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복합적층판의 고유진동수에 대한 하중 크기의 영향
Influence of Loading Sizes on Natural Frequency of Composite Laminates

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ABSTRACT

A method of calculating natural frequencies corresponding to the modes of vibration of beams and tower structures with irregular cross sections and arbitrary boundary conditions was developed. The result is compared with that of the beam theory. Finite difference method is used for this purpose. The influence of the D_{22} stiffness on the natural frequency is rigorously investigated. In this paper, the relation between the applied loading sizes and the natural frequency of vibration of some structural elements is presented. The results of application of this method to steel bridge and reinforced concrete slab bridge by using specially orthotropic plate theory is presented.

요 지

임의의 단면과 지점을 갖고 임의의 하중을 받는 보나 탑의 진동해석 방법이 발표된 바가 있다. 이러한 진동해석을 위하여 처짐의 영향을 고려한 다양한 방법이 검토되었다. 본 연구에서 얻은 결과를 보 이론과 비교하였다. 이러한 목적으로 본 논문에서는 유한차분법을 사용하였다. 고유진동수에 대한 D_{22} 탄성계수의 영향을 철저히 검토하였다. 본 논문에서는 구조부재의 고유진동수와 적용 하중의 크기에 대한 관련성을 연구하였으며 그 결과를 제시하였다. 본 논문에서는 특별직교이방성 판이론 이용하여 강교량과 철근콘크리트 슬래브 교량에 적용하여 을 해석하였으며 그 결과를 제시하였다.

Key Words : specially orthotropic plate theory(특별직교이방성 판이론), influence of loading sizes(하중크기의 영향), natural frequencies(고유진동수), finite difference method(유한차분법)

1. INTRODUCTION

The advanced composite materials can be used economically and efficiently in broad civil engineering applications when standards and processes for analysis, design, fabrication, construction and quality control are established. The problem of deteriorating infrastructures is very serious in our country.

The advanced composite materials can be effectively used for repairing such structures. Because of the advantages of these materials, such repair job can fulfill two purposes :

(1) Repair of existing damage caused by corrosion, impact, earthquake, and others.
(2) Reinforcing the structure against anticipated future situation which will require increasing the load beyond the design parameters used for this structure.

Before making any decision on repair work, reliable non-destructive evaluation is necessary. One of the dependable methods is to evaluate the in-situ stiffness of the structure by means of obtaining the natural frequency. By comparing the in-situ stiffness with the one obtained at the design stage, the degree of damage can be estimated rather accurately.

The reinforced concrete slab can be assumed as a [0, 90, 0]r type specially orthotropic plate as a close approximation, assuming that the influence of B_{16}, B_{26}, D_{16} and D_{26} stiffness are negligible. Many of the bridge and building floor systems, including the girders and cross beams, also behave as similar specially orthotropic plates. Such plates are subject to the concentrated mass/masses in the form of traffic loads, or the test equipments such as the accelerator in addition to their own masses. Analysis of such problems is usually very difficult.

The most of the design engineers for construction has academic background of bachelors degree. Theories for advanced composite structures are too difficult for such engineers and some simple but accurate enough methods are necessary.

Most of the civil structures are large in sizes and the numbers of laminae are large, even though the thickness to

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length ratios are small enough to allow to neglect the transverse shear deformation effects in stress analysis. For such plates, the fiber orientations given above behave as specially orthotropic plates and simple formulas developed by the reference [Kim 1995, Han & Kim 2001, 2003] can be used.

Most of the bridge and building slabs on girders have large aspect ratios. For such cases further simplification is possible by neglecting the effect of the longitudinal moment terms (M_x) on the relevant partial differential equations of equilibrium [Han & Kim, 2001]. In this paper, the result of the study on the subject problem is presented. Even with such assumption, the specially orthotropic plate with boundary conditions other than Navier or Levy solution types, or with irregular cross section, or with nonuniform mass including point masses, analytical solution is very difficult to obtain. Numerical method for eigenvalue problems are also very much involved in seeking such a solution [Han & Kim, 2001, 2003, 2009, Han & Suk, 2010, Kim, 1995].

The method of vibration analysis used is the one developed by the author. He developed and reported a simple but exact method of calculating the natural frequency of beam and tower structures with irregular cross sections and attached mass/masses. This method has been extended to two dimensional problems with several types of given conditions and has been reported at several international conferences.

2. METHOD OF ANALYSIS

The equilibrium equation for the specially orthotropic plate is :

$$D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}=q(x,y) \qquad (1)$$

where $D_1=D_{11}, D_2=D_{22}, D_3=D_{12}+2D_{66}$

The assumptions needed for this equation are :

(1) The transverse shear deformation is neglected.
(2) Specially orthotropic layers are arranged so that no coupling terms exist, i.e. $B_{ij}=0, (\ )_{16}=(\ )_{26}=0$.
(3) No temperature or hygrothermal terms exist.

The purpose of this paper is to demonstrate, to the practicing engineers, how to apply this equation to the slab systems made of plate girders and cross beams.

In case of an orthotropic plate with boundary conditions other than Navier or Levy solution type, or with irregular cross section, or with nonuniform mass including point masses, analytical solution is very difficult to obtain. Numerical methods for eigenvalue problems are also very much involved in seeking such a solution. Finite difference method (F.D.M) is used in this paper. The resulting linear algebraic equations can be used for any cases with minor modifications at the boundaries, and so on.

The problem of deteriorating infrastructures is very serious all over the world. Before making any decision on repair work, reliable non-destructive evaluation is necessary. One of the dependable methods is to evaluate the in-situ stiffness of the structure by means of obtaining the natural frequency. By comparing the in-situ stiffness with the one obtained at the design stage, the degree of damage can be estimated rather accurately.

The basic concept of the Rayleigh method, the most popular analytical method for vibration analysis of a single degree of freedom system, is the principle of conservation of energy ; the energy in a free vibrating system must remain constant if no damping forces act to absorb it. In case of a beam, which has an infinite number of degree of freedom, it is necessary to assume a shape function in order to reduce the beam to a single degree of freedom system(Clough 1995). The frequency of vibration can be found by equating the maximum strain energy developed during the motion to the maximum kinetic energy. This method, however, yields the solution either equal to or larger than the real one. Recall that Rayleigh's quotient ≥ 1 (Kim, 1995). For a complex beam, assuming a correct shape function is not possible. In such cases, the solution obtained is larger than the real one.

Design engineers need to calculate the natural frequencies of such element but obtaining exact solution to such problems is very much difficult. Pretlove reported a method of analysis of beams with attached masses using the concept of effective mass. This method, however, is useful only for certain simple types of beams. Such problems can be easily solved by presented method.

A simple but exact method of calculating the natural frequency corresponding to the first mode of vibration of beam and tower structures with irregular cross sections and attached mass/masses was developed and reported by Kim in 1974. This method consists of determining the deflected mode shape of the member due to the inertia force under resonance condition. Beginning with initially "guessed" mode shape, "exact" mode shape is obtained by the process similar to iteration. Recently, this method was extended to two dimensional problems including composite laminates, and has been applied to composite plates with various boundary conditions with/without shear deformation effects and reported at several international conferences including the Eighth

Structures Congress (1990) and Fourth Materials Congress (1996) of American Society of Civil Engineers.

This method is used for vibration analysis in this paper.

A natural frequency of a structure is the frequency under which the deflected mode shape corresponding to this frequency begins to diverge under the resonance condition. From the deflection caused by the free vibration, the force required to make this deflection can be found, and from this force, resulting deflection can be obtained. If the mode shape as determined by the series of this process is sufficiently accurate, then the relative deflections (maximum) of both the converged and the previous one should remain unchanged under the inertia force related with this natural frequency. Vibration of a structure is a harmonic motion and the amplitude may contain a part expressed by a trigonometric function. Considering only the first mode as a start, the deflection shape of a structural member can be expressed as

$$w = W(x,y)F(t) = W(x,y)\sin\omega t \tag{2}$$

where

W : maximum amplitude

ω : circular frequency of vibration

t : time

By Newton's second law, the dynamic force of the vibrating mass, m, is

$$F = m\frac{\partial^2 w}{\partial t^2} \tag{3}$$

Substituting (2) into this,

$$F = -m\,(\omega)^2\, W\sin\omega t \tag{4}$$

In this expression, ω and W are unknowns. In order to obtain the natural circular frequency ω, the following process is taken.

The magnitudes of the maximum deflection at a certain number of points are arbitrarily given as

$$w(i,j)(1) = W(i,j)(1) \tag{5}$$

where (i,j) denotes the point under consideration. This is absolutely arbitrary but educated guessing is good for accel erating convergence. The dynamic force corresponding to this (maximum) amplitude is

$$F(i,j)(1)=m(i,j)\{\omega(i,j)(1)\}^2 w(i,j)(1) \tag{6}$$

The “new” deflection caused by this force is a function of F and can be expressed as

$$w(i,j)(2)=f\{m(k,l)\{\omega(i,j)(1)\}^2\, w(k,l)(1)\}= \sum^{k.l} \triangle(i,j,k,l)\{m(k,l)\{\omega(i,j)(1)\}^2\, w(k,l)(1)\} \tag{7}$$

where △ is the deflection influence surface. The relative (maximum) deflections at each point under consideration of a structural member under resonance condition, w(i,j)(1) and w(i,j)(2), have to remain unchanged and the following condition has to be held :

$$w(i,j)(1)\ /\ w(i,j)(2)=1. \tag{8}$$

From this equation, w(i,j)(1) at each point of (i,j) can be obtained, but they are not equal in most cases. Since the natural frequency of a structural member has to be equal at all points of the member, i.e., w(i,j) should be equal for all (i,j), this step is repeated until sufficient equal magnitude of w(i,j) is obtained at all (i,j) points.

However, in most cases, the difference between the maximum and the minimum values of w(i,j) obtained by the first cycle of calculation is sufficiently negligible for engineering purposes. The accuracy can be improved by simply taking the average of the maximum and the minimum, or by taking the value of w(i,j) where the deflection is the maximum. For the second cycle, w(i,j)(2) in

$$w(i,j)(3) = f\{m(i,j)\,[\,\omega(i,j)(2)\,]^2 w(i,j)(2)\} \tag{9}$$

the absolute numerics of w(i,j)(2) can be used for convenience.

In case of a structural member with irregular section including composite one, and non-uniformly distributed mass, regardless of the boundary conditions, it is convenient to consider the member as divided by finite number of elements. The accuracy of the result is proportional to the accuracy of the deflection calculation.

For practical design purposes, it is desirable to simplify the vibration analysis procedure. One of the methods is to neglect the weight of the structural elem- ent. The effect of neglecting the weight (thus mass) of the plate is studied as follow. If a weightless plate is acted upon by a concentrated load, $P=N\cdot q\cdot a\cdot b$, the critical circular frequency of this plate is

$$\omega_n = \sqrt{\frac{g}{\delta_{st}}} \tag{10}$$

where δ_{st} is the static deflection.

Similar result can be obtained by the use of Eqs. (7) and

(8).

$$[\omega(i,j)]^2 = \frac{1}{[\triangle(i,j,i,j) \cdot \frac{P(i,j)}{g}]} \qquad (11)$$

where,

$$P(i,j) = N \cdot q \cdot a \cdot b \qquad (12)$$

In case of the plate with more than one concentrated loads,

$$[\omega(i,j)]^2 = \frac{1}{[\sum^{k,l} \triangle(i,j,k,l) \cdot \frac{P(k,l)}{g}]} \qquad (13)$$

If we consider the mass of the plate as well as the concentrated loads,

$$\begin{aligned} & w(i,j)(1) = w(i,j)(2) \\ & = \{ \sum^{k,l} \triangle (i,j,k,l) \cdot m(k,l) \cdot w(k,l)(1) \\ & + \sum^{m,n} \triangle (i,j,m,n) \cdot \frac{P(m,n)}{g} \cdot w(m,n)(1) \} \\ & \times [\omega(i,j)(1)]^2 \end{aligned} \qquad (14)$$

where (m,n) is the location of the concentrated loads. The effect of neglecting the weight of the plate can be found by simply comparing Eqs. (13) and (14).

3. FINITE DIFFERENCE METHOD

Since no reliable analytical method is available for the subject problem, F.D.M. is applied to the governing equation of the special orthotropic plates.

The number of the pivotal points required in the case of the order of error $\triangle^2$, where $\triangle$ is the mesh size, is five for the central differences of the fourth order single derivative terms. This makes the procedure at the boundaries complicated. In order to solve such problem, the three simultaneous partial differential equations of equilibrium with three dependent variables, w, Mx, and My, are used instead of Eq.(1) for the bending of the specially orthotropic plate.

$$D_{11}\frac{\partial^2 Mx}{\partial x^2} + 4D_{66}\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^2 My}{\partial y^2} = -q(x,y) + kw(x,y) \qquad (15)$$

$$M_x = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} \qquad (16)$$

$$M_y = -D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} \qquad (17)$$

If F.D.M. is applied to these equations, the resulting matrix equation is very large in sizes, but the tridiagonal matrix calculation scheme used by Kim[Kim, 1965, 1967] is very efficient to solve such equations.

In order to confirm the accuracy of the F.D.M., [A/B/A]r type laminate with aspect ratio of a/b=1m/1m=1 is considered. The material properties are :

$E_1 = 67.36\ GPa$, $E_2 = 8.12\ GPa$,
$G_{12} = 3.0217\ GPa$,
$\nu_{12} = 0.272$, $\nu_{21} = 0.0328$,

The thickness of a ply is 0.005m. As the r increases, B_{16}, B_{26}, D_{16}, and D_{26} decrease and the equations for special orthotropic plates can be used. For simplicity, it is assumed that $A = 0^o$, $B = 90^o$ and r=1. Then D_{22}=18492 N-m.

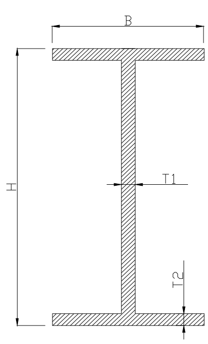
Since one of the few efficient analytical solutions of the special orthotropic plate is Navier solution, and this is good for the case of the four edges simple supported, F.D.M. is used to solve this problem and the result is compared with the Navier solution.

The mesh size is Δx=a/10=0.1m, Δy=b/10=0.1m. The deflection at (x, y), under the uniform load of 100N/m², the origin of the coordinates being at the corner of the plate, is obtained, and the ratio of the Navier solution to the F.D.M solution is 1.005~1.00028.

4. NUMERICAL EXAMPLES

4.1 Steel Bridges

The steel bridges under consideration is as given in Figure 1 and 2.



Es=200,000MPa
υ=0.3
S=2m

	Girder(mm)	Cross Section (mm)
H_L	500	300
B_L	200	150
$T1_L$	15	10
$T2_L$	20	18

Fig. 1 Cross Section

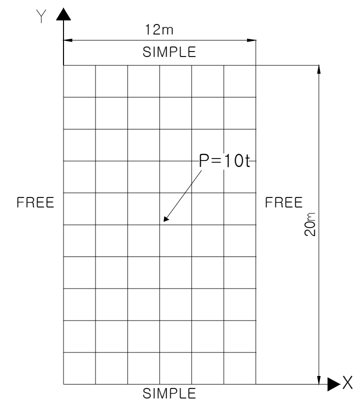


Fig. 2 Girder, Beam and Loading Point

The stiffnesses are given in Table 1.

Table 1 Stiffness

D_{ij} (N・m)	Plate	Beam
D_{11}	101,199,927.65	101,199,927.65
D_{22}	21,757,837.94	0.00

Analysis is carried out and the result is given in Table 2.

Table 2 Deflection at the center (m)

	Plate	Beam	Plate/Beam
δ(m)	0.6765E-01	0.1646E+00	2.43

4.2 Reinforced Concrete Slab Bridges

The bridge is as shown in Figure 3 and 4.

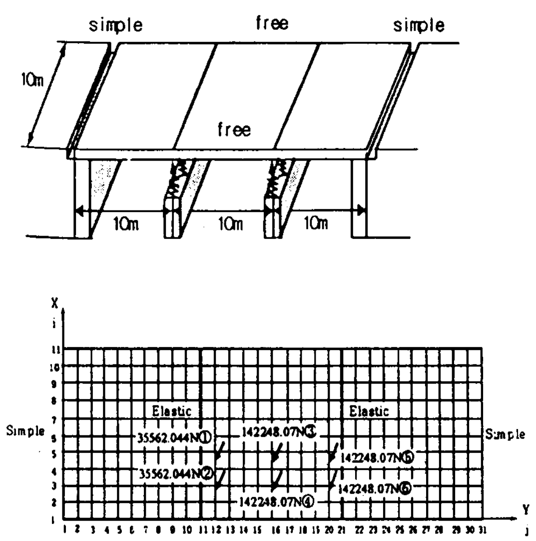


Fig. 3 Concrete Slab Bridge and Loading Points

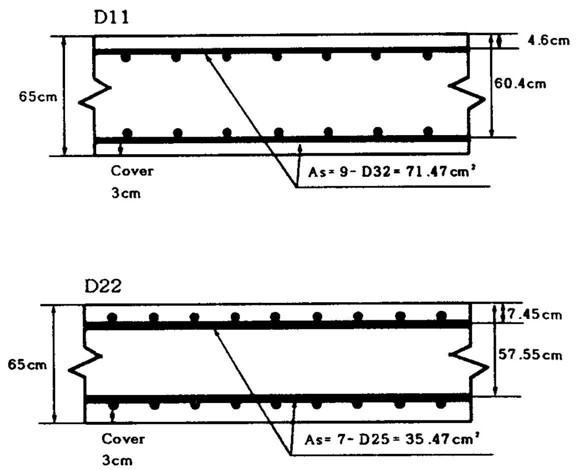


Fig. 4 Cross Section of the Slab with Unit Width.

Figure 4 shows the cross section of the slab with unit width.

$f_{ck} = 21MPa$ and $E_c = 15000\sqrt{f_{ck}} = 21.3GPa$.

Poissons ratio $\nu_{12} = \nu_{21} = 0.18$ for concrete.

The stiffness and deflections are given in Tables 3 and 4.

Table 3 Flexural Stiffnesses (N・m)

Stiffness	N・m
D_{11}	323,428,383.7
D_{22}	151,828,300.8
D_{12}	90,690,632.4
D_{66}	206,573,097.2

Table 4 Deflections at Wheel Loading Points (m)

Load Point	Deflection (m)
1	0.2955E-03
2	0.2458E-03
3	0.2300E-02
4	0.2054E-02
5	0.4155E-03
6	0.3504E-03

5. CONCLUSION

A natural frequency of a structure is the frequency under which the deflected mode shape corresponding to this frequency begins to diverge under the resonance condition. From the deflection caused by the free vibration, the force required to make this deflection can be found and from this force, the resulting deflection can be obtained. For practical design purposes, it is desirable to simplify the vibration analysis procedure. One of the methods is to neglect the weight of the beam.

In this paper, the relation between the applied loading sizes and the natural frequency of vibration of some structural elements is presented. Many practicing engineers get confused on such relations. It is hoped that this paper gives some guideline to such practicing engineers. The purpose of this paper is to demonstrate, to the practicing engineers, how to apply the specially orthotropic plate theory to the slab systems made of plate girders and cross beams.

In this paper, results of analysis for design of both plate girders and reinforced concrete slabs for bridges are presented. It is concluded that the existing design methods with beam strip concept gives us too far off results from the safe and economic bridges.

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