotone f of $P = (X, \leq_{sk(F)})$ with $n = |X|$ is $[n]$, we call such f a *fuzzy labeling* of F. The *fuzzy tightness* of an isotone f on an F-poset F, written as $T_f(F)$, is the maximum difference between the values of f of incomparable pairs in $sk(F)$. We define $T_f(F) = 0$ for an F-poset F whose skeleton is a chain. The *fuzzy linear discrepancy* of $F = (X, L, G, I)$, written as $ld(F)$, is the minimum fuzzy tightness over all isotones on F, i.e.,

$$\begin{aligned} ld(F) &= \min_{f\in\mathcal{F}} T_f(F) \\ &= \min_{f\in\mathcal{F}} \max\{I(x,y)|f(x)-f(y)| : \\ &\qquad I(x,y) > \tfrac{1}{2} \text{ and } x, y \in X\} \end{aligned}$$

where $\mathcal{F}$ is the set of all fuzzy isotones of F. A fuzzy isotone f on F is called *fuzzy optimal* if $T_f(F) = ld(F)$. For an optimal fuzzy isotone f on F, if $f(y) - f(x) = ld(F)$ for some $(x, y) \in X$, then (x, y) is called an *fuzzy ld-pair* of F.

Remark 2.1. In the notation $ld(R)$, if R is a poset, then $ld(R)$ means the linear discrepancy of a poset R. If R is an F-poset, then $ld(R)$ implies that the fuzzy linear discrepancy of an F-poset R. The same argument can be applied to $T_f(R)$.

The following example shows the linear discrepancy of a poset and the fuzzy linear discrepancy of an F-poset.

Example 2.2. Let $X = \{a, b, c, d\}$. Suppose that a F-poset $F = (X, L, G, I)$ is defined as follows: Then, from

	(a,a)	(a,b)	(a,c)	(a,d)	(b,a)	(b,b)	(b,c)	(b,d)
L	0.5	0.2	1	0.1	0.2	0.5	0.1	1
G	0.5	0.2	0	0.1	0.1	0.5	0.1	0
I	0	0.6	0	0.8	0.7	0	0.8	0
	(c,a)	(c,b)	(c,c)	(c,d)	(d,a)	(d,b)	(d,c)	(d,d)
L	0	0.1	0.5	0.2	0.1	0	0.1	0.5
G	1	0	0.5	0.2	0.1	1	0.2	0.5
I	0	0.9	0	0.6	0.8	0	0.7	0

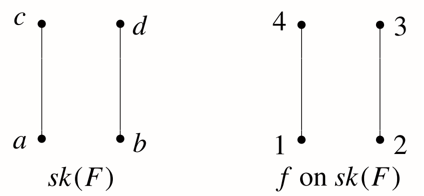

Figure 2: $\mathbf{2}+\mathbf{2}$ and an isotone f on $sk(F)$

Example 1.11, $sk(F)$ is $\mathbf{2}+\mathbf{2}$.

(1) An isotone $f : X \to [4]$ is defined as $f(a) = 1$, $f(b) = 2$, $f(c) = 4$ and $f(d) = 3$. Then f is an optimal isotone on $sk(F)$ so that $ld(sk(X)) = 2$. Moreover, (a, d) and (b, c) are ld-pairs.

(2) An fuzzy isotone $f : X \to [4]$ is defined as $f(a) = 1$, $f(b) = 2$, $f(c) = 4$ and $f(d) = 3$. Then $T_f(F) = 1.6$. In fact, f is a fuzzy optimal isotone on F so that $ld(F) = 1.6$. Moreover, (a, d) and (b, c) are fuzzy ld-pairs.

These are illustrated in Figure 2.

By the definition of a skeleton of an F-poset, we can easily obtain the following lemma.

Lemma 2.3. For a given F-poset $F = (X, L, G, I)$ and x, $y \in X$, we have $I(x, y) > \frac{1}{2}$ if and only if $x \| y$ in $sk(F)$.

Proposition 2.4. For a given F-poset $F = (X, L, G, I)$, we have $ld(F) \leq ld(sk(F))$.

Proof. Let f be an isotone on $sk(X)$. Since $I(x, y) < 1$, we have

$$I(x,y)|f(x)-f(y)| \leq |f(x)-f(y)|$$

for all x, $y \in X$. Hence, we have $T_f(F) \leq T_f(sk(F))$. This implies that

$$ld(F) \leq ld(sk(F)).$$

□

Proposition 2.5. Let $F = (X, L, G, I)$ be an F-poset whose skeleton is an antichain. If $|X| \leq 3$, then $ld(F) \leq \min\{I(x, y) : x, y \in X\}(|X| - 1)$.

Proof. Let $X = \{x_1, \cdots, x_n\}$ and let $I(x_1, x_n)$ be a minimal value of I. Suppose that $n \leq 3$. Firstly, if $n = 2$, then it is clear that $ld(F) \leq \min\{I(x, y) : x, y \in X\}(|X| - 1)$. Suppose that $n = 3$, i.e., $X = \{x_1, x_2, x_3\}$, and $I(x_1, x_3)$ is a minimal value. Then $ld(sk(X)) = 2$. Define an fuzzy isotone f as $f(x_i) = i$ for $i = 1, 2, 3$. Then we have

$$I(x_1,x_3)|f(x_1)-f(x_3)| > \frac{1}{2}(n-1) = 1$$

since $I(x_1, x_3) > \frac{1}{2}$. By Proposition 2.4, we have

$$I(x_1,x_3)|f(x_1)-f(x_3)| \geq I(x,y)|f(x)-f(y)|$$

for $x, y \in X$, i.e.,

$$T_f(F) = \min\{I(x,y) : x, y \in X\}(|X|-1).$$

Since $ld(F) \leq T_f(F)$, we conclude that

$$ld(F) \leq \min\{I(x,y) : x, y \in X\}(|X|-1).$$

□

In general, although f is optimal on $sk(F)$, it can not be fuzzy optimal on an F-poset $F = (X, L, G, I)$. The following example shows this.

Example 2.6. Let $X = \{a, b, c, d\}$. Suppose that a F-poset $F = (X, L, G, I)$ is defined as follows: Then, from Example 1.11, $sk(X)$ is $\mathbf{2}+\mathbf{2}$.

(1) An map $f : X \to [4]$ is defined as $f(a) = 1$, $f(b) = 2$, $f(c) = 4$ and $f(d) = 3$. Then f is an fuzzy isotone on $sk(F)$ so that $T_f(F) = 2$.

	(a,a)	(a,b)	(a,c)	(a,d)	(b,a)	(b,b)	(b,c)	(b,d)
L	0.5	0.2	1	0.3	0.2	0.5	0	1
G	0.5	0.2	0	0.19	0.1	0.5	0	0
I	0	0.6	0	0.51	0.7	0	1	0
	(c,a)	(c,b)	(c,c)	(c,d)	(d,a)	(d,b)	(d,c)	(d,d)
L	0	0.1	0.5	0.2	0.19	0	0.1	0.5
G	1	0	0.5	0.2	0.3	1	0.2	0.5
I	0	0.9	0	0.6	0.51	0	0.7	0

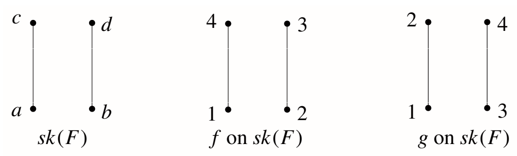

Figure 3: $\mathbf{2}+\mathbf{2}$ and an isotone f and g on $sk(F)$

(2) An map $g : X \to [4]$ is defined as $g(a) = 1$, $g(b) = 3$, $g(c) = 2$ and $g(d) = 4$. Then g is a fuzzy optimal isotone on F so that $T_g(F) = 1.53$. In fact, g is a fuzzy optimal isotone on F, and moreover, (a, d) is a fuzzy ld-pairs.

Hence, f defined in (1) is optimal on $sk(F)$ but not a fuzzy optimal on F. These are illustrated in Figure 3

The following theorem tell us that, in the case of $ld(sk(F)) = 1$, if f is a fuzzy optimal isotone, then f is optimal, and vice versa.

Theorem 2.7. For a given F-poset $F = (X, L, G, I)$ with $ld(sk(F)) = 1$, a map f is an optimal isotone on $sk(F)$ if and only if f is also a fuzzy optimal isotone on F.

Proof. Let f be a fuzzy optimal isotone on F. Suppose not, i.e., f is not optimal on $sk(F)$. Then $T_f(sk(F)) \geq 2$, and there are x_0, $x_0' \in X$ such that $x_0 \| x_0'$ in $sk(F)$ and

$$|f(x_0) - f(x_0')| \geq 2.$$

Since $x_0 \| x_0'$ in $sk(F)$, we have

$$I(x_0, x_0') > \frac{1}{2}$$

so that

$$I(x_0, x_0')|f(x_0) - f(x_0')| > 1,$$

i.e., $T_f(F) > 1$. Since f is fuzzy optimal on F, we have $ld(F) > 1$, however, $ld(F) \leq ld(sk(F)) = 1$. Hence, we have

$$1 < ld(F) \leq ld(sk(X)) = 1.$$

This is a contradiction. Therefore, if f is fuzzy optimal on F, then f is optimal on $sk(F)$.

Let f be an optimal isotone on $sk(F)$. Suppose not, i.e., f is not a fuzzy optimal on F. Let g be a fuzzy optimal isotone on F. Then g is an optimal isotone on $sk(F)$ and $T_g(F) < T_f(F)$. Since g is optimal on $sk(X)$ and $ld(sk(F)) = 1$, we have

$$|g(x) - g(y)| = 1$$

for all x, $y \in X$ with $x \| y$ in $sk(F)$. Similarly, we have

$$|f(x) - f(y)| = 1$$

for x, $y \in X$ with $x \| y$ in $sk(F)$ since f is optimal on $sk(F)$. Hence, we have

$$|g(x_1) - g(x_1')| = |f(x_1) - f(x_1')|$$

so that

$$I(x, y)|g(x_1) - g(x_1')| = I(x, y)|f(x_1) - f(x_1')|.$$

This implies that $T_f(F) = T_g(F) = ld(F)$. This is a contradiction. Therefore, if f is optimal on $sk(F)$, then f is fuzzy optimal on F. □

From Theorem 2.7, for an F-poset $F = (X, L, G, I)$ with $ld(sk(F)) = 1$, fuzzy optimal isotones are exactly same to optimal isotones.

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