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Preconnectedness Degree of L-Fuzzy Topological Spaces

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Abstract

In this paper, we use L-fuzzy preopen operator to introduce the degree of pre-separatedness and the degree of preconnectedness in L-fuzzy topological spaces. Many characterizations of the degree of preconnectedness are presented in L-fuzzy topological spaces.

Key Words: L-fuzzy topology; L-fuzzy preclosure operator; preseparatedness; preconnectedness.

1. Introduction

It is well known that after the introduction of the L-fuzzy topological space by Kubiak [1] and Šostak [2] in 1985, a large number of mathematicians have taken great interests in generalizing and extending different concepts of set topology and Chang's fuzzy topology [3] into L-fuzzy topology. The concept of connectedness along with some of its allied forms is one of the directions that have hitherto been ventured with meticulous attention. However, the results obtained in connection with different contexts like fuzzy connectedness, semi-connectedness, preconnectedness etc. in L-fuzzy topological spaces are seen to be quite parallel and analogous. This is chiefly due to the fact that the study of these variations of the concept of fuzzy connectedness has been effected only by replacing open L-subsets by the r-level cut of fuzzy open L-subset or the like. It can thus be conjectured that the use of a suitable operator should obviates the use of r-level cut in L-fuzzy topological spaces.

In [4], Shi introduced the notion of L-fuzzy preopen operator $\mathcal{T}_p$ in L-fuzzy topological spaces as a generalization of preopen L-subsets, where L completely distributive DeMorgan algebra. $\mathcal{T}_p(A)$ can be regarded as the degree to which A is preopen. So that, actually $\mathcal{T}_p$ reflects the essence of L-fuzzy topology.

In this paper, we introduce and characterize the degree of preseparatedness and the degree of preconnectedness in L-fuzzy topological spaces. The results in our paper is a generalization to the results of [5].

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2. Preliminaries

Throughout this paper, $(L, \leq, \bigwedge, \bigvee, ')$ denotes a completely distributive DeMorgan algebra, X is a nonempty set. The smallest element and the largest element in L are denoted by 0 and 1, respectively. The set of all nonzero co-prime elements of L is denoted by $M(L)$.

We say that a is wedge below b in L, denoted by $a \ll b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \bigvee D$ always implies the existence of $d \in D$ with $a \leq d$ [6]. A complete lattice L is completely distributive if and only if $b = \bigvee\{a \in L | a \ll b\}$ for each $b \in L$. For any $b \in L$, define $\beta(b) = \{a \in L | a \ll b\}$.

For a nonempty set X, the set of all nonzero coprime elements of L^X is denoted by $M(L^X)$. It is easy to see that $M(L^X)$ is exactly the set of all fuzzy points x_λ ($\lambda \in M(L)$). The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$, respectively.

An L-topological space is a pair $(X, \mathfrak{T})$, where $\mathfrak{T}$ is a subfamily of L^X which contains $\underline{0}$; $\underline{1}$ and is closed for any suprema and finite infima. $\mathfrak{T}$ is called an L-topology on X. Members of $\mathfrak{T}$ are called open L-subsets and their complements are called closed L-subsets. The closure and the interior of L-subset A are denoted by $Cl(A)$ and $Int(A)$, respectively. An L-subset A in $(X, \mathfrak{T})$ is called preopen [7] iff $A \leq Int(Cl(A))$. The preclosure operator of A in $(X, \mathfrak{T})$ is denoted by $pCl(A)$.

In an L-topological space $(X, \mathfrak{T})$, two L-subsets A, B are called preseparated [5] if $pCl(A) \wedge B = A \wedge pCl(B) = \underline{0}$. An L-subset C is called preconnected if C can not be represented as a union of two preseparated non-null L-subsets.

Definition 2.1. [1, 2]An L-fuzzy topology on a set X is a map $\mathcal{T} : L^X \to L$ such that

(O1) $\mathcal{T}(\underline{0}) = \mathcal{T}(\underline{1}) = 1$;

(O2) for all A, $B \in L^X$, $\mathcal{T}(A \wedge B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$;

(O3) for all $A_j \in L^X$, $j \in J$, $\mathcal{T}(\bigvee_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathcal{T}(A_j)$.

$\mathcal{T}(A)$ can be interpreted as the degree to which A is an open set. $\mathcal{T}^*(A) = \mathcal{T}(A')$ will be called the degree of closedness of A. The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space.

A mapping $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is said to be continuous with respect to L-fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ if $\mathcal{T}_1(f_L^{\leftarrow}(B)) \geq \mathcal{T}_2(B)$ holds for all $B \in L^Y$, where $f_L^{\leftarrow}$ is defined by $f_L^{\leftarrow}(B)(x) = B(f(x))$.

Theorem 2.1. [8] Let $\mathcal{T} : L^X \to L$ be a function. Then the following conditions are equivalent:

(1) $\mathcal{T}$ is an L-fuzy topology on X;

(2) $\mathcal{T}_{[a]}$ is an L-topology on X, for each $a \in M(L)$.

Definition 2.2. [9] An L-fuzzy closure operator on X is a mapping $Cl : L^X \to L^{M(L^X)}$ satisfying the following conditions:

(C1) $Cl(A)(x_\lambda) = \bigwedge_{\mu \ll \lambda} Cl(A)(x_\mu)$, for all $x_\lambda \in M(L^X)$;

(C2) $Cl(\underline{0})(x_\lambda) = 0$ for any $x_\lambda \in M(L^X)$;

(C3) $Cl(A)(x_\lambda) = 1$ for any $x_\lambda \leq A$;

(C4) $Cl(A \vee B) = Cl(A) \vee Cl(B)$;

(C5) for all $a \in L_0$, $(Cl(\bigvee(Cl(A))_{[a]}))_{[a]} \subset (Cl(A))_{[a]}$.

$Cl(A)(x_\lambda)$ is called the degree to which x_λ belongs to the closure of A.

Lemma 2.1. [9] Let $(X, \mathcal{T})$ be an L-fuzzy topological space and let Cl be the L-fuzzy closure operator induced by $\mathcal{T}$. Then for all $x_\lambda \in M(L^X)$ and $A \in L^X$,

$$Cl(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq D \geq A} (\mathcal{T}(D'))'.$$

Definition 2.3. [4] Let $(X, \mathcal{T})$ be an L-fuzzy topological space. For $A \in L^X$, define the mapping $\mathcal{T}_p : L^X \to L$ by

$$\mathcal{T}_p(A) = \bigwedge_{x_\lambda \ll A} \bigvee_{x_\lambda \ll B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \ll B} \bigwedge_{y_\mu \not\leq D \geq A} (\mathcal{T}(D'))' \right\}.$$

Then $\mathcal{T}_p$ is called L-fuzzy preopen operator induced by $\mathcal{T}$, where $\mathcal{T}_p(A)$ can be regarded as the degree to which A is preopen and $\mathcal{T}_p^*(A) = \mathcal{T}_p(A')$ can be regarded as the degree to which A is preclosed.

Theorem 2.2. [4] Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $A \in L^X$. Then $A \in (\mathcal{T}_p)_{[a]}$ if and only if A is preopen in $\mathcal{T}_{[a]}$, where $a \in M(L)$ and $(\mathcal{T}_p)_{[a]} = \{A \in L^X \mid \mathcal{T}_p(A) \geq a\}$.

Lemma 2.2. Let $\mathcal{T}_p : L^X \to L$ be An L-fuzzy preopen operator induced by L-fuzzy topology $\mathcal{T}$. Then $\mathcal{T}_p$ satisfies the following conditions:

(1) $\mathcal{T}_p(\underline{0}) = \mathcal{T}_p(\underline{1}) = 1$;

(2) for all $A_j \in L^X$, $j \in J$, $\mathcal{T}_p(\bigvee_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathcal{T}_p(A_j)$.

Proof. Straightforward. □

Definition 2.4. An L-fuzzy preclosure operator on X is a mapping $pCl : L^X \to L^{M(L^X)}$ satisfying the following conditions:

(P1) $pCl(A)(x_\lambda) = \bigwedge_{\mu \ll \lambda} pCl(A)(x_\mu)$, for all $x_\lambda \in M(L^X)$;

(P2) $pCl(\underline{0})(x_\lambda) = 0$ for any $x_\lambda \in M(L^X)$;

(P3) $pCl(A)(x_\lambda) = 1$ for any $x_\lambda \leq A$;

(P4) for all $a \in L_0$, $(pCl(\bigvee(pCl(A))_{[a]}))_{[a]} \subset (pCl(A))_{[a]}$.

$pCl(A)(x_\lambda)$ is called the degree to which x_λ belongs to the preclosure of A.

Theorem 2.3. Let $\mathcal{T}_p$ be the L-fuzzy preopen operator on X and let $pCl^{\mathcal{T}_p}$ be the L-fuzzy preclosure operator induced by $\mathcal{T}_p$. Then for each $x_\lambda \in M(L^X)$ and $A \in L^X$,

$$pCl^{\mathcal{T}_p}(A)(x_\lambda) = \bigwedge_{x \not\leq D \geq A} (\mathcal{T}_p(D'))'.$$

Proof. Straightforward. □

3. Preseparatedness Degree in L-Fuzzy Topological Spaces

Definition 3.1. Let $(X, \mathcal{T})$ be an L-fuzzy topological space and A, $B \in L^X$. Define

$$\mathcal{P}(A, B) = \Big(\bigwedge_{x_\lambda \leq A} (pCl(B)(x_\lambda))' \Big) \wedge \Big(\bigwedge_{y_\mu \leq B} (pCl(A)(y_\mu))' \Big),$$

Then $\mathcal{P}(A, B)$ is said to be the preseparatedness degree of A and B.

Proposition 3.1. Let $\mathcal{T} : L^X \to \{0, 1\}$ be an L-fuzzy topology on X and A, $B \in L^X$. Then $\mathcal{P}(A, B) = 1$ if and only if A and B are preseparated in $(X, \mathcal{T})$.

Lemma 3.1. Let $(X, \mathcal{T})$ be an L-fuzzy topological space and A, $B \in L^X$. If $A \wedge B \neq \underline{0}$, then $\mathcal{P}(A, B) = 0$.

Proof. Let $z_\mu \in M(L^X)$ such that $x_\mu \leq A \wedge B$. Thus we have

$$\begin{aligned} \mathcal{P}(A, B) &= \Big(\bigwedge_{x_\lambda \leq A} (pCl(B)(x_\lambda))' \Big) \wedge \Big(\bigwedge_{x_\lambda \leq B} (pCl(A)(x_\lambda))' \Big) \\ &\leq \big(pCl(B)(z_\mu)\big)' \wedge \big(pCl(A)(z_\mu)\big)' = 1' \wedge 1' = 0. \end{aligned}$$

□

Lemma 3.2. Let $(X, \mathcal{T})$ be an L-fuzzy topological space, and A, B, C, $D \in L^X$. If $C \leq A$ and $D \leq B$, then $\mathcal{P}(A, B) \leq \mathcal{P}(C, D)$.

Proof. If $C \le A$ and $D \le B$, then $pCl(C) \le pCl(A)$ and $pCl(D) \le pCl(B)$. Hence we have

$$\begin{aligned}\mathcal{P}(A,B) &= \Big(\bigwedge_{x_\lambda \le A} (pCl(B)(x_\lambda))'\Big) \wedge \Big(\bigwedge_{y_\beta \le B} (pCl(A)(y_\beta))'\Big)\\ &\le \Big(\bigwedge_{x_\lambda \le A} (pCl(D)(x_\lambda))'\Big) \wedge \Big(\bigwedge_{y_\beta \le B} (pCl(C)(y_\beta))'\Big)\\ &\le \Big(\bigwedge_{x_\lambda \le C} (pCl(D)(x_\lambda))'\Big) \wedge \Big(\bigwedge_{y_\beta \le D} (pCl(C)(y_\beta))'\Big)\\ &= \mathcal{P}(C,D).\end{aligned}$$

□

Lemma 3.3. Let $(X,\mathcal{T})$ be an L-fuzzy topological space, A, $B \in L^X$ and $a \in M(L)$. Then $(\mathcal{P}(A,B))' \ge a$ if and only if there exist D, $E \in L^X$ such that $D \ge A$, $E \ge B$, $D \wedge B = E \wedge A = \underline{0}$ and $(\mathcal{T}_p(D'))' \vee (\mathcal{T}_p(E'))' \not\ge a$.

Proof. Suppose that $(\mathcal{P}(A,B))' \ge a$. Then $(\mathcal{P}(A,B))' \ge b$ for some $b \in \beta^*(a)$. Then

$$\bigvee_{x_\lambda \le A} pCl(B)(x_\lambda) \vee \bigvee_{y_\beta \le B} pCl(A)(y_\beta) \not\ge b.$$

Moreover, we have

$$\bigvee_{x_\lambda \le A}\ \bigwedge_{x_\lambda \not\le E \ge B} (\mathcal{T}_p(E'))' \vee \bigvee_{y_\beta \le B}\ \bigwedge_{y_\beta \not\le D \ge A} (\mathcal{T}_p(D'))' \not\ge b.$$

Hence for any $x_\lambda \le A$ and for any $y_\beta \le B$, there exist D_{y_β}, $E_{x_\lambda} \in L^X$ such that $x_\lambda \not\le E_{x_\lambda} \ge B$, $y_\beta \not\le D_{y_\beta} \ge A$ and $(\mathcal{T}_p(D'_{y_\beta}))' \vee (\mathcal{T}_p(E'_{x_\lambda}))' \not\ge b$. Let $E = \bigwedge_{x_\lambda \le A} E_{x_\lambda}$ and $D = \bigwedge_{y_\beta \le B} D_{y_\beta}$. Then, we have that $D \ge A$, $E \ge B$, $D \wedge B = E \wedge A = \underline{0}$ and

$$\begin{aligned}(\mathcal{T}_p(D'))' \vee (\mathcal{T}_p(E'))' &= (\mathcal{T}_p(\bigvee_{y_\beta \le B} D'_{y_\beta}))' \vee (\mathcal{T}_p(\bigvee_{x_\lambda \le B} E'_{x_\lambda}))'\\ &\le \bigvee_{y_\beta \le B} (\mathcal{T}_p(D'_{y_\beta}))' \vee \bigvee_{x_\lambda \le A} (\mathcal{T}_p(E'_{x_\lambda}))'\\ &\not\ge a.\end{aligned}$$

Conversely if there exist D, $E \in L^X$ such that $D \ge A$, $E \ge B$, $D \wedge B = E \wedge A = \underline{0}$ and $(\mathcal{T}_p(D'))' \vee (\mathcal{T}_p(E'))' \not\ge a$. Since

$$\begin{aligned}(\mathcal{P}(A,B))' &= \bigvee_{x_\lambda \le A} pCl(B)(x_\lambda) \vee \bigvee_{y_\beta} pCl(A)(y_\beta)\\ &= \bigvee_{x_\lambda \le A}\ \bigwedge_{x_\lambda \not\le G \ge B} (\mathcal{T}_p(G'))'\\ &\vee \bigvee_{y_\beta \le B}\ \bigwedge_{y_\beta \not\le H \ge A} (\mathcal{T}_p(H'))'\\ &\le (\mathcal{T}_p(D'))' \vee (\mathcal{T}_p(E'))'.\end{aligned}$$

Then $(\mathcal{P}(A,B))' \not\ge a$. □

4. Preconnectedness Degree in L-Fuzzy Topological Spaces

Definition 4.1. Let $(X,\mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. Define

$$\mathcal{PC}(G) = \bigwedge \Big\{\mathcal{P}(A,B)' : \ A, B \in L^X \backslash \{\underline{0}\},\ G = A \vee B\Big\}.$$

Then $\mathcal{PC}(G)$ is said to be the preconnectedness degree of G.

From Definition 3.1, we have

$$\mathcal{PC}(G) = \bigwedge_{\substack{A,B \in L^X \backslash \{\underline{0}\},\\ G = A\vee B}} \left\{\bigvee_{x_\lambda \le A} pCl(B)(x_\lambda) \vee \bigvee_{y_\beta \le B} pCl(A)(y_\beta)\right\}.$$

Proposition 4.1. Let $\mathcal{T} : L^X \to \{0,1\}$ be an L-topology on X and $G \in L^X$. Then $\mathcal{PC}(G) = 1$ if and only if G is preconnected in $(X,\mathcal{T})$.

Theorem 4.1. Let $(X,\mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. Then

$$\mathcal{PC}(G) = \bigwedge_{\substack{G\wedge A \ne \underline{0}, G \wedge B \ne \underline{0},\\ G\wedge A \wedge B \ne \underline{0}, G \le A \vee B}} \{(\mathcal{T}_p(A'))' \vee (\mathcal{T}_p(B'))'\}.$$

Proof. $(\Rightarrow)$: $\mathcal{PC}(G)$

$$\begin{aligned}&= \bigwedge_{\substack{A,B \in L^X \backslash \{\underline{0}\},\\ G = A\vee B}} \Big\{\bigvee_{x_\lambda \le A} pCl(B)(x_\lambda) \vee \bigvee_{y_\beta \le B} pCl(A)(y_\beta)\Big\}\\ &= \bigwedge_{\substack{A,B \in L^X \backslash \{\underline{0}\},\\ G = A\vee B}} \Big\{\bigvee_{x_\lambda \le A}\ \bigwedge_{x_\lambda \not\le D \ge B} (\mathcal{T}_p(D'))' \vee \bigvee_{y_\beta \le B}\ \bigwedge_{y_\beta \not\le E \ge A} (\mathcal{T}_p(E'))'\Big\}\\ &= \bigwedge_{\substack{G\wedge A \ne \underline{0},\\ G\wedge B \ne \underline{0},\\ G \wedge A \wedge B = \underline{0},\\ G = A \vee B}} \Big\{\bigvee_{x_\lambda \le G \wedge A}\ \bigwedge_{\substack{x_\lambda \not\le D\\ D \ge G \wedge B}} (\mathcal{T}_p(D'))' \vee \bigvee_{y_\beta \le G \wedge B}\ \bigwedge_{\substack{y_\beta \not\le E\\ E \ge G\wedge A}} (\mathcal{T}_p(E'))'\Big\}\\ &\le \bigwedge_{\substack{G\wedge A \ne \underline{0}, G \wedge B \ne \underline{0},\\ G\wedge A \wedge B = \underline{0}, G = A \vee B}} \Big\{\bigvee_{x_\lambda \le G \wedge A} (\mathcal{T}_p(D'))' \vee \bigvee_{y_\beta \le G\wedge B} (\mathcal{T}_p(E'))'\Big\}\\ &= \bigwedge_{\substack{G\wedge A \ne \underline{0}, G \wedge B \ne \underline{0},\\ G\wedge A \wedge B = \underline{0}, G = A \vee B}} \Big\{(\mathcal{T}_p(D'))' \vee (\mathcal{T}_p(E'))'\Big\}\end{aligned}$$

$(\Leftarrow)$: Suppose that $\mathcal{PC}(G) \not\ge a$ where $a \in M(L)$. Then there exist A, $B \in L^X \backslash \{\underline{0}\}$ such that $G = A \vee B$ and $(\mathcal{P}(A,B))] \not\ge a$. By Lemma 3.3, there exist D, $E \in L^X$ such that $D \ge A$, $E \ge B$, $D \wedge B = E \wedge A = \underline{0}$ and $(\mathcal{T}_p(D'))' \vee (\mathcal{T}_p(E'))' \not\ge a$. Hence we have

$$\bigwedge_{\substack{G\wedge A \ne \underline{0}, G \wedge B \ne \underline{0},\\ G\wedge A \wedge B = \underline{0}, G \le A \vee B}} \{(\mathcal{T}_p(B'))' \vee (\mathcal{T}_p(A'))'\} \not\ge a.$$

Therefore

$$\mathcal{PC}(G) \ge \bigwedge_{\substack{G\wedge A \ne \underline{0}, G \wedge B \ne \underline{0},\\ G\wedge A \wedge B = \underline{0}, G \le A \vee B}} \{(\mathcal{T}_p(B'))' \vee (\mathcal{T}_p(A'))'\}$$

and this completes the proof. □

Corollary 4.2. Let $(X,\mathcal{T})$ be an L-fuzzy topological space. Then

$$\mathcal{PC}(\underline{1}) = \bigwedge_{\substack{A\neq\underline{0},A\wedge B=\underline{0},\\ A\vee B=\underline{1}}} \{(\mathcal{T}_p(A))' \vee (\mathcal{T}_p(B))'\}.$$

Theorem 4.3. For any $x_\lambda \in M(L^X)$, it follows that $\mathcal{PC}(x_\lambda) = 1$.

Proof. Straightforward. □

Theorem 4.4. For any $G \in L^X$, we have

$$\bigvee_{b\in M(L^X)} \mathcal{PC}(\bigvee (pCl(G))_{[b]}) \geq \mathcal{PC}(G).$$

Proof. Let $a \leq \mathcal{PC}(G)$ where $a \in M(L)$ and suppose that $\bigvee_{b\in M(L)} \mathcal{PC}(\bigvee (pCl(G))_{[b]}) \not\geq a$. Then $\mathcal{PC}(\bigvee (pCl(G))_{[a]}) \not\geq a$. By using Theorem 4.1, there exist A, $B \in L^X$ such that $(\bigvee (pCl(G))_{[a]}) \wedge A \neq \underline{0}$, $(\bigvee (pCl(G))_{[a]}) \wedge B \neq \underline{0}$, $(\bigvee (pCl(G))_{[a]}) \wedge A \wedge B = \underline{0}$, $\bigvee (pCl(G))_{[a]} \leq A \wedge B$ and $(\mathcal{T}_p(B'))' \vee (\mathcal{T}_p(A'))' \not\geq a$.

Since $(\bigvee (pCl(G))_{[a]}) \wedge A \neq \underline{0}$, there exists $x_\lambda \leq A$ such that $pCl(G)(x_\lambda) \geq a$. Furthermore, Since $(\bigvee (pCl(G))_{[a]}) \wedge A \wedge B = \underline{0}$, we have $x_\lambda \not\leq B$.

If $G \wedge A \neq \underline{0}$, then $G \leq \bigvee (pCl(G))_{[a]} \leq A \vee B$ we have $G \leq B$, hence it follows that

$$a \leq pCl(G)(x_\lambda) = \bigwedge_{x_\lambda \not\leq E \geq G} (\mathcal{T}_p(E'))' \leq (\mathcal{T}_p(B'))'$$

which is a contradiction. Analogously, we can prove $G \wedge B \neq \underline{0}$. Thus by $G \wedge A \neq \underline{0}$, $G \wedge B \neq \underline{0}$, $G \wedge A \wedge B = \underline{0}$, $G \leq A \vee B$, $(\mathcal{T}_p(B'))' \vee (\mathcal{T}_p(A'))' \not\geq a$ and Theorem 4.1, we know that $\mathcal{PC}(G) \not\geq a$, contradicting $\mathcal{PC}(G) \geq a$. It is proved that

$$\bigvee_{b\in M(L^X)} \mathcal{PC}(\bigvee (pCl(G))_{[b]}) \geq \mathcal{PC}(G).$$

□

Theorem 4.5. For any G, $H \in L^X$, we have

$$\mathcal{PC}(G \vee H) \geq (\mathcal{P}(G,H))' \wedge \mathcal{PC}(G) \wedge \mathcal{PC}(H).$$

Proof. Let $a \leq (\mathcal{P}(G,H))' \wedge \mathcal{PC}(G) \wedge \mathcal{PC}(H)$ where $a \in M(L)$ and suppose that $\mathcal{PC}(G\vee H) \not\geq a$. Then by using Theorem 4.1, there exist A, $B \in L^X$ such that $G \vee H) \wedge A \neq \underline{0}$, $(G \vee H) \wedge B \neq \underline{0}$, $(G \vee H) \wedge A \wedge B = \underline{0}$, $G \vee H \leq A \vee B$ and $(\mathcal{T}_p(B'))' \vee (\mathcal{T}_p(A'))' \not\geq a$. Since $(G \vee H) \wedge A \neq \underline{0}$, we have $G \wedge A \neq \underline{0}$ and $H \wedge A \neq \underline{0}$. Suppose that $G \wedge A \neq \underline{0}$ (The case of $H \wedge A \neq \underline{0}$ is analogous). Then we have $G \wedge B = \underline{0}$, otherwise if $G \wedge B \neq \underline{0}$, then by $G \wedge A \neq \underline{0}$, $G \wedge B \neq \underline{0}$, $G \wedge A \wedge B = \underline{0}$, $G \leq A \vee B$ and $(\mathcal{T}_p(B'))' \vee (\mathcal{T}_p(A'))' \not\geq a$, we know that $\mathcal{PC}(G) \not\geq a$, which is a contradiction. In this case by $(G \vee H) \wedge B \neq \underline{0}$ we know that $H \wedge B \neq \underline{0}$. Analogously we can prove $H \wedge A = \underline{0}$. Thus by $G \vee H \leq A \vee B$ we can obtain that $G \leq A$ and $H \leq B$. Hence by $G \leq A$, $H \leq B$, $G \wedge B = H \wedge A = \underline{0}$, $(\mathcal{T}_p(B'))' \vee (\mathcal{T}_p(A'))' \not\geq a$ and Lemma 3.3, we have $(\mathcal{P}(G,H))' \not\geq a$, which is a contradiction. This shows that $\mathcal{PC}(G \vee H) \geq a$ and this completes the proof. □

Corollary 4.6. Let $(X,\mathcal{T})$ be an L-fuzzy topological space and G, $H \in L^X$. If $A \wedge B \neq \underline{0}$, then

$$\mathcal{PC}(G \vee H) \geq \mathcal{PC}(G) \wedge \mathcal{PC}(H).$$

Theorem 4.7. Let $(X,\mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. Then

$$\mathcal{PC}(G) = \bigwedge_{x_\lambda, y_\beta \leq G} \bigvee \{\mathcal{PC}(D_{x_\lambda,y_\beta}) : x_\lambda, y_\beta \leq D_{x_\lambda,y_\beta} \leq G\}.$$

Proof. Suppose that $\bigwedge_{x_\lambda,y_\beta\leq G} \bigvee \{\mathcal{PC}(D_{x_\lambda,y_\beta}) : x_\lambda, y_\beta \leq D_{x_\lambda,y_\beta} \leq G\} \geq a$ where $a \in M(L)$. Take a $x_\lambda \leq G$ fixed. Then for any $y_\beta \leq G$, there exists $D_{x_\lambda,y_\beta} \in L^X$ such that x_λ, $y_\beta \leq D_{x_\lambda,y_\beta} \leq G$ and $\mathcal{PC}(D_{x_\lambda,y_\beta}) \geq a$. Let $D_{x_\lambda} = \bigvee_{y_\beta\leq G} D_{x_\lambda,y_\beta}$. Then $D_{x_\lambda} = G$ and $\bigwedge_{y_\beta\leq G} D_{x_\lambda,y_\beta} \neq \underline{0}$. By using Corollary 4.6, we have $\mathcal{PC}(G) = \mathcal{PC}(D_{x_\lambda}) \geq \bigwedge_{y_\beta\leq G} \mathcal{PC}(D_{x_\lambda,y_\beta}) \geq a$. This shows that

$$\mathcal{PC}(G) \geq \bigwedge_{x_\lambda, y_\beta \leq G} \bigvee \{\mathcal{PC}(D_{x_\lambda,y_\beta}) : x_\lambda, y_\beta \leq D_{x_\lambda,y_\beta} \leq G\}.$$

Since

$$\mathcal{PC}(G) \leq \bigwedge_{x_\lambda, y_\beta \leq G} \bigvee \{\mathcal{PC}(D_{x_\lambda,y_\beta}) : x_\lambda, y_\beta \leq D_{x_\lambda,y_\beta} \leq G\}$$

is clear, then we have

$$\mathcal{PC}(G) = \bigwedge_{x_\lambda, y_\beta \leq G} \bigvee \{\mathcal{PC}(D_{x_\lambda,y_\beta}) : x_\lambda, y_\beta \leq D_{x_\lambda,y_\beta} \leq G\}.$$

□

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