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Preservice teachers' Key Developmental Understandings (KDUs) for fraction multiplication

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The concept of pedagogical content knowledge (PCK) has been developed and expanded to identify essential components of mathematical knowledge for teaching (MKT) by Ball and her colleagues (2008). This study proposes an alternative perspective to view MKT focusing on key developmental understandings (KDUs) that carry through an instructional sequence, that are foundational for learning other ideas. In this study we provide constructive components of KDUs in fraction multiplication by focusing on the constructs of '*three-level-of-units structure*' and '*recursive partitioning operation*'. Expecially, our participating preservice elementary teacher, Jane, demonstrated that recursive partitioning operations with her length model played a significant role as a KDU in fraction multiplication.

Key Words: key developmental understanding, fraction multiplication, recursive partitioning operation, three-levels-of-units structure

I. INTRODUCTION

Teacher knowledge related to student learning was conceptualized by Shulman (1986) as being comprised of a series of interconnected knowledge types. In particular, Shulman's concept of pedagogical content knowledge has led to ongoing discussion about the unique knowledge that teachers need in order to support student learning. Since its proposal, researchers have built from Shulman's proposition for different kinds of knowledge that teachers need in order to be successful in their practice. In mathematics education, Ball and colleagues (Ball, 1993; Ball, Thames, & Phelps, 2008, Hill, Rowan, & Ball, 2005) have sought to define the content knowledge teachers need that the 'man on the street' does not in their work around mathematical knowledge for teaching (MKT). MKT is comprised of specialized content knowledge (purely mathematical, yet specific to the work of teaching) and pedagogical content knowledge (knowledge of how students learn content or of ways to teach specific topics), each of

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which is further categorized into three sub-knowledge elements. This expansion highlights increasing awareness that teachers need not only content knowledge that many educated adults have, but also knowledge specialized for teaching mathematics to students.

In recent research, Silverman and Thompson (2008) suggested another perspective to view mathematical knowledge for teaching (MKT). Rather than focusing on identifying the mathematical reasoning, insight, understanding and skill needed in teaching mathematics as in Ball's MKT, they focused on the mathematical understandings "that play into a network of ideas that does significant work in students' reasoning" (Thompson, 2008, p.1). They stipulated a teacher's MKT as being grounded in a personally powerful understanding of particular mathematical concepts and as being transformed into 'pedagogically' powerful understanding. It implies that to build a personally powerful understanding could be the first and crucial step for teachers to construct their MKT.

Aligned with Silverman's and Thompson's views of teachers' MKT, Simon(2006) designated personally powerful understanding as key developmental understanding (KDU). He characterized a KDU as a conceptual advance or a "change in students' ability to think about and/or perceive particular mathematical relationships" (p. 362). Understanding can be developed through connections between different internal representations and being able to reason among the representations. However, greater numbers of representations available will not necessarily result in any great restructuring of our understanding. Being able to include certain key representations for a concept in our understanding, and being able to reason between this and other representations that we already have, will result in a greater restructuring of our understanding (Barmby, Harries, Higgins, & Suggate, 2009). Simon's idea of KDU can be related to this view of 'understanding.' In the fraction example, Simon indicated understanding that equal partitioning creates specific units of quantity could be a key understanding in fraction because it allowed students to conceive of and act with fractions in powerful ways. Nevertheless, not on fraction alone, on any domain-specific subjects few studies have been attempted to address fine-grained analysis concerning KDUs for supporting mathematics teachers to develop their MKT. Therefore, a qualitative study to deeply look into one teacher's KDU would shed light on mathematics education research with regard to mathematics teachers' needed knowledge in their classrooms, which initiated this study of investigating preservice teachers' KDUs for fraction multiplication.

II. RESEARCH QUESTIONS

Since Simon(2006) did not explicitly mention KDUs in fraction multiplication, our interest in preservice teachers′ key developmental understanding for teaching fraction multiplication led to the following research questions:

1. What could be KDUs in fraction multiplication?
2. How did the participant preservice teachers conceive of and solve fraction multiplication problems in terms of KDUs?

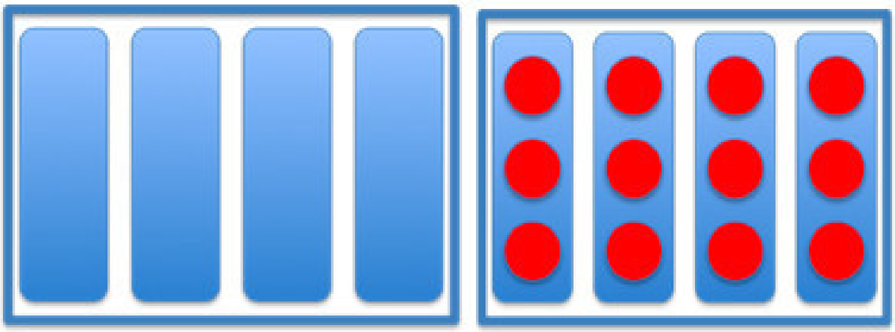
III. KEY DEVELOPMENTAL UNDERSTANDING IN FRACTION MULTIPLICATION

Steffe (2002) studied children's reasoning of fraction using TIMA stick, the computer software which was designed to serve the experiment which allowed students to construct various area or length models on the screen without having to worry about drawing accurately. He used a teaching experiment methodology[3] to explore children's thinking, In their study, learner's ability to abstract composite unit was important in fraction. For example, students might think of 3 candies as one group of three candies and three groups of one candy. In his study of fraction multiplication, he revealed that students should coordinate at least two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit. Steffe (2003) defined it as a units-coordinating operation (which was built when studying students′ construction of whole number multiplication knowledge), and stressed the recursive partitioning based on a units-coordinating operation as a fundamental operation in fraction multiplication.

We will use '1/3 x 1/4' to illustrate how the recursive partitioning operation functions in learners' solving fraction multiplication problem (from an operational view) and differences in their abilities to do recursive partitioning based on their unitizing operations. Most of all, it is important to discuss the distinction between two and three levels of units. A child who forms two levels of units can understand a whole number, say, seven not only as seven separate units (one level of unit) but also as one group of seven conceived as a single entity by unitizing (the whole group is a second level of unit). Steffe (2003) states that such a child have formed composite units. He further elaborates that a child produces three levels of units by nesting composite units within composite units through interiorization and coordination of those composite units. For instance, a child who constructed a three-levels-of-units structure can assimilate an array of 12 candies as a single group of 12 (the whole group is one level), being

3) Although Steffe's & Thompson's teaching experiment methodology is derived from Piaget's clinical interview, it is distinguished from the clinical interview in that the former involves experimentation with the ways and means of influencing students' mathematical knowledge (Steffe & Thompson, 2000).

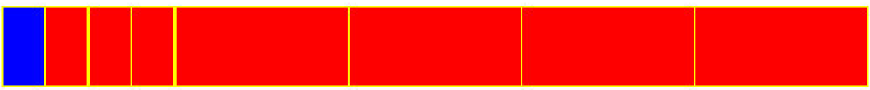
composed of four units (a second level), where each of the four units is also a composite unit composed of three separate units (a third level) (See Figure 1).



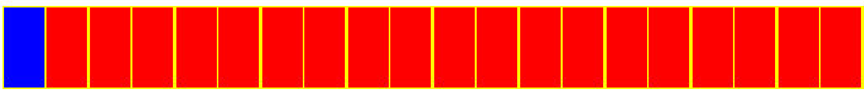
[Figure 1] (a) The number '4' understood with two levels of units. (b) The number '12' understood with three levels of units.

With the distinction between two levels of units (or composite units) and three levels of units in our mind, consider the situation where you are asked to quantify the result of taking 1/4 of 1/5. You might first construct the requested quantity by partitioning a unit length into five parts (reasoning with two levels of units), narrowing down your attention to the first part of the five pieces, and re-partitioning the first part into four smaller pieces (reasoning with two levels of units second time) (See Figure 2a). Students would determine the size of the resulting piece in more than one way. Students may iterate the resulting piece 20 times and count to see if their copies exhaust the whole bar (Figure 2b). In addition to the previous reasoning with two levels of units twice, this solution involves another reasoning with two levels of units, which in total, such students used three times because one unit is no more than conceived as one unit containing 20 twentieths.

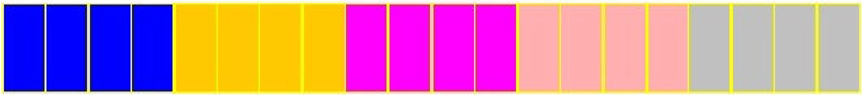
Alternatively, students might recursively partition by subdividing each of the remaining fifths into four pieces (Figure 2c). In contrast to the previous solution, the recursive partitioning operation involves reasoning with three levels of units in which four of the smaller units are nested within each mid-level unit (which is each one-fifth bar in the Figure 2c), and so there must be 4 x 5 = 20 units in the whole unit.



(a) Constructing part of part



(b) Using iteration with two levels of units



(c) Using recursive partitioning with three levels of units.
[Figure 2] Figuring out the quantity of 1/4 of 1/5

IV. METHODOLOGY

1. Context and Participants

To acquire initial certification from the state of Georgia to teach children from prekindergarten to grade 5, preservice teachers need to take two method and content courses from department of mathematics education. We have selected two pairs of preservice teacher participants from the first method session of which the primary focus was on learning to teach number and operations involving prenumber concepts, whole numbers, and rational numbers. The course was taught by a professor who had taught the course more than thirty years at the time. The first author was a teaching assistant of the course.

Among 30 preservice elementary teachers who were enrolled in the program, four of them volunteered to participate in our study. Although most of the students were juniors in the class, three of the four volunteers were sophomore. The participants were not from mathematics or mathematics education department but from elementary education department. It is wellknown that US elementary teachers are very insecure of their mathematical knowledge which often prevents them from expressing their thinking out loud. We deliberately paired them because we knew the pairs were acquainted to each other. We also did not want them to get embarrassed working alone surrounded by two cameras and interviewers knowing that most elementary teachers feel insecure of their mathematical knowledge.

Each pair participated in three interviews on May 2007. There were two parts to the first interview: 1) 7 self-reporting questions of their beliefs about mathematics, teacher, and fraction knowledge; 2) 3 tasks on fractions. The second interviews were conducted three days after the first interviews and the questions were formed based on the first interviews.

All meetings were videotaped using two cameras - one to capture the participants' facial expressions, and the other to capture their hand gestures and written works. These two sources were then mixed to create a restored view of the event (Hall,2000). From the restored view, we created lesson graphs, a document that parsed the lessons and the interviews into episodes. From the lesson graphs, we noted emerging characteristics of the preservice teachers' knowledge as we went through each video episode. Then we coded the lesson graphs using an emergent set of categories to generate hypotheses and united them into comprehensive accounts. Interviews were all conducted by the first author and were semi-structured in terms that interviews were pre-planned, but follow-up interview questions were improvised during the interview for the clarifications (Zazkis & Hazzan, 1999)

For the present study, we focus on one preservice elementary teacher, Jane, whose problem solving activities were exemplary to indicate key developmental understanding in fraction multiplication.

V. FINDINGS

1. Recursive Partitioning Afforded by Flexibilities with Three levels of Units

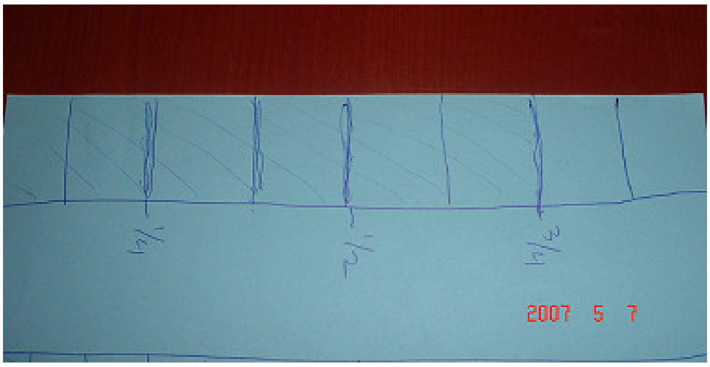
Before we asked Jane fraction multiplication tasks, we gave her the following proble m[4]:

Given that six pieces of candies are equally divided into three-fourths of the bar, how many candies will there be in the whole bar?

In order to understand if recursive partitioning was available, we asked the question to Jane. As soon as she listened to the problem, she started to draw a long strip on the paper and partitioned it into four parts, and labeled 1/4, 1/2, and 3/4 respectively. After about 10 seconds, she subdivided each of the three fourths into two parts. When we asked her why she did that, she said that she tried to make three fourths into six pieces as the problem stated, and she knew that each fourth needs to be equally subdivided if necessary. So the only way was to subdivide each fourth into two pieces. After she divided each fourth into two pieces, she automatically divided the last fourth into two parts and stated that there were two parts between 3/4 and 4/4 since she had two parts for each of the fourth[5] up to 3/4 at the time. Thus the answer was eight pieces of a candy (See Figure 3)

4) This problem was adapted from L. P. Steffe (Professor of the University of Georgia)'s problem sets that were used in his class for preservice mathematics teachers.

5) We will call Jane's drawing 'length model' from now on because she seemed to consider her drawing as a line or a string although it had a rectangular two-dimensional shape.

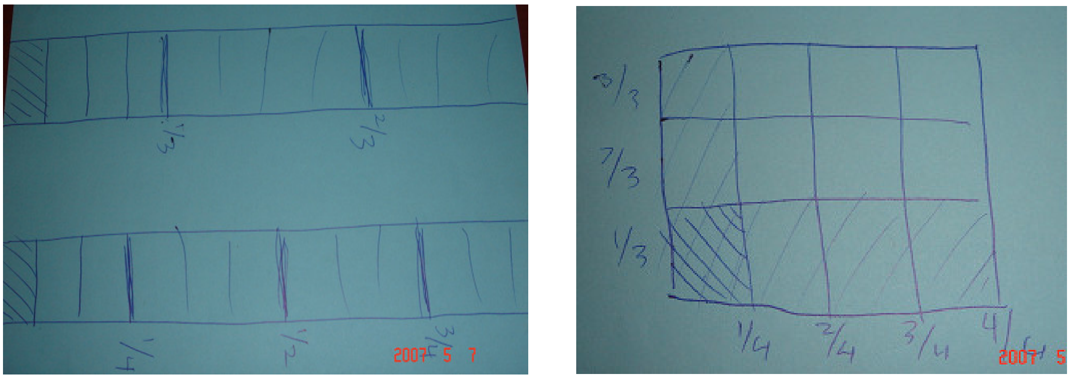


[Figure 3] Jane's length model for finding the number of candies in the whole bar

Based on Jane's problem solving activities for the question, we assumed that she was afforded by her flexibility with three levels of units. In other words, she was able to visualize a whole of four fourths where each of the fourths was composed of two parts as a unit structure. In addition, the fact that she quickly sub-divided each of the three fourths into two parts indicated her use of recursive partitioning, which was another evidence of flexibilities with three levels of units.

Since she was able to produce three levels of units with a length model with the candy bar problem, we wondered how she would approach to model "1/3 times 1/4." Finally, Jane drew us the fraction multiplication situation in three ways: two with length models and one with an area model. As soon as she drew two length models as Figure 4a, she said,

> On the first line, I split the one strip into thirds, and then once I had one third I split by fourths cause you are kind of saying one third of one fourth. So I found where one third was then I found one fourth just about one third which is you kept doing all your divisions it will end up being one twelfth. And then I also did it other way. I checked myself, and I wanted to find one third of one fourth I guess the opposite way. So I split into one fourth and then found one third of that one piece lined up in the right way. I did rest of this (inaudible) to fill the strip.



[Figure 4] (a) Jane's recursive partitioning operation to figure out “1/3 times 1/4”
(b) Jane's use of overlapping strategy with an area model[6)]

Jane used a part of part interpretation of fraction multiplication and described two alternative ways (using commutative law of multiplication) to represent 1/3 times 1/4: one as 1/4 of 1/3 (see the upper model of Figure 4a) and the other as 1/3 of 1/4 (see the lower model of Figure 4a). Jane was explicitly using three levels of units for both upper and lower models in Figure 4a. For the upper model, the largest unit of quantity was the unit of one, the second level of unit was three thirds, and the third level of unit was four fourths of a third in each of three thirds of one whole. She used recursive partitioning in providing the answer '1/12' because she knew that there were four fourths of one-third in the rest two thirds each when she shaded the one fourth of the first one-third in the upper model. The same analysis can be applied to the lower model except that she used four fourths as the second level and three thirds of each of four fourths of one whole as the third level of unit. Although she produced three-levels-of-units structures with her length model, she relied on the overlapping strategy for the area model. She said,

> So...on one axis (referring to the length in Figure 4b model), you would split it by one third and then...on the other one (referring to the width in Figure 4b model) you would split it by one fourth ...but then (it took about 5 seconds) one third times one fourth is... it is just harder for me to understand what one third times one fourth is doing it this way I guess. I don't understand why the overlap works but it shows us one twelfth. One third of one forth makes more sense to me than just knowing how to do this way I guess.

Even though she could use her part of part concept of fraction with the area model as she did with the length models, and took a fourth of a third of a whole (three levels

6) It seems clear that Jane dealt the rectangular shape like a two-dimensional figure in 4(b), which led us to call her drawing as 'area model'.

of units), or vice versa, she did not understand how or why the overlapping strategy worked. One possible explanation for such result is that she watched Shelly[7]'s using overlapping strategy prior to her attempt to model the problem with the area model, and it might have influenced her way of modeling the problem because it gave her the answer 1/12. The second reason seems to apply to both Jane and Shelly. They used overlapping strategies because they somehow felt the necessity to see the quantities of 1/3, 1/4, and 1/12 and the overlapping strategy enabled them to express those quantities in their drawing. Even though the strategy did not make sense to them, they accepted it as a possible way if it provided them with the correct answer. The fact that they accepted the overlapping strategy as a feasible way to solve fraction multiplication with the area model indicated their preference towards getting a correct answer in mathematics and it took priority over their putting value about knowing mathematics in a meaningful and conceptual way. To look more deeply into their overlapping strategy, the first author asked both of them why they used overlapping, and gave them a more complicated fraction multiplication problem, 1/3 × 2/5 to probe their thinking. Surprisingly to build an area model for the more complex problem Jane adapted her conception of fraction multiplication as "part of part of whole" as in her modeling with a length model. As stated, we focus on describing Jane's developing knowledge of fraction multiplication under the following section.

2. Construction of Three Levels of Units with an Area Model

Jane did not like to use an area model since she could not see the way overlapping strategy worked. She was treating the strategy as if it was a correct method just because it gave her with the correct answer. However, she kept questioning herself how and why the overlapping part of crossing two fractions provided her the answer of the problem. As she struggled so hard, the first author guided her by asking the following question,

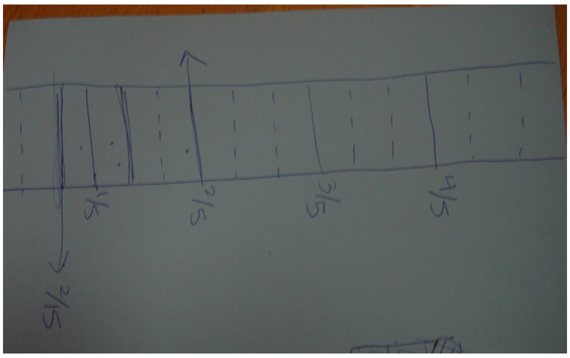
> ('Int' stands for the interviewer.)
> Int: Do you think what you did for using length model is kind of compatible...I mean the reasoning is kind of compatible to what you did for area model?

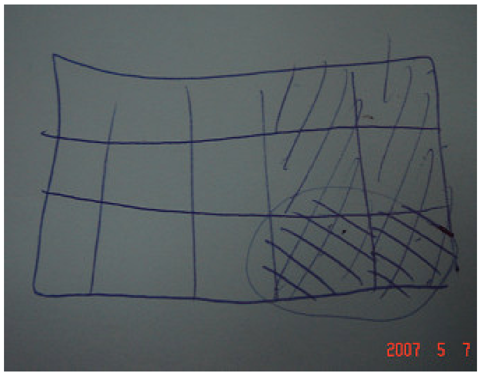
As soon as the interviewer asked her the question, she answered,

> Jane: I think that is what I am trying to figure out. Cause this reasoning makes so much sense to me. So I was trying to figure out how this could also represent one third of one fourth or one fourth of one third the way that I think about it this way.

7) Shelly was another participating preservice teacher paired with Jane in the interview.

Jane's effort was not in vain since she could provide nice account for representing 1/3 × 2/5 using part of part conception of fractions multiplication with both area and length models.





[Figure 5] Jane's models for showing "1/3 of 2/5" (a) recursive partitioning with length model (b) recursive partitioning with area model

She gave us the description of the length model and the area model respectively (See Figure 5a and 5b). With the length model, as seen in Figure 5a, she first divided the strip into five parts and labeled each part as 1/5, 2/5, 3/5, and 4/5 respectively. After she put an arrow above the 2/5, she divided the two-fifths into three parts by estimation. To figure out how much the third of the two-fifths was, she subdivided each third in two fifths into two parts, and the rest of the three fifths into three parts, then she told me that a third of two-fifths (which was the question she was asked) was two-fifteenths. That is, she produced the three-levels-of-units structure where the whole was the largest unit, five fifths as the mid-level unit, and fifteen-fifteenths as the smallest unit. Her flexibility with three levels of units afforded her to solve the following area model problem as well. We will talk about Jane's method by comparing it with her pair partner, Shelly's one.

Shelly used overlapping as her algorithm to get outcome for the product. In other words, she cross-partitioned a piece into three by five pieces and shaded two-thirds horizontally and one-third vertically, then she stated the overlapping part of the two was the answer to the problem 1/3 of 2/5. When we asked her why the method worked for the case, she said it was the way she was taught in school.

Jane suggested a different way to solve the problem. As seen in Figure 5b, first she split a bar into five columns and shaded two-fifths of the whole rectangle. Then she divided the bar horizontally into three parts and shaded only the third of the two-fifths that she had already shaded, and continued to state that one-third of two-fifths was two-fifteenths. In contrast to Shelly, she only double-shaded on two shaded cells which were one-third of the six shaded area. We think that her effort to make a connection between the reasoning for the area model and the length model enabled her to conduct such activities emerging at this episode. It would mean that she was aware of the fifteen smaller units as a partitioned whole and two of the fifteen pieces as one-third of two-fifths at the same time.

VI. CONCLUSION

In the article, we elaborate the construct of KDU in fraction multiplication by focusing on the constructs of 'three-level-of-units structure' and 'recursive partitioning operation'. The preservice elementary teacher, Jane, demonstrated recursive partitioning operations with her length model, which would play a role as a KDU in fraction multiplication. However, the fact that Jane accepted the overlapping strategy as a feasible way to solve fraction multiplication with the area model warns us that preference for getting a correct answer using algorithmic methods without any reason might lead to ignoring a KDU in a particular topic, which finally results in preservice teachers' teaching the topic in their classroom away from the KDU. Fortunately, Jane's putting value on meaningful and conceptual mathematics kept her from accepting the overlapping strategy as an appropriate way to model the fraction multiplication, and made her continuously question why the strategy would work for the area model. When the first author scaffolded her by asking if the overlapping strategy was a correct way to model the fraction multiplication, and if she could apply the way she used to model the length model, she was able to reorganize her knowledge of modeling fraction multiplication by adapting her part of part conception of fraction multiplications with the length model to the area model.

Even though studies (Olive, 1999; Olive & Steffe, 2002; Steffe, 2002, 2003, 2004; Steffe & Olive, 2010) are abound in research on children's knowledge of fractions with conceptual units and operations, a few (e.g., Behr et al., 1997) was done in the field of preservice teacher education. Most studies of preservice teachers knowledge of fraction multiplication have not looked at teachers' knowledge any closer than simply saying that teachers' reasoning about fraction multiplication is insufficient or based on procedural knowledge. We have showed in this study how one preservice teacher (Jane) could reason with conceptual units accompanied by partitioning operations in order to represent her thinking on the paper using the length or area model. Although it can not be applied or generalized to other cases in the same way only by the observations of one preservice teacher's mathematical behaviors, we argue that the idea of examining teachers' knowledge in terms of KDU, especially close analysis of operating on conceptual units, could be an important contribution to the field of mathematics teacher education. This study is only a beginning step toward that understanding. Again, note that this study considers only one teacher and that we make no claims of generalizability of the patterns of reasoning observed. We certainly believe that teachers may follow a variety of paths in learning to reason about these concepts. In this study, we considered only those ways of reasoning that emerged from our participant.

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예비교사의 분수 곱셈을 위한 '발달에 핵심적인 이해'에 관한 연구

이수진[8]·신재홍[9]

초 록

'교수학적 내용 지식(Pedagogical Content Knowledge)'의 개념은 '교수활동을 위한 수학 내용 지식(Mathematical Knowledge for Teaching: MKT)'의 핵심 요소들을 밝히기 위한 연구의 일환으로 많은 연구자들에 의해 확장, 발전되어 왔다. 특히 Ball(1993)은 교수활동에서 가시적으로 드러나는 교사가 알아야 할 수학에 관해 초점을 맞추어 왔는데, 본 연구에서는 MKT를 바라보는 또 하나의 대안적 관점으로서 '발달에 핵심적인 이해 (Key Developmental Understanding: KDU)'라는 개념을 제안하고 있다. Simon (2006)은 KDU란 일련의 교수활동을 통해 수행되고 다른 수학적 아이디어의 학습에 기초가 되는 이해 또는 개념이며, '등분할 조작'이 분수 개념의 KDU가 될 수 있음을 주장하였다. 본 연구에서는 예비 초등교사와의 면담을 통하여 '반복 분할 조작'과 '세 수준의 단위 구조'의 구성이 분수 곱셈에 대한 KDU가 될 수 있음을 제시하고 있다.

주요용어 : 발달에 핵심적인 이해, 분수 곱셈, 반복 분할 조작, 세 수준의 단위 구조

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