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A RELATIONSHIP BETWEEN THE LIPSCHITZ CONSTANTS APPEARING IN TAYLOR'S FORMULA

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Abstract. Taylor's formula is a powerful tool in analysis. In this study, we assume that an operator is m-times Fréchet-differentiable and satisfies a Lipschitz condition. We then obtain some Taylor formulas using only the Lipschitz constants. Applications are also provided.

1. Introduction

Taylor's formula has been used for a long time as a powerful tool in analysis to study the convergence of iterative processes but also in other areas [1]–[4].

In this study we assume that operator F is m-times (m a natural number) Fréchet-differentiable on a non-empty subset D of a Banach space X with values in a Banach space Y. Furthermore, we assume that operator $F^{(m)}$ is Lipschitz continuous on D. Then, the constant is used to relate the corresponding Lipschitz constants for operators $F^{(i)}, i = 1, 2, ..., m$. Applications are also provided in this study.

2. Taylor Formulas

We need the following results on Taylor's formula for m-Fréchet-differentiable operators.

Theorem 2.1. *Let $G : D \subseteq X \to Y$ be a m-times ($m \in N$) Fréchet-differentiable operators defined on a non-empty subset D of a Banach space X with values in a Banach space Y.*

Assume: (a) there exist a constant $e_{m+1} > 0$, and a convex subset D_0 of D such that for all $x, y \in D_0$

$$\|G^{(m)}(x) - G^{(m)}(y)\| \leq e_{m+1}\|x - y\|_X. \tag{2.1}$$

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Then, for all $x, y \in D_0$, the following estimate holds:

$$\|G(x) - G(y) - \sum_{i=1}^{m} \frac{G^{(i)}(y)}{i!}(x-y)^i\|_Y \leq \frac{e_{m+1}}{(m+1)!}\|x-y\|_X^{m+1}. \tag{2.2}$$

(b) If $D_0 = U(x_0, R) = \{x \in X : \|x - x_0\| \leq R\} \subseteq D$ for some $x_0 \in D$, $R > 0$, and

$$\|G^{(m)}(x) - G^{(m)}(x_0)\| \leq e_{m+1}^0 \|x - x_0\|_X \quad \textit{for some } e_{m+1}^0 > 0 \tag{2.3}$$

holds true on D_0, then

$$\|G^{(k)}(x)\| \leq e_k \quad k = 0, 1, ..., m, \tag{2.4}$$

and

$$\|G^{(k-1)}(x) - G^{(k-1)}(y)\| \leq e_k \|x - y\|_X, \quad k = 1, ..., m, \tag{2.5}$$

where, $e_m = \|G^{(m)}(x_0)\| + e_{m+1}^0 R$,

$$e_k = \|G^{(k)}(x_0)\| + e_{k+1}^0 R, \quad k = 0, 1, ..., m-1. \tag{2.6}$$

(c) Under hypotheses of part (b) the following hold for all $x, y \in D_0$, and $k = 1, 2, ..., m$:

$$\|G(x) - G(y) - \sum_{i=1}^{k} \frac{G^{(i)}(y)}{i!}(x-y)^i\|_Y \leq \frac{e_{k+1}}{(k+1)!}\|x-y\|_X^{k+1}. \tag{2.7}$$

Proof. (a) Let us denote by $\alpha \in Y$ the element given by

$$\alpha = G(x) - G(y) - \sum_{i=1}^{m} \frac{G^{(i)}(y)}{i!}(x-y)^i. \tag{2.8}$$

It is well known [3], [4] that there exists $\beta \in L(Y, \mathbf{R})$ the space of bounded linear operators from Y into $\mathbf{R}$ so that

$$\|\beta\|_X = 1, \quad \textit{and} \ \ \beta(\alpha) = \|\alpha\|_Y. \tag{2.9}$$

It then follows from (2.8), and (2.9) that

$$\beta(\alpha) = |\beta(G(x)) - \beta(G(y)) - \sum_{i=1}^{m} \frac{\beta(G^{(i)}(y)(x-y)^i)}{i!}|. \tag{2.10}$$

Let us define on $[0, 1]$ the real function

$$\gamma(\theta) = \beta(G(y + \theta(x - y))). \tag{2.11}$$

In view of the convexity of D_0, $y+\theta(x-y)\in D_0$ if $x,y\in D_0$. That is function γ is well defined. It follows from the existence of the Fréchet-derivatives of operator G that functions

$$\gamma^{(k)}(\theta)=\beta(G^{(k)}(y+\theta(x-y))(x-y)^k) \quad k=0,1,...,m \tag{2.12}$$

are well defined.

Using the integral form of Taylor's formula [3], [4], we have:

$$\gamma(1)=\gamma(0)+\sum_{i=1}^{m}\gamma^{(i)}(0)+\frac{1}{(m-1)!}\int_0^1\gamma^{(m)}(\theta)(1-\theta)^{p-1}d\theta. \tag{2.13}$$

We also need the estimates:

$$\int_0^1(1-\theta)^{m-1}d\theta=\frac{1}{m}, \tag{2.14}$$

and

$$\int_0^1\theta(1-\theta)^{m-1}d\theta=\int_0^1(1-\theta)^{m-1}d\theta-\int_0^1(1-\theta)^{m}d\theta=\frac{1}{m}-\frac{1}{m+1}=\frac{1}{m(m+1)}. \tag{2.15}$$

We then have:

$$\begin{aligned}\|\alpha\|_Y &= \tfrac{1}{(m-1)!}\left|\int_0^1\gamma^{(m)}(\theta)(1-\theta)^{m-1}d\theta-\int_0^1\gamma^{(m)}(0)(1-\theta)^{m-1}d\theta\right|\\ &\le \tfrac{1}{(m-1)!}\int_0^1|\gamma^{(m)}(\theta)-\gamma^{(m)}(0)|(1-\theta)^{m-1}d\theta,\end{aligned} \tag{2.16}$$

but

$$\begin{aligned}|\gamma^{(m)}(\theta)-\gamma^{(m)}(0)| &= |\beta(G^{(m)}(y+\theta(x-y))(x-y)^m)-\beta(G^{(m)}(y))(x-y)^m|\\ &= |\beta([G^{(m)}(y+\theta(x-y))(x-y)^m)-(G^{(m)}(y)])(x-y)^m|\\ &\le \|\beta\|\|G^{(m)}(y+\theta(x-y))(x-y)^m)-(G^{(m)}(y)\|\|x-y\|_X^m\end{aligned} \tag{2.17}$$

and consequently,

$$\begin{aligned}\|\alpha\|_Y &= \tfrac{e_{m+1}}{(m-1)!}\|x-y\|_X^{m+1}\int_0^1\theta(1-\theta)^{m-1}d\theta\\ &\le \tfrac{e_{m+1}}{(m-1)!m(m+1)}\|x-y\|_X^{m+1}.\end{aligned} \tag{2.18}$$

That completes the proof of part (a).

(b) It follows from (2.3)

$$\|G^{(m)}(x)\|\le\|G^{(m)}(x_0)\|+e_{m+1}^0\|x-x_0\|=e_m. \tag{2.19}$$

By Langrange's theorem applied to $G^{(m-1)}:D_0\to L(X^{m-1},Y)$ we get

$$\|G^{(m-1)}(x)-G^{(m-1)}(y)\|\le\|G^{(m)}(y+\theta(x-y))\|\|x-y\|_X \quad \theta\in(0,1). \tag{2.20}$$

If $x, y \in D_0$, and $\theta \in (0,1)$ we get $y + \theta(x-y) \in D_0$ and consequently

$$\|G^{(m)}(y + \theta(x-y))\| \leq e_m,$$

and

$$\|G^{(m-1)}(x) - G^{(m-1)}(y)\| \leq e_m \|x - y\|_X.$$

Estimates (2.3) and (2.4) are obtained by continuing the same way.

(c) This part follows immediately from parts (a), and (b).

That completes the proof of the theorem. □

3. Applications

The first application involves the most popular iterative process which is Newton's method.

We need a result on majorizing sequences for Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad (n \geq 0),\ (x_0 \in D) \tag{3.1}$$

for generating a sequence $\{x_n\}$ approximating a solution $x^\star$ of equation

$$F(x) = 0. \tag{3.2}$$

Lemma 3.1 ([3, Lemma 1.1.2, p. 14])**.** *Assume:*
there exist constants $L_0 \geq 0$, $L \geq 0$, with $L_0 \leq L$, and $\eta \geq 0$, such that:

$$q_0 = \overline{L}\ \eta \begin{cases} \leq \dfrac{1}{2}, & if \quad L_0 \neq 0, \\ < \dfrac{1}{2}, & if \quad L_0 = 0, \end{cases} \tag{3.3}$$

where,

$$\overline{L} = \frac{1}{8} \left(L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L} \right). \tag{3.4}$$

Then, sequence $\{t_k\}$ $(k \geq 0)$ given by

$$t_0 = 0, \quad t_1 = \eta, \quad t_{k+1} = t_k + \frac{L\ (t_k - t_{k-1})^2}{2\ (1 - L_0\ t_k)} \qquad (k \geq 1), \tag{3.5}$$

is well defined, nondecreasing, bounded from above by $t^{\star\star}$ and converges to its unique least upper bound $t^\star \in [0, t^{\star\star}]$, where

$$t^{\star\star} = \frac{2\ \eta}{2 - \delta}, \tag{3.6}$$

$$1 \leq \delta = \frac{4\ L}{L + \sqrt{L^2 + 8\ L_0\ L}} < 2 \quad \text{for } L_0 \neq 0. \tag{3.7}$$

Moreover, the following estimates hold:

$$L_0\, t^\star \leq 1, \tag{3.8}$$

$$0 \leq t_{k+1} - t_k \leq \frac{\delta}{2}\,(t_k - t_{k-1}) \leq \cdots \leq \left(\frac{\delta}{2}\right)^k \eta, \quad (k \geq 1), \tag{3.9}$$

$$t_{k+1} - t_k \leq \left(\frac{\delta}{2}\right)^k (2\, q_0)^{2^k-1}\,\eta, \quad (k \geq 0), \tag{3.10}$$

$$0 \leq t^\star - t_k \leq \left(\frac{\delta}{2}\right)^k \frac{(2\, q_0)^{2^k-1}\,\eta}{1-(2\, q_0)^{2^k}}, \quad (2\, q_0 < 1), \quad (k \geq 0). \tag{3.11}$$

We also need the result related to Lemma 3.1.

Lemma 3.2. *Let $m \geq 2$ be a natural number; α_i non-negative numbers, $i = 2,...,m+1$, $\eta > 0$ and define functions $P, \overline{L}_0, \overline{L}, H$ on $(0,+\infty)$ by*

$$P(r) = \frac{\alpha_{m+1}}{(m+1)!} r^{m+1} + \frac{\alpha_m}{m!} r^m + \ldots + \frac{\alpha_2}{2!} r^2 - r + \eta, \tag{3.12}$$

$$\overline{L}_0(r) = \frac{1+p'(r)}{r} = \frac{\alpha_{m+1}}{m!} r^{m-1} + \frac{\alpha_m}{(m-1)!} r^{m-2} + \ldots + \alpha_2, \tag{3.13}$$

$$\overline{L}(r) = P''(r) = \frac{\alpha_{m+1}}{(m-1)!} r^{m-1} + \frac{\alpha_m}{(m-2)!} r^{m-2} + \ldots + \alpha_2 \tag{3.14}$$

and

$$H(r) = (\overline{L}(r) + 4\overline{L}_0(r) + \sqrt{\overline{L}^2(r) + 8\overline{L}_0(r)\overline{L}(r)})\eta - 4. \tag{3.15}$$

Assume:

$$H(\eta) < 0. \tag{3.16}$$

Then, function H has a unique positive zero r_0 such that

$$r_0 > \eta. \tag{3.17}$$

Moreover, for a fixed $r^\star \in (\eta, r_0]$, set

$$L_0 = \overline{L}_0(r^\star), \quad \text{and} \quad L = \overline{L}(r^\star). \tag{3.18}$$

Then, the conclusions of Lemma 3.1 hold for iteration $\{t_n\}$.

Proof. Function H is well defined, since $\overline{L}_0$, and $\overline{L}$ are positive functions. Moreover, H is increasing, since $H'(r) > 0$ for $r > 0$. It then follows from (3.12) that $H(r) > 0$ for sufficiently large $r > 0$. The existence, uniqueness of r_0 follows from the intermediate value theorem, and the monotonicity of function H, respectively. Using the definition of L_0, L, and H, we deduce that estimate (3.17) holds.

That completes the proof of lemma. □

Hypothesis (3.16) can be replaced by the weaker, and more general

$$\text{Function } H \text{ has a minimal positive zero } r_0. \tag{3.19}$$

We can show the following semilocal convergence result for Newton's method (3.1).

Proposition 3.3. *Let $F : D \subseteq X \to Y$ be a m-times Fréchet-differentiable operator defined on a non-empty, open and convex subset D of a Banach space X with values in a Banach space Y. Assume there exists $x_0 \in D$, $e_{m+1} > 0$ such that*

$$F'(x_0)^{-1} \in D; \tag{3.20}$$

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq e_{m+1}\|x - y\| \quad for\ all\ x, y \in D; \tag{3.21}$$

hypotheses of Lemmas 2.1, 2.2 hold for

$$G = F'(x_0)^{-1}F, \quad \alpha_i = e_i \quad i = 1, 2, \ldots, m+1; \tag{3.22}$$

$$\overline{U}(x_0, \alpha^\star) \subseteq D \tag{3.23}$$

and

$$\alpha^\star = t^\star \text{ or } t^{\star\star} < r_0; \tag{3.24}$$

where $t^\star, t^{\star\star}$ are given in Lemma 3.1, and r_0 is in Lemma 3.2.

Then, sequence $\{x_n\}$ generated by Newton's method (3.1) is well-defined, remains in $\overline{U}(x_0, \alpha^\star)$ for all $n \geq 0$, and converges to a unique solution $x^\star \in \overline{U}(x_0, \alpha^\star)$ of equation $F(x) = 0$.

Moreover, the following estimates hold for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{3.25}$$

and

$$\|x_n - x^\star\| \leq t^\star - t_n. \tag{3.26}$$

Proof. Simply repeat the proof of Theorem by Argyros in [2], but use (2.2) for $x = x_{n+1}, y = x_n$ to obtain:

$$\|F'(x_n)^{-1}F'(x_0)\| \leq \frac{1}{1 - L\|x_n - x_0\|} \leq \frac{1}{1 - Lt_n}, \tag{3.27}$$

$$\|F'(x_0)^{-1}F'(x_n)\| \leq \frac{L}{2}\|x_{n+1} - x_n\|^2 \leq \frac{L}{2}(t_{n+1} - t_n)^2, \tag{3.28}$$

and by (3.1)
(3.29)

$$\|x_{n+1} - x_n\| \leq \|F'(x_n)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_n)\| \leq \frac{L(t_{n+1} - t_n)^2}{2(1 - L_0 t_n)} = t_{n+1} - t_n.$$

That completes the proof of the proposition. □

As a second application, consider $X = Y = \mathbf{R}$, $D = \overline{U}(0, 1)$, and define function F on D by

$$F(x) = e^x. \tag{3.30}$$

Then, for any $m \geq 1$,

$$a_{m+1} = e, \quad \|F'(x_0)^{-1}F^{(m)}(x_0)\| = 1, \quad a^0_{m+1} = e - 1, \quad a_m = 1 + (e - 1)R, \tag{3.31}$$

and

$$a_k = 1 + eR, \quad k = 0, 1, 2, \ldots, m - 1. \tag{3.32}$$

Estimates (2.2) and (2.7) can now be obtained with these choices.

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