J. KSIAM Vol.15, No.3, 201-222, 2011

OPTIMAL HOMOTOPY ASYMPTOTIC METHOD SOLUTION OF UNSTEADY SECOND GRADE FLUID IN WIRE COATING ANALYSIS

REHAN ALI SHAH[1*], S. ISLAM[1*], A. M. SIDDIQUI[2] AND T. HAROON[1]

[1]DEPARTMENT OF MATHEMATICAL SCIENCES, COMSATS INSTITUTE OF INFORMATION TECHNOLOGY, H-8/1, ISLAMABAD, PAKISTAN
E-mail address : mmrehan79@yahoo.com, saeed@comstas.edu.pk, tahiraharoon@comstas.edu.pk

[2]DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, YARK CAMPUS, EDECOMB 1703, USA.
E-mail address: ams5@psu.edu

ABSTRACT. In the present work, the mathematical model of wire coating in a straight annular die is developed for unsteady second grade fluid in the form of partial differential equation. The Optimal Homotopy Asymptotic Method (OHAM) is applied for obtaining the solution of the model problem.. This method provides us a suitable way to control the convergence of the series solution using the auxiliary constants which are optimally determined.

KEYWORDS: Unsteady flow, Second grade fluid, Wire coating, straight annular die, Optimal Homotopy Asymptotic Method

1. INTRODUCTION

Interest in the study of non-Newtonian fluids has been mainly motivated by their importance in most of the problems arising from engineering practice and chemical industry. In non-Newtonian fluids the non-linear relation between the stress and the strain developed the non-linearity in equations. The exact solutions for these equations have rare in the literature.

The particular class of non-Newtonian fluids for which the exact solution is reasonably possible is the class of viscoelastic fluids, which was first introduced by Rivlin and Ericksen [1]. For creeping flow Rajagopal [2] established the exact solution, and for unidirectional flow Rajagopal [3] gives the exact solution. Hayat et al. [4, 5] and Siddiqui at el. [6] extended this idea to periodic

Received by the editors April 27, 2011; Revised July 23, 2011; Accepted in revised form August 22, 2011.
2000 *Mathematics Subject Classification*. 93B05.
Key words and phrases. Unsteady flow, Second grade fluid, Wire coating, Optimal Homotopy Asymptotic Method
*Corresponding author

flows. Rajagopal and Gupta [7] also discussed the exact flow between the rotating parallel plates. We extend this idea to the problem of wire coating in cylindrical die with second order fluid.

Wire coating is used for the purpose of high and low voltage and protection against corrosion. The process is performed by dragged the wire in coating unit filled with molten polymer. The experimental set-up of wire coating process is given in Fig. 1. The uncoated wire unwinds at the payoff reel, firstly passes through a straightener, secondly through a preheater, and then a cross head die where the wire meets the melt polymer coming from the extruder and is coated. The refined product passes through a cooling water trough, a capstan and a tester and finally, the take-up reel wound the coated wire on the rotating reel.

Wire coating is an important chemical process in which different types of polymer is used. The coating of wire depends on geometry of the die, the viscosity of the fluid, the temperature of the wire and polymer used for coating the wire.

Han and Rao [8] discussed the Rheology of wire coating extrusion. Akter and Hashmi [9, 10] have studied wire coating using power law fluid and investigated the effect of the change in viscosity. Siddiqui et al. [11] studied the wire coating extrusion in a pressure-type die in flow of third grade fluid. Fenner and Williams [12] carried out an analysis of the flow in the tapering section of a pressure type die. Sajjid et al. [13] studied the wire coating with Oldroyd 8- constant fluid using the Homotopy Analyses Method (HAM), and give the solution for velocity field in the form of series. Mitsoulis [14] have studied fluid flow and heat transfer in wire coating.

In this paper, the new mathematical model arises in the study of wire coating for unsteady incompressible second grade fluid in cylindrical die is solved by Optimal Homotopy Asymptotic Method (OHAM) [15-16]. In a series of papers Marinca et al. [17-19] and Islam et al. [20-21] have not only applied this method to nonlinear differential equations but have shown that it is reliable and powerful tool than other perturbation tools for non linear differential equations.

Recently, S. Iqbal et al. [24] have applied this method to partial differential equation for solution of the Klein-Gordon equations. We use this idea for the solution of partial differential equation arising in wire coating analysis and give some related examples to our problem for stability measurements. According to best of our knowledge this study has not been previously investigated in wire coating process.

The plan of the paper is as follows: Section 2 develops the fundamental governing equations of the unsteady second grade fluid flow between wire and die. Section 3 gives the formulation of the problem. Section 4 describes the basic idea of OHAM and Section 4.1 is reserved for the solution of the problem. In Section 5 some examples related to our problem are solved using OHAM. Results and discussion are given in Section 6. Finally, the conclusion is made in Section 7.

2. BASIC EQUATIONS

Basic equations governing the flow of an incompressible fluid neglecting the thermal effects are:

$$\nabla \cdot \underline{u} = 0 , \tag{2.1}$$

$$\rho \frac{D\underline{u}}{Dt} = div\underline{T} + \rho \underline{f} , \tag{2.2}$$

where $\underline{u}$ is the velocity vector of the fluid, $\underline{T}$ the Cauchy stress tensor, ρ the constant density, $\underline{f}$ the body force per unit mass and $\frac{D}{Dt}$ is the material derivative.

For second grade fluid the stress tensor $\underline{T}$ is defined as

$$\underline{T} = -p\underline{I} + \mu \underline{A}_1 + \alpha_1 \underline{A}_2 + \alpha_2 \underline{A}_1, \tag{2.3}$$

in which p is the pressure, $\underline{I}$ the identity tensor, μ the coefficient of viscosity of the fluid, α_1, α_2 are the normal stress moduli and $\underline{A}_1, \underline{A}_2$ are the line kinematic tensors defined by

$$\underline{A}_1 = (\nabla \underline{u}) + (\nabla \underline{u})^T, \tag{2.4}$$

$$\underline{A}_2 = \frac{D\underline{A}_1}{Dt} + \underline{A}_1 (\nabla \underline{u}) + (\nabla \underline{u})^T \underline{A}_1. \tag{2.5}$$

3. PROBLEM FORMULATION

Let us consider an incompressible second grade fluid flow in straight annular die in wire coating process. The geometry of wire coating in a die is shown in Fig. 2 in which R_w and R_d are the radii of the wire and die respectively, where the wire and die are concentric. At time $t = 0^+$ the wire is oscillated and translated in its plane in a stationary die. The coordinate system is chosen at the centre of the wire, in which the axial direction z is taken in the direction of the fluid flow due to the oscillation and translation of wire in that direction, where r is taken perpendicular to the fluid flow.

Boundary conditions corresponding to the cosine oscillation of the boundary are:

$$\text{At } r = R_w, \quad w = U_w(1 + a\cos\omega t), \qquad \forall \; t \ge 0$$

$$\text{and at } r = R_d, \quad w = 0, \quad \forall \; t \ge 0, \tag{3.1}$$

where a is amplitude and ω is frequency of oscillation of wire.

Initial condition

$$w = 0, \quad \text{at } t = 0, \; R_w \le r \le R_d, \tag{3.2}$$

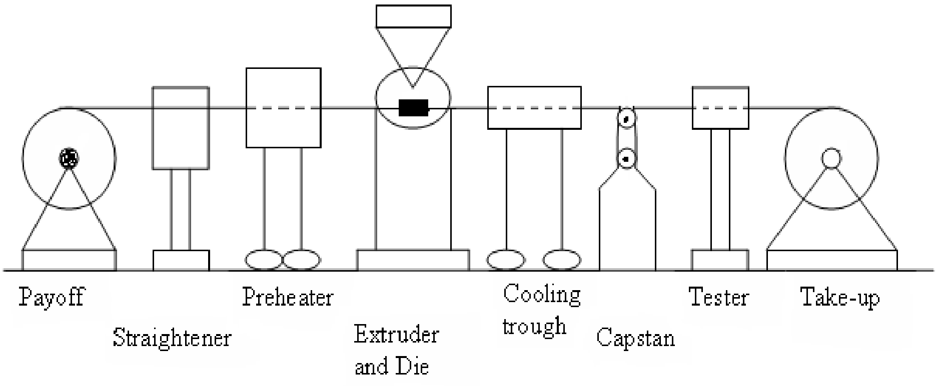

FIGURE 1. A typical wire coating line.

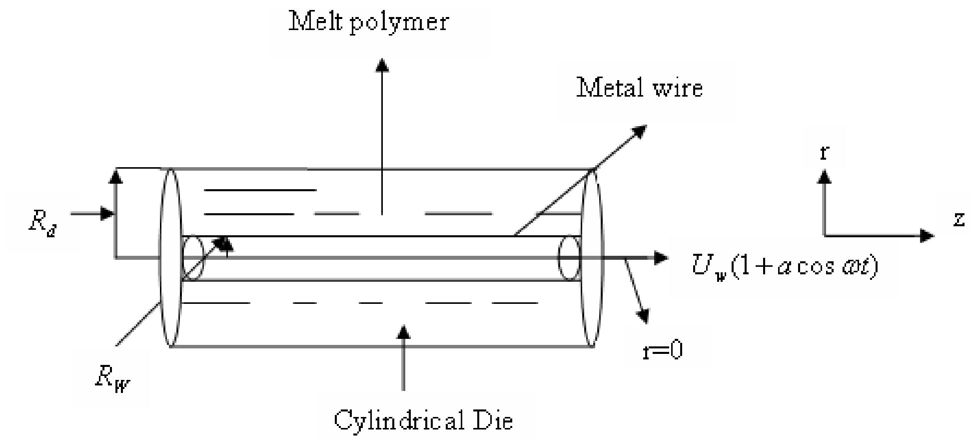

FIGURE 2. Schematic profile of wire coating in a straight annular die.

For the problem under consideration, we shall seek the velocity field and pressure distribution as

$$\underline{u} = [0,0,w(r,t)], \qquad p = p(r,t). \tag{3.3}$$

Under the consideration of velocity field given in equation (3.3), the continuity equation (2.1) is satisfied identically.
On substituting equations (2.3-2.5) and (3.3) into the balance of momentum (2.2), one obtains component form momentum equation in the absence of body forces as:

$$0 = -\frac{\partial p}{\partial r} + \alpha_1 \left(\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^2 + 4\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} \right) + \alpha_2 \left(\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} \right), \tag{3.4}$$

$$\frac{\partial p}{\partial \theta} = 0, \tag{3.5}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \right) + \frac{\alpha_1}{\rho}\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \right), \tag{3.6}$$

where $\alpha = \frac{\alpha_1}{\rho}, \nu = \frac{\mu}{\rho}$.

Assume that there is no pressure gradient along the axial direction and the flow is only due to drag of wire. Hence, equation (3.6) with $\partial p / \partial z = 0$ yields:

$$\frac{\partial w}{\partial t} = \nu \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \right) + \alpha \frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \right). \tag{3.7}$$

The volume flow rate of coating is

$$Q = \pi U_w \left(R_c^{\,2} - R_w^{\,2} \right), \tag{3.8}$$

where R_c is the radius of the coated wire. On the other hand at the cross-section, within the die, the volume flow rate is

$$Q = \int_{R_W}^{R_d} 2\pi\, r\, w(r) dr . \tag{3.9}$$

The thickness of the coated wire can be obtained from equations (3.8) and (3.9).
The force on the total wire surface in the die is

$$\underline{F} = 2\pi R_w L S_{rz}\big|_{r=R_W} . \tag{3.10}$$

Equation (3.7) with the appropriate boundary conditions (3.1) is solved with the help of OHAM to obtain the approximate solution for velocity field. The pressure distribution function can be then obtained from equation (3.4).

4. BASIC IDEA OF OHAM

Here, we present the basic idea of OHAM, for this consider the boundary value problem of the form

$$K(w(r,t)) + h(r,t) = 0, \qquad r \in \Omega, \qquad B\left(w, \frac{\partial w}{\partial t}\right) = 0, \quad r \in \Im, \tag{4.1}$$

where K is a differential operator and B is a boundary operator, $w(r,t)$ is the unknown function, r and t denotes the spatial and time independent variables, respectively, $\Im$ is the boundary of the domain Ω and $h(r,t)$ is a known analytic function. In general form the operator K can be written as

$$K = L + N, \tag{4.2}$$

where L is a linear operator and N is a nonlinear operator.

According to OHAM, one can construct a Homotopy $\Psi(r,t;p) : \Omega \times [0,1] \to \Re$ which satisfies

$$H(\Psi(r,t;p), p) = (1-p)\{L(\Psi(r,t;p)) + h(r,t)\} - H(p)\{K(w(r,t)) + h(r,t)\} = 0, \tag{4.3}$$

where $p \in [0,1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$ and $H(0) = 0$. Obviously, when $p = 0$ and $p = 1$, we have $\Psi(r,t;0) = w_0(r,t)$ and $\Psi(r,t;1) = w(r,t)$, respectively.

Thus as p varies from 0 to 1, the solution $\Psi(r,t;p)$ approaches from $w_0(r,t)$ to $w(r,t)$, where $w_0(r,t)$ is obtained from equation (4.3) when $p = 0$ giving

$$L(\Psi(r,t;0)) + h(r,t) = 0, \quad B\left(w_0, \frac{\partial w_0}{\partial t}\right) = 0. \tag{4.4}$$

The auxiliary function $H(p,c_i)$ depends either upon some constants [15-19] or upon some functions depending on a physical parameter [22, 23]. It was shown in [22, 23] that a more complex function $H(p,c_i)$ leads to more accurate results.

Next, we choose the auxiliary function of the form

$$H(p) = pC_1 + p^2C_2 + p^3C_3 + \ldots, \tag{4.5}$$

where $C_1, C_2, C_3, \ldots,$ are constants to be determined later.

To get an approximate solution, we expand $\Psi(r,t;p,C_i)$ in Taylor's series about p in the following manner:

$$\tilde{w}(r,t;p,C_i) = w_0(r,t) + \sum_{k=1}^{\infty} w_k(r,t,C_1,C_2,C_3,\ldots,C_k)p^k. \tag{4.6}$$

Substituting equation (4.6) into equation (4.3) and equating the coefficient of like powers of p, we obtain the following linear equations.

Zeroth order problem is given by equation (4.4) and the first and second order problems are given by equations (4.7) and (4.8) respectively:

$$L(w_1(r,t)) + h(r,t) = C_1 N_0(w_0(r,t)), \quad B\left(w_1, \frac{\partial w_1}{\partial t}\right) = 0$$
$$L(w_2(r,t)) - L(w_1(r,t)) = C_2 N_0(w_0(r,t)) + C_1\left[L(w_1(r,t)) + N_1(w_1(r,t))\right],$$
$$B\left(w_2, \frac{\partial w_2}{\partial t}\right) = 0 \tag{4.7}$$

The general governing equations for $w_k(r,t)$ are given by:

$$L(w_k(r,t)) - L(w_{k-1}(r,t)) = C_k N_0(w_0(r,t))$$
$$+ \sum_{i=1}^{k-1} C_i\left[L(w_{k-i}(r,t)) + N_{k-i}(w_0(r,t), w_1(r,t), ..., w_{k-1}(r,t))\right], \quad k = 2,3,...,$$
$$B\left(w_k, \frac{\partial w_k}{\partial t}\right) = 0 \tag{4.8}$$

where $N_{k-i}(w_0(r,t), w_1(r,t), ..., w_{k-1}(r,t))$ is the coefficient of p^{k-i} in the expansion of $N(\psi(r,t;p))$ about the embedding parameter p [15-19].

$$N(\psi(r,t;p)) = N_0(w_0(r,t)) + \sum_{k-i=1}^{\infty} N_{k-i}(w_0, w_1, w_2, ..., w_{k-i})p^{k-i}. \tag{4.9}$$

It has been convenient that the convergence of the series (4.6) depends upon the auxiliary constants $C_1, C_{2,} ...$.

If it is convergent at $p = 1$,

$$\tilde{w}(r,t,C_1,C_2,C_3,...,C_{k-i}) = w_0(r,t) + \sum_{k-i=1}^{\infty} w_i^{k-i}(r,t,C_1,C_2,C_3,...,C_{k-i}). \tag{4.10}$$

Substitution of equation (4.10) into equation (3.7), results the following expression for residual:

$$R(r,t,C_1,C_2,C_3,...,C_{k-i}) = L(\tilde{w}(r,t,C_1,C_2,C_3,...,C_{k-i})) + h(r,t)$$
$$+ N(\tilde{w}(r,t,C_1,C_2,C_3,...,C_{k-i})). \tag{4.11}$$

If $R = 0$, then we recover the exact solution of the problem. Usually it doesn’t happen, particularly in non-linear problems.

Numerous methods like Method of Least Squares, Galerkin's Method, Ritz Method, and Collocation Method are used to find the optimal values of C_i, $i = 1, 2, 3, \ldots$. We apply the Method of Least Squares in our problem as given below:

$$J\left(C_1, C_2, C_3, \ldots, C_{k-i}\right) = \int_a^b R^2\left(r, t, C_1, C_2, C_3, \ldots, C_{k-i}\right) dr\ dt, \tag{4.12}$$

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \frac{\partial J}{\partial C_3} = \ldots = \frac{\partial J}{\partial C_{k-i}} = 0, \tag{4.13}$$

where a and b are properly chosen numbers from the domain of the problem to hit upon the desired C_i $\left(i = 1, 2, \ldots, k - i\right)$. Finally, from these known constants, the approximate solution (of order $k - i$) is well-determined.

4.1 SOLUTION OF THE PROBLEM

Construct a homotopy for equation (3.7) with the corresponding boundary conditions given in equation (3.1) according to equation (4.3).

We obtain zeroth, first order and second order problem. For solution of the problem the we take $R_d = 1$ and the radius of the wire $R_w = \delta,\quad 0 < \delta < 1.$

$$p^0 : \frac{\partial^2 w_0}{\partial r^2} + \frac{1}{r}\frac{\partial w_0}{\partial r} = 0, \tag{4.14}$$

subject to the boundary conditions

$$w_0\left(1, t\right) = 0, \quad w_0\left(\delta, t\right) = U_w\left(1 + a\cos\omega t\right), \tag{4.15}$$

$$\begin{aligned} p^1 : &\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} + \lambda\ \ C_1\frac{\partial w_0}{\partial t} - \frac{1}{r}\frac{\partial w_0}{\partial r} - \frac{C_1}{r}\frac{\partial w_0}{\partial r} - \frac{\alpha' C_1}{r}\frac{\partial}{\partial t}\left(\frac{\partial w_0}{\partial r}\right) \\ &- \ \frac{\partial^2 w_0}{\partial r^2} - C_1\frac{\partial^2 w_0}{\partial r^2} - \alpha' C_1\frac{\partial}{\partial t}\left(\frac{\partial^2 w_0}{\partial r^2}\right) = 0, \end{aligned} \tag{4.16}$$

subject to boundary conditions

$$w_1\left(1, t\right) = 0, \quad w_1\left(\delta, t\right) = 0, \tag{4.17}$$

$$p^2: \frac{\partial^2 w_2}{\partial r^2} + \frac{1}{r}\frac{\partial w_2}{\partial r} + \lambda \ C_2 \frac{\partial w_0}{\partial t} + \lambda \ C1 \frac{\partial w_1}{\partial t} - \frac{C_2}{r}\frac{\partial w_0}{\partial r} - \frac{\alpha^{'} C_2}{r}\frac{\partial}{\partial t}\left(\frac{\partial w_1}{\partial r}\right) - \frac{1}{r}\frac{\partial w_1}{\partial r}$$
$$-\frac{C_1}{r}\frac{\partial w_1}{\partial r} - \frac{\alpha^{'} C_1}{r}\frac{\partial}{\partial t}\left(\frac{\partial w_1}{\partial r}\right) - C_1 \frac{\partial^2 w_1}{\partial r^2} - C_2 \frac{\partial^2 w_0}{\partial r^2} - \frac{\partial^2 w_1}{\partial r^2} - \alpha^{'} C_2 \frac{\partial}{\partial t}\left(\frac{\partial^2 w_0}{\partial r^2}\right) \tag{4.18}$$
$$-\alpha^{'} C_1 \frac{\partial}{\partial t}\left(\frac{\partial^2 w_1}{\partial r^2}\right) = 0,$$

subject to the boundary conditions

$$w_2(1,t) = 0, \quad w_2(\delta,t) = 0 \tag{4.19}$$

where $\lambda = \dfrac{1}{\nu}$ and $\alpha^{'} = \dfrac{\alpha}{\nu}$.

Zeroth order problem given by equations (4.14) and (4.15) are the following solution:

$$w_0 = U_w(1 + a\cos\omega t)\frac{\ln r}{\ln\delta}, \tag{4.20}$$

If equation (4.20) is substituted into equation (4.16), and solving subject to the boundary conditions (4.17) gives the first order solution as bellow:

$$w_1 = \frac{1}{8}\left[\lambda C_1 U_w a\omega \sin\omega t - \lambda C_1 U_w a\omega r^2 \sin\omega t + \sigma_{11}\ln r \sin\omega t + r^2 \lambda C_1 U_w a\omega \ln r \sin\omega t\right], \tag{4.21}$$

Similarly the second order solution obtains from equations (4.18) and (4.19) is as follows:

$$w_2 = \sigma_{12}\sin\omega t + \sigma_{13}r^2\sin\omega t + \sigma_{14}\ln r\sin\omega t + \sigma_{15}r^2\ln r\sin\omega t + \sigma_{16}\cos\omega t + \sigma_{17}\ln r\cos\omega t$$
$$+\sigma_{18}r^2\cos\omega t + \sigma_{19}r^2\ln r\cos\omega t + \Lambda_{11}r^4\cos\omega t + \Lambda_{12}r^4\ln r\cos\omega t, \tag{4.22}$$

Finally, the second order approximate solution is

$$w = U_w\left(1 + a\cos\omega t\right)\frac{\ln r}{\ln\delta} + \frac{1}{8}\lambda C_1 U_w a\omega\sin\omega t - \frac{1}{8}\lambda C_1 U_w a\omega r^2\sin\omega t$$
$$+\frac{1}{8}\sigma_{11}\ln r\sin\omega t + \frac{1}{8}r^2\lambda C_1 U_w a\omega\ln r\sin\omega t + \sigma_{12}\sin\omega t$$
$$+\sigma_{13}r^2\sin\omega t + \sigma_{14}\ln r\sin\omega t + \sigma_{15}r^2\ln r\sin\omega t + \sigma_{16}\cos\omega t \qquad (4.23)$$
$$+\sigma_{17}\ln r\cos\omega t + \sigma_{18}r^2\cos\omega t + \sigma_{19}r^2\ln r\cos\omega t + \Lambda_{11}r^4\cos\omega t$$
$$+\Lambda_{12}r^4\ln r\cos\omega t,$$

where $\sigma_{11},\sigma_{12},\sigma_{13},\sigma_{14},\sigma_{15},\sigma_{16},\sigma_{17},\sigma_{18},\sigma_{19},\Lambda_{11}$ and Λ_{12} are constants involving the auxiliary constants C_1, C_2 are given as bellow:

$$\sigma_{11} = \frac{\delta^2}{\ln\delta}\left(1 - \ln\delta - \frac{1}{\delta^2}\right)\lambda a\omega C_1 U_w,$$

$$\sigma_{12} = \frac{1}{6}\left(C_1 + C_1^2 + C_2\right)\lambda a\omega U_w,$$

$$\sigma_{13} = -\frac{1}{6}\left(C_1 + C_1^2 + C_2\right)\lambda a\omega U_w,$$

$$\sigma_{14} = -\frac{1}{12}\left(C_1 + C_1\delta^2 + C_1^2 + C_1^2\delta^2 + C_2 + C_2\delta^2 + \frac{2}{\lambda}\alpha' C_2\right)\lambda a\omega U_w,$$

$$\sigma_{15} = \frac{1}{6}\left(C_1 + C_1^2 + C_2\right)\lambda a\omega U_w,$$

$$\sigma_{16} = \frac{1}{48}\left(\lambda C_1^2 - \lambda\delta^2 C_1^2 + 8\alpha' C_1 + \frac{5}{4}\lambda C_1^2 + 2\lambda\delta^2 C_1^2 + 3\alpha' C_1 C_2\right)\lambda a\omega^2 U_w,$$

$$\sigma_{17} = \Big(-\frac{1}{48\ln\delta}(\lambda C_1^2 - 2\lambda\delta^2 C_1^2 + \lambda\delta^4 C_1^2) - \frac{1}{12}\alpha' C_1^2 - \frac{1}{12}\alpha'\delta^2 C_1^2 - \frac{1}{414}\lambda C_1^2$$
$$-\frac{1}{48}\lambda\delta^2 C_1^2\frac{13}{414}\delta^4\lambda C_1^2 - \frac{3}{96}\delta^4\lambda C_1^2 + \frac{3}{48}\alpha' C_1 C_2 - \frac{9}{48}\alpha'\delta^2 C_1 C_2\Big)\lambda a\omega^2 U_w,$$

$$\sigma_{18} = -\frac{1}{48}\left(C_1^2\lambda - C_1^2\lambda\delta^2 + 8\alpha' C_1^2 + 2\lambda C_1^2 + 2\lambda\delta^2 C_1^2 + 4\alpha' C_1 C_2\right)\lambda a\omega^2 U_w,$$

$$\sigma_{19} = \frac{1}{48}\left(C_1^2\lambda - C_1^2\lambda\delta^2 + 8\alpha' C_1^2 + 2\lambda C_1^2 + 2\lambda\delta^2 C_1^2 + 4\alpha' C_1 C_2\right)\lambda a\omega^2 U_w,$$

$$\Lambda_{11} = \frac{1}{64}\lambda^2 a\omega^2 C_1^2 U_w,$$

$$\Lambda_{12} = -\frac{1}{96}\lambda^2 a\omega^2 C_1^2 U_w$$

Table 1. Shows velocity distribution of fluid flow between the wire and die for different values of time by taking $\omega = 0.2, \alpha' = 0.02, \delta = 0.2, U_w = 2, \lambda = 0.5, a = 0.01$ and $C_1 = -0.5924838150, C_2 = -0.09024558924$.

r	velocity distribution			
	$t = 1$	$t = 10$	$t = 20$	$t = 30$
0.2	2.0196	2.01081	1.99168	1.98020
0.3	1.51081	1.50426	1.48995	1.48133
0.4	1.14982	1.14484	1.13396	1.12738
0.5	0.869806	0.866054	0.85782	0.852834
0.6	0.641019	0.63826	0.632192	0.628511
0.7	0.447581	0.445658	0.441421	0.438847
0.8	0.280017	0.278815	0.276165	0.274553
0.9	0.132214	0.131647	0.130396	0.129634
1.0	0	0	0	0

Table 2. Shows velocity distribution of fluid at various domain points at different time level by taking $\omega = 0.2, \alpha' = 0.02, \delta = 0.2, U_w = 2, \lambda = 0.5, a = 0.01$ and $C_1 = -0.5924838150, C_2 = -0.09024558924$.

t	velocity distribution			
	$r = 0.2$	$r = 0.22$	$r = 0.24$	$r = 0.26$
0	2.02	1.90038	1.79117	1.69071
1	2.0196	1.9	1.79082	1.69038
2	2.01842	1.8989	1.78978	1.6894
3	2.01651	1.8971	1.78808	1.6878
4	2.01393	1.89468	1.7858	1.68565
5	2.01081	1.89174	1.78303	1.68304
6	2.00725	1.88839	1.77988	1.68006
7	2.0034	1.88477	1.77647	1.67684
8	1.99942	1.88102	1.77294	1.67351
9	1.99546	1.8773	1.76942	1.67019
10	1.99168	1.87374	1.76607	1.66703

Table 3. Shows velocity distribution of fluid at various orders along the domain at $t = 10$ time level by taking $\omega = 0.2, \alpha' = 0.02, \delta = 0.2, U_w = 2, \lambda = 0.5, a = 0.01$ and $C_1 = -0.5924838150, C_2 = -0.09024558924$.

r	velocity distribution		
	Zeroth order	**First order**	**Second order**

0.2	2.00848	2.00848	2.00848
0.3	1.50249	1.50247	1.50245
0.4	1.14348	1.14345	1.14343
0.5	0.865007	0.864975	0.864956
0.6	0.63748	0.637452	0.637434
0.7	0.445109	0.445086	0.445073
0.8	0.27847	0.278454	0.278445
0.9	0.131484	0.131476	0.131471
1.0	0	0	0

Table 4. Shows velocity distribution of fluid flow at different values of time by using $\omega = 0.2, \alpha' = 0.02, \delta = 0.2, U_w = 2, \lambda = 0.8, a = 0.5$ and $C_1 = -0.3296806629, C_2 = -0.306008832$.

r	velocity distribution			
	$t = 1$	$t = 5$	$t = 10$	$t = 15$
0.2	2.98007	2.5403	1.58385	1.01001
0.3	2.22989	1.90284	1.18755	0.755984
0.4	1.69744	1.44974	0.9055	0.575615
0.5	1.28429	1.09767	0.68605	0.435604
0.6	0.946615	0.80953	0.50623	0.321125
0.7	0.661032	0.565557	0.35381	0.224274
0.8	0.413589	0.353968	0.221506	0.140335
0.9	0.195292	0.167173	0.104633	0.0662685
1.0	0	0	0	0

Table 5. Shows velocity distribution at different time level by taking $\omega = 0.2, \alpha' = 0.02, \delta = 0.2, U_w = 2, \lambda = 0.8, a = 0.5$ and $C_1 = -0.3296806629, C_2 = -0.306008832$.

t	velocity distribution			
	$r = 0.2$	$r = 0.22$	$r = 0.24$	$r = 0.26$
0	3.0	2.82234	2.66015	2.51095
2	2.92106	2.74839	2.59074	2.44569
4	2.69671	2.53759	2.39229	2.25859
6	2.36236	2.22322	2.09614	1.9792
8	1.9708	1.85491	1.74905	1.65161
10	1.58385	1.4908	1.4058	1.32756
12	1.26261	1.18839	1.12059	1.05819
14	1.05778	0.995411	0.938457	0.886046
16	1.00171	0.942339	0.888144	0.838293
18	1.10324	1.03755	0.9776	0.922476
20	1.34636	1.26601	1.1927	1.1253

Table 6. Shows velocity distribution at various orders at $t = 5$ time level by taking $\omega = 0.2, \alpha' = 0.02, \delta = 0.2, U_w = 2, \lambda = 0.8, a = 0.5$ and $C_1 = -0.3296806629, C_2 = -0.306008832$.

r	velocity distribution		
	Zeroth order	**First order**	**Second order**
0.2	1.58385	1.58385	1.58385
0.3	1.18483	1.18581	1.18755
0.4	0.901725	0.903092	0.9055
0.5	0.682128	0.683551	0.68605
0.6	0.502705	0.503985	0.50623
0.7	0.351005	0.352024	0.35381
0.8	0.219596	0.220289	0.221506
0.9	0.103686	0.104029	0.104633
1.0	0	0	0

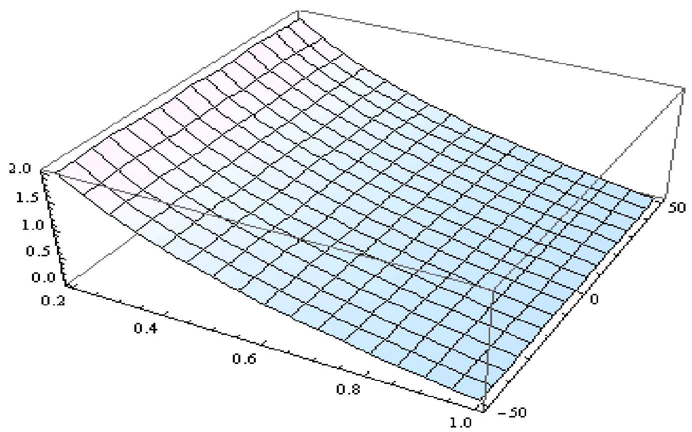

FIGURE 3. Velocity profile for $\omega = 0.2, \alpha' = 0.02, \delta = 0.2, U_w = 2, \lambda = 0.5, a = 0.01$ and $C_1 = -0.5924838150, C_2 = -0.09024558924$.

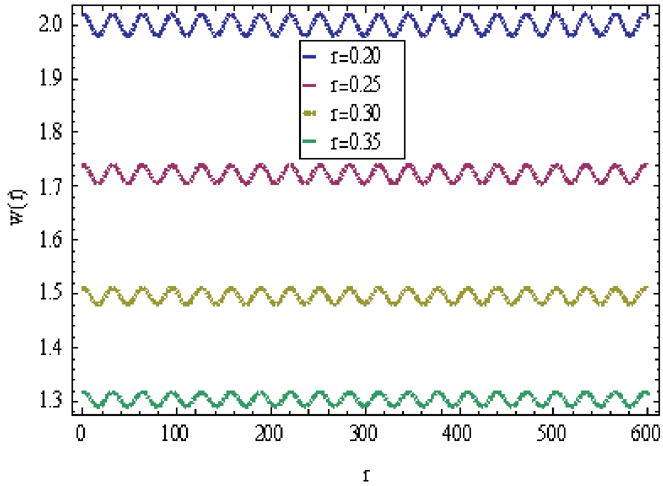

FIGURE 4. Velocity profile for at different values of r by taking $\omega = 0.2, \alpha' = 0.02, \delta = 0.2, U_w = 2, \lambda = 0.5, a = 0.01$ and $C_1 = -0.5924838150, C_2 = -0.09024558924$.

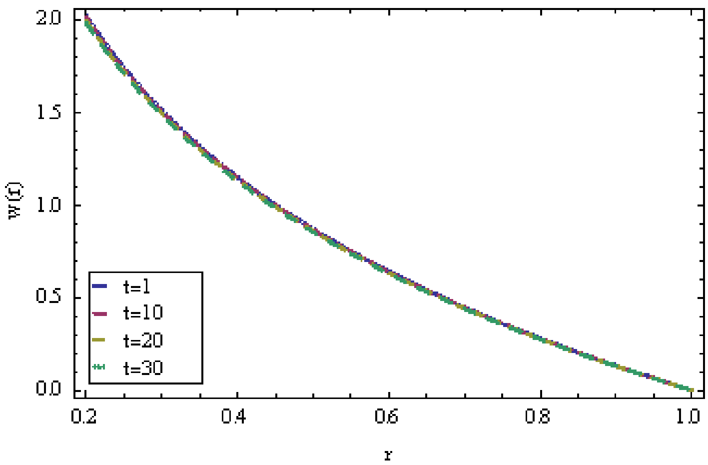

FIGURE 5. Velocity profile for at different values of t by taking $\omega = 0.2, \alpha' = 0.02, \delta = 0.2, U_w = 2, \lambda = 0.5, a = 0.01$ and $C_1 = -0.5924838150, C_2 = -0.09024558924$.

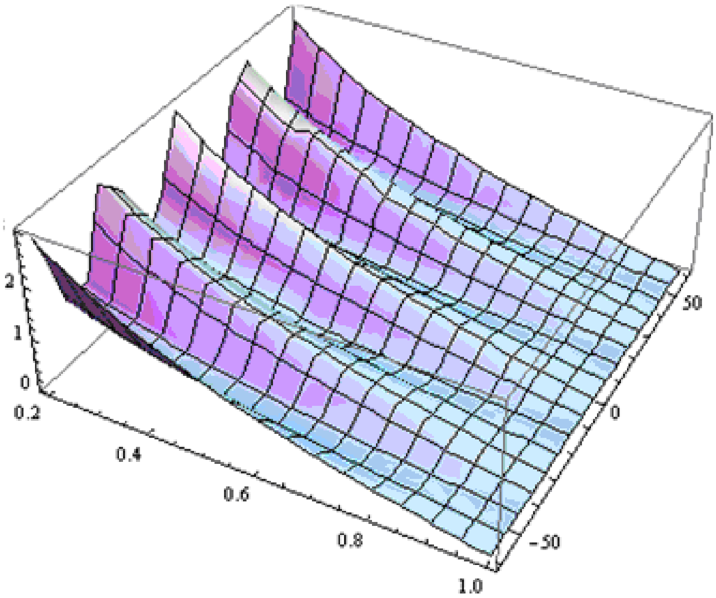

FIGURE 6. Velocity distribution of fluid flow at t=2 by taking
$\omega = 0.2, \alpha' = 0.02, \delta = 0.2, U_w = 2, \lambda = 0.8, a = 0.5$ and
$C_1 = -0.3296806629, C_2 = -0.306008832$.

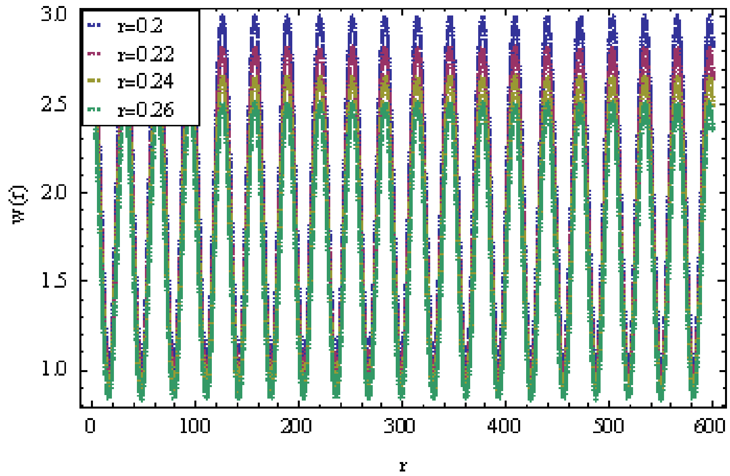

FIGURE 7. Velocity distribution of fluid at different values of r by taking $t = 0.5$
$\omega = 0.2, \alpha' = 0.02, \delta = 0.2, U_w = 2, \lambda = 0.8, a = 0.5$ and
$C_1 = -0.3296806629, C_2 = -0.306008832$.

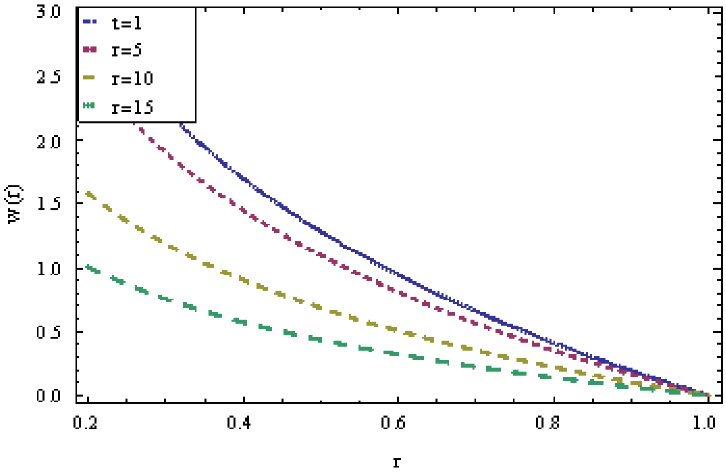

FIGURE 8. Velocity profile for at different values of t by taking $\omega = 0.2, \alpha' = 0.02, \delta = 0.2, U_W = 2, \lambda = 0.8, a = 0.5$ and $C_1 = -0.3296806629, C_2 = -0.306008832$.

SECTION 5

To investigate the stability and convergence of OHAM, we make an effort to solve some linear and non-linear partial differential equations with known exact solution.

Example 5.1

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial r^2}, \; 0 \le r \le 1, \tag{5.1}$$

with the boundary and initial conditions

$$w(r,0) = e^r, \; w(0,t) = e^t, \; w(1,t) = e^{1+t}. \tag{5.2}$$

The exact solution of equation (5.1) with the corresponding boundary condition (5.2) is as follows

$$w(r,t) = e^{r+t}. \tag{5.3}$$

Here, we have

$$L(w(r,t)) = \frac{\partial^2 w}{\partial r^2}, \; N(w(r,t)) = -\frac{\partial w}{\partial t}. \tag{5.4}$$

Following the procedure of OHAM, we obtain the solution to the given problem up to third order approximation with

$C_1 = -0.9724175167,$

$C_2 = -0.00186254006,$

$C_3 = 0.0018848878.$

The absolute error is presented in the form of numerical data in Table 7 and Table 8.

Example 5.2

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial r^2} + w\frac{\partial w}{\partial r} - e^{2(r+t)}, \ \ 0 \le r \le 1, \tag{5.5}$$

with the boundary and initial conditions

$$w(r,0) = e^r, \quad w(0,t) = e^t, \quad w(1,t) = e^{1+t}. \tag{5.6}$$

Having the exact solution

$$w(r,t) = e^{r+t}. \tag{5.7}$$

Here, we have

$$L(w(r,t)) = \frac{\partial^2 w}{\partial r^2}, \quad N(w(r,t)) = -\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial r}, \ h(r,t) = -e^{2(r+t)}, \tag{5.8}$$

Handling, the problem with OHAM as discussed earlier, we obtain the third order approximate solution with

$C_1 = -0.555334661,$

$C_2 = -0.321611989,$

$C_3 = 0.43579432289.$

The absolute error of Example 2 is presented in the form of numerical data in Table 9 and Table 10.

Example 5.3

$$\frac{\partial w}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial r}\right)\right), \ \ 0 \le r \le 1, \tag{5.9}$$

with the boundary and initial conditions

$$w(r,0) = r, \quad w(0,t) = t^2, \quad w(1,t) = 1 + t^2, \tag{5.10}$$

The exact solution to the problem is as bellow

$$w(r,t) = r + t^2. \tag{5.11}$$

In this case, we have

$$L(w(r,t)) = \frac{\partial^2 w}{\partial r^2}, \quad N(w(r,t)) = -2\frac{\partial w}{\partial t} + \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial r}\right), \tag{5.12}$$

Applying OHAM as discussed in previous section, we obtain the third order approximate solution with

$C_1 = 0.8698907118147899,$

$C_2 = -1.5239850805438448,$

$C_3 = 1.1597421393001897.$

The absolute error of Example 3 can be observed from the numerical data in Tables 11 and Table 12.

TABLE 7. The absolute error (Example 1) at different orders of approximation at $t = 2$.

r	Absolute error $\left\|w_{exact} - w_{OHAM}\right\|$			
	Zeroth order order	**First order**	**Second order**	**Third**
0.0	0	0	0	0
0.1	0.0814154	0.00570537	0.249482×10^{-5}	7.56031×10^{-9}
0.2	0.149321	0.0110026	0.481734×10^{-4}	1.52796×10^{-8}
0.3	0.202296	0.0152718	0.666759×10^{-4}	2.31512×10^{-8}
0.4	0.23877	0.0180423	0.780313×10^{-4}	3.11826×10^{-8}
0.5	0.257007	0.0190087	0.808035×10^{-4}	3.85911×10^{-8}
0.6	0.25509	0.0180482	0.747926×10^{-4}	4.34906×10^{-8}
0.7	0.2309	0.0152395	0.611×10^{-4}	4.32616×10^{-8}
0.8	0.182092	0.0108845	0.41991×10^{-4}	3.56143×10^{-8}
0.9	0.106079	0.00553118	0.205352×10^{-4}	2.01492×10^{-8}
1.0	0	0	0	0

TABLE 8. The absolute error (Example 1) of third order approximation by OHAM at different time level shown in table.

r	Absolute error $\left\|w_{exact} - w_{OHAM}\right\|$		
	$t = 1$	$t = 3$	$t = 5$
0.0	0	0	0
0.1	6.25207×10^{-9}	6.84085×10^{-9}	1.02053×10^{-9}
0.2	1.26356×10^{-8}	1.38256×10^{-8}	2.06253×10^{-8}
0.3	1.91451×10^{-8}	2.09481×10^{-8}	3.12508×10^{-8}
0.4	2.57868×10^{-8}	2.82152×10^{-8}	4.20921×10^{-8}
0.5	3.19133×10^{-8}	3.49187×10^{-8}	5.20926×10^{-8}
0.6	3.59649×10^{-8}	3.93519×10^{-8}	5.87061×10^{-8}
0.7	3.57756×10^{-8}	3.91447×10^{-8}	5.8397×10^{-8}
0.8	2.94516×10^{-8}	3.22251×10^{-8}	4.80743×10^{-8}
0.9	1.66625×10^{-8}	1.82317×10^{-8}	2.71985×10^{-8}
1.0	0	0	0

TABLE 9. The absolute error (Example 2) at different orders of approximation at $t = 0.5$.

r	Absolute error $\lvert w_{exact} - w_{OHAM} \rvert$			
	Zeroth order order	**First order**	**Second order**	**Third**
0.0	0	0	0	0
0.1	0.00673272	0.00284105	0.000107105	0.000036837
0.2	0.0123482	0.00491567	0.000238502	0.000070689
0.3	0.016729	0.00629913	0.000373329	0.000102489
0.4	0.0197453	0.00705434	0.000492424	0.000130985
0.5	0.0212534	0.00723054	0.000578245	0.00015322
0.6	0.0210949	0.0068619	0.000614799	0.00016496
0.7	0.0190945	0.00596588	0.000587582	0.00016111
0.8	0.0150583	0.00454153	0.000483547	0.000136094
0.9	0.00877234	0.00256752	0.00029108	0.000084219
1.0	0	0	0	0

TABLE 10. The absolute error (Example 2) of third order approximation by OHAM at different time level shown in table.

r	Absolute error $\lvert w_{exact} - w_{OHAM} \rvert$		
	$t = 0.2$	$t = 0.7$	$t = 1.2$
0.0	0	0	0
0.1	0.000036837	0.0000379589	0.0000445452
0.2	0.000070689	0.0000728427	0.0000854817
0.3	0.000102489	0.00010561	0.000123934
0.4	0.000130985	0.000134974	0.000158394
0.5	0.00015322	0.000157886	0.000185281
0.6	0.00016496	0.000169984	0.000199478
0.7	0.00016111	0.000166016	0.000194822
0.8	0.000136094	0.000140239	0.000164572
0.9	0.0000842191	0.0000867839	0.000101842
1.0	0	0	0

TABLE 11. The absolute error (Example 3) at different orders of approximation at $t = 2$.

r	Absolute error $\lvert w_{exact} - w_{OHAM} \rvert$			
	Zeroth order	**First order**	**Second order**	**Third order**

0.0	0	0	0	0
0.1	0.0285	0.000134087	0.111993×10^{-4}	0.402434×10^{-5}
0.2	0.048	0.00102312	0.199196×10^{-4}	0.805369×10^{-5}
0.3	0.0595	0.00308721	0.245956×10^{-4}	0.115625×10^{-4}
0.4	0.064	0.0055984	0.245932×10^{-4}	0.139915×10^{-4}
0.5	0.0625	0.00803744	0.201174×10^{-4}	0.148591×10^{-4}
0.6	0.056	0.0098477	0.122106×10^{-4}	0.138719×10^{-4}
0.7	0.0455	0.0104624	0.280969×10^{-5}	0.110492×10^{-4}
0.8	0.032	0.00933614	0.517164×10^{-5}	0.687801×10^{-5}
0.9	0.0165	0.0059802	0.777698×10^{-5}	0.250961×10^{-5}
1.0	0	0	0	0

TABLE 12. The absolute error (Example 3) of third order approximation by OHAM at different time level shown in table.

r	Absolute error $\left\|w_{exact}-w_{OHAM}\right\|$			
	$t=0.5$	$t=1.5$	$t=2.5$	$t=3.0$
0.0	0	0	0	0
0.1	0.208495×10^{-4}	0.197658×10^{-4}	0.186455×10^{-5}	0.17491×10^{-5}
0.2	0.376054×10^{-4}	0.35773×10^{-4}	0.338665×10^{-5}	0.318911×10^{-5}
0.3	0.479629×10^{-4}	0.459034×10^{-4}	0.437308×10^{-5}	0.414529×10^{-5}
0.4	0.517735×10^{-4}	0.49929×10^{-4}	0.479772×10^{-5}	0.458895×10^{-5}
0.5	0.503948×10^{-4}	0.491237×10^{-4}	0.476926×10^{-5}	0.461118×10^{-5}
0.6	0.459324×10^{-4}	0.452381×10^{-4}	0.443965×10^{-5}	0.434159×10^{-5}
0.7	0.394581×10^{-4}	0.392185×10^{-4}	0.338482×10^{-5}	0.384126×10^{-5}
0.8	0.303416×10^{-4}	0.303955×10^{-4}	0.30385×10^{-5}	0.31313×10^{-5}
0.9	0.169306×10^{-4}	0.171267×10^{-4}	0.172969×10^{-5}	0.174421×10^{-5}
1.0	0	0	0	0

6. RESULTS AND DISCUSIONS

The formulation presented in Section 4 and illustration of the formulation in the examples given in Section 5 provides accurate solution without discretization of the problem domain. Examples 1-3 gives the numerical solution of zeroth, 1st, 2nd and 3rd order problem in Tables 7,9 and 11 which shows that as the order of OHAM increase the accuracy of the solution also increase, which confirms the convergence of OHAM. As the fluid flow is due to the oscillation and translation of the wire so the velocity of the fluid will be high at the surface of the wire as compared to remaining domain and will be decrease for the fluid away from the surface of wire, these phenomena can be observed from Tables 1, 2 and Tables 4, 5. Tables 3 and 6 illustrate the solution of different order problems at time $t=10$, and $t=5$ respectively for different parameters which show that the effect of nonlinearity in the problem is less effective because the absolute errors between different order problems are very less. It is evident from Tables 7-12 that OHAM can be applied for large time domain and the accuracy remains almost consistent.

7. CONCLUSION

In this paper, we model the unsteady second grade fluid flow between wire and die with one oscillating boundary and other stationary in the form of partial differential equation. The model problem is solved by OHAM using the boundary conditions only and obtained satisfactory results. For stability of OHAM some time dependent linear and non-linear problems are solved having exact solutions. The obtained results verify that OHAM is convergent to the exact solution as the order increases. Furthermore, this method provides a convenient way to control the convergence by optimally determining the auxiliary constants. This work extends the idea of OHAM that it is not only use for the solution of linear and non-linear differential equations but also can be applied for linear and non-linear partial differential equations.

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