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# CONTROLLABILITY OF IMPULSIVE FRACTIONAL EVOLUTION INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. According to fractional calculus theory and Banach's fixed point theorem, we establish the sufficient conditions for the controllability of impulsive fractional evolution integrodifferential equations in Banach spaces. An example is provided to illustrate the theory.

## 1. INTRODUCTION

The purpose of this paper is to establish the sufficient conditions for the controllability of impulsive fractional evolution integrodifferential equation of the form

$$\frac{d^q x(t)}{dt^q} = A(t)x(t) + Bu(t) + f(t, x(t)) + \int_0^t g(t, s, x(s))ds, \quad t \in J = [0, T],\ t \neq t_k,\ k = 1, 2, \dots, m, \tag{1.1}$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \tag{1.2}$$

$$x(0) = x_0, \tag{1.3}$$

where the state variable $x(\cdot)$ takes values in a Banach space $\mathbb{X}$ and control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with $U$ as a Banach space. Here $0 < q \leq 1$ and $A(t)$ is a bounded linear operator on a Banach space $\mathbb{X}$. Further $f : J \times \mathbb{X} \to \mathbb{X},\ g : \Omega \times \mathbb{X} \to \mathbb{X},\ I_k : \mathbb{X} \to \mathbb{X},\ \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, for all $k = 1, 2, \dots, m;\ 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T;\ \Omega = \{(t, s),\ 0 \leq s \leq t \leq T\}$.

The study of impulsive differential equation is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. Now impulsive integrodifferential equations have become an important object of investigation in recent years stimulated by their numerous applications

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to problems arising in mechanics, electrical engineering, medicine, biology, ecology, etc., For instance, we refer [16, 23, 29, 30, 39, 34, 48].

Fractional differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc., (see[19, 31, 2, 27, 41, 51]). There has been significant development in fractional differential equations in recent years; see the monographs of Kilbas et al. [32], Miller and Ross [42], Podlubny [47], Lakshmikantham et al. [36] and the papers [33, 28, 40, 3, 50, 4, 6, 7, 8, 17, 22, 35, 37, 38, 45, 52, 53, 13] and the references therein. Among previous research, little is concerned with differential equations with fractional order with impulses. Recently, Benchohra et al. [1, 14] establish sufficient conditions for the existence of solutions for a class of initial value problem for impulsive fractional differential equations involving the Caputo fractional derivative of order $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$. In [5], B. Ahmad et al. give some existence results for two-point boundary value problems involving nonlinear impulsive hybrid differential equations of fractional order $1 < \alpha \leq 2$ where as Mophou [46] discussed the existence and uniqueness results for impulsive fractional differential equations. Very recently, K. Balachandran et al. [12, 13] studied the existence results for impulsive fractional differential and integrodifferential equations in Banach spaces by using standard fixed point theorems.

On the other hand, the most important qualitative behavior of a dynamical system is controllability. It is well known that the issue of controllability plays an important role in control theory and engineering [9, 10, 18, 24] because they have close connections to pole assignment, structural decomposition, quadratic optimal control and observer design etc., In recent years, the problem of controllability for various kinds of fractional differential and integrodifferential equations have been discussed in [11, 15, 25, 49]. In [50], the authors established the sufficient conditions for the controllability of fractional order impulsive neutral functional integrodifferential equations with infinite delay in Banach spaces.

Motivated by the works [12, 50], the main aim of this paper is to establish the controllability results for impulsive fractional evolution integrodifferential systems in Banach spaces by using the semigroup theory, fractional calculus and fixed point theorem.

## 2. PRELIMINARIES

In this section, we give some basic definitions and properties of fractional calculus which are used throughout this paper.

Let $C(J,\mathbb{X})$ be the Banach space of continuous functions $x(t)$ with $x(t) \in \mathbb{X}$ for $t \in J = [0,T]$ and $\|x\|_{C(J,\mathbb{X})} = \max_{t\in J} \|x(t)\|$. Also consider the Banach space

$$PC(J,\mathbb{X}) = \{x : J \to \mathbb{X} : x \in C((t_k,t_{k+1}],\mathbb{X}),\ k = 0,1,\ldots,m \quad \text{and there exist } x(t_k^-) \text{ and } x(t_k^+),\ k = 1,2,\ldots,m \quad \text{with } x(t_k^-) = x(t_k)\},$$

with the norm $\|x\|_{PC} = \sup_{t\in J} \|x(t)\|$. Denote $J' = [0,T] - \{t_1,t_2,\ldots,t_m\}$.

**Definition 1.** *A real function $f(t)$ is said to be in the space $C_\alpha,\ \alpha \in R$ if there exists a real number $p > \alpha$, such that $f(t) = t^p g(t)$, where $g \in C[0,\infty)$ and it is said to be in the space $C_\alpha^m$ iff $f^{(m)} \in C_\alpha,\ m \in N$.*

**Definition 2.** *The Riemann-Liouville fractional integral operator of order $\beta > 0$ of function $f \in C_\alpha,\ \alpha \geq -1$ is defined as*

$$I^\beta f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{\beta-1} f(s) ds$$

*where $\Gamma(\cdot)$ is the Euler Gamma function.*

**Definition 3.** *If the function $f \in C_{-1}^m$ and $m$ is a positive integer then we define the fractional derivative of $f(t)$ in the Caputo sense as*

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^m(s) ds, \quad m-1 < \alpha \leq m.$$

If $0 < \alpha \leq 1$, then

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds,$$

where $f'(s) = \frac{df(s)}{ds}$ and $f$ is an abstract function with values in $\mathbb{X}$.

**Definition 4.** *The impulsive integrodifferential system (1.1)-(1.3) is said to be controllable on the interval $J = [0,T]$ if for every $x_0, x_1 \in \mathbb{X}$, there exists a control $u \in L^2(J,U)$ such that the solution $x(\cdot)$ of (1.1)-(1.3) satisfies $x(T) = x_1$.*

It is easy to prove that [12] the equation (1.1)-(1.3) is equivalent to the following integral equation

$$\begin{aligned}
x(t) &= x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)x(s)ds + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} A(s)x(s)ds \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \Big[Bu(s) + f(s,x(s))\Big] ds + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} \Big[Bu(s) \\
&\quad + f(s,x(s))\Big] ds + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s g(s,\tau,x(\tau)) d\tau ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} \int_0^s g(s,\tau,x(\tau)) d\tau ds + \sum_{0<t_k<t} I_k(x(t_k^-))).
\end{aligned}$$

Let $B_r = \{x \in \mathbb{X} : \|x\| \leq r\}$ for some $r > 0$. We assume the following conditions to prove the controllability of the system (1.1)-(1.3).

(H1) $A(t)$ is a bounded linear operator on $\mathbb{X}$ for each $t \in J$ and the function $t \to A(t)$ is continuous in the uniform operator topology.

(H2) The linear operator $W : L^2(J,U) \to \mathbb{X}$ defined by

$$Wu = \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} Bu(s)ds$$

has an inverse operator $W^{-1}$, which takes values in $L^2(J,U)/KerW$ and there exists a positive constant $K > 0$ such that $\|BW^{-1}\| \le K$ for every $x \in B_r$.

(H3) The nonlinear function $f : J \times \mathbb{X} \to \mathbb{X}$ is continuous and there exist constants $L_1 > 0, \widetilde{L}_1 > 0$, such that

$$\|f(t,x) - f(t,y)\| \le L_1\|x-y\|, \quad \text{for } x,y \in B_r$$

and $\widetilde{L}_1 = \max_{t\in J} \|f(t,0)\|$.

(H4) The nonlinear function $g : \Omega \times \mathbb{X} \to \mathbb{X}$ is continuous and there exist constants $L_2 > 0, \widetilde{L}_2 > 0$, such that

$$\|g(t,s,x) - g(t,s,y)\| \le L_2||x-y\|, \quad \text{for } x,y \in B_r$$

and $\widetilde{L}_2 = \max_{t\in J} \|g(t,s,0)\|$.

(H5) The functions $I_k : \mathbb{X} \to \mathbb{X}$ are continuous and there exist constants $L_3 > 0, \widetilde{L}_3 > 0$, such that

$$\|I_k(x) - I_k(y)\| \le L_3\|x-y\|, \quad \text{for } x,y \in B_r \text{ and } k = 1,2,\dots,m,$$

and $\widetilde{L}_3 = \|I_k(0)\|$.

(H6) There exists a constant $r > 0$ such that

$$\|x_0\| + (m+1)\gamma\Big[(M + L_1 + TL_2)r + (\widetilde{K} + \widetilde{L}_1 + T\widetilde{L}_2)\Big] + m(L_3r + \widetilde{L}_3) \le r,$$

where

$$\widetilde{K} = K\Big[\|x_1\| + \|x_0\| + (m+1)\gamma\big[(M + L_1 + TL_2)r + (\widetilde{L}_1 + T\widetilde{L}_2)\big] + m(L_3r + \widetilde{L}_3)\Big]$$

with $\gamma = \frac{T^q}{\Gamma(q+1)}$.

## 3. Controllability Result

In this section, we present and prove the controllability results for the system (1.1)-(1.3).

**Theorem 3.1.** *If the hypotheses (H1)-(H6) are satisfied, then the impulsive fractional integrodifferential system (1.1)-(1.3) is controllable on J provided*

$$p = \Big[(m+1)\gamma\Big[K\big[(m+1)\gamma(M + L_1 + TL_2) + mL_3\big] + [M + L_1 + TL_2]\Big] + mL_3\Big] < 1.$$

*Proof.* Using the hypothesis (H2) for an arbitrary function $x(\cdot)$, define the control

$$
\begin{aligned}
u(t) = W^{-1}\Big[ & x_1 - x_0 - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)x(s)ds \\
& - \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} A(s)x(s)ds - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s))ds \\
& - \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} f(s, x(s))ds \\
& - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds \\
& - \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds - \sum_{0<t_k<T} I_k(x(t_k^-)) \Big](t).
\end{aligned}
$$

We have to show that when using this control, the operator $\Phi : PC(J, B_r) \to PC(J, B_r)$ defined by

$$
\begin{aligned}
(\Phi x)(t) = x_0 & + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)x(s)ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s)x(s)ds \\
& + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \eta)^{q-1} BW^{-1}\Big[ x_1 - x_0 \\
& - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)x(s)ds - \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} A(s)x(s)ds \\
& - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s))ds - \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} f(s, x(s))ds \\
& - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds \\
& - \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds - \sum_{0<t_k<T} I_k(x(t_k^-)) \Big](\eta)d\eta \\
& + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-\eta)^{q-1} BW^{-1}\Big[ x_1 - x_0 \\
& - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)x(s)ds - \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} A(s)x(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s))ds - \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} f(s, x(s))ds \\
&- \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds - \sum_{0<t_k<T} I_k(x(t_k^-))\Big](\eta)d\eta \\
&+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s))ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, x(s))ds \\
&+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds \\
&+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds + \sum_{0<t_k<t} I_k(x(t_k^-))
\end{aligned}
$$

has a fixed point. Since all the functions involved in the operator are continuous therefore $\Phi$ is continuous. For our convenience, let us take

$$
\begin{aligned}
F(\eta, x) = BW^{-1}\Big[&x_1 - x_0 - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)x(s)ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} A(s)x(s)ds - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s))ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} f(s, x(s))ds \\
&- \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds - \sum_{0<t_k<T} I_k(x(t_k^-))\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
F(\eta, y) = BW^{-1}\Big[&x_1 - x_0 - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)y(s)ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} A(s)y(s)ds - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, y(s))ds
\end{aligned}
$$

$$- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} f(s, y(s)) ds - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s g(s, \tau, y(\tau)) d\tau ds$$
$$- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} \int_0^s g(s, \tau, y(\tau)) d\tau ds - \sum_{0<t_k<T} I_k(y(t_k^-)) \Big]$$

From our assumptions, we have

$$\begin{aligned} \|F(\eta, x)\| &\leq K \Big[ \|x_1\| + \|x_0\| + (m+1)\gamma \big[ (M + L_1 + TL_2) r + (\widetilde{L}_1 + T\widetilde{L}_2) \big] \\ &\quad + m(L_3 r + \widetilde{L}_3) \Big] \\ &= \widetilde{K}. \end{aligned}$$

and

$$\|F(\eta, x) - F(\eta, y)\| \leq K \Big[ (m+1)\gamma \big[ M + L_1 + TL_2 \big] + mL_3 \Big] \|x - y\|.$$

First we show that $\Phi$ maps $PC(J, B_r)$ into $PC(J, B_r)$. Now

$$\begin{aligned} &\|(\Phi x)(t)\| \\ &\leq \|x_0\| + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|A(s)\| \|x(s)\| ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|A(s)\| \|x(s)\| ds + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \eta)^{q-1} \|F(\eta, x)\| d\eta \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-\eta)^{q-1} \|F(\eta, x)\| d\eta + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \big[ \|f(s, x(s)) \\ &\quad - f(s, 0)\| + \|f(s, 0)\| \big] ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \big[ \|f(s, x(s)) - f(s, 0)\| + \|f(s, 0)\| \big] ds \\ &\quad + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s \big[ \|g(s, \tau, x(\tau)) - g(s, \tau, 0)\| + \|g(s, \tau, 0)\| \big] d\tau ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \int_0^s \big[ \|g(s, \tau, x(\tau)) - g(s, \tau, 0)\| + \|g(s, \tau, 0)\| \big] d\tau ds \\ &\quad + \sum_{0<t_k<t} \big[ \|I_k(x(t_k^-)) - I_k(0)\| + \|I_k(0)\| \big] \\ &\leq \|x_0\| + (m+1)\gamma \Big[ (M + L_1 + TL_2) r + (\widetilde{K} + \widetilde{L}_1 + T\widetilde{L}_2) \Big] + m(L_3 r + \widetilde{L}_3) \\ &\leq r. \end{aligned}$$

From assumption (H6), one gets $\|(\Phi x)(t)\| \leq r$. Therefore $\Phi$ maps $PC(J, B_r)$ into $PC(J, B_r)$. Moreover, if $x, y \in PC(J, B_r)$, then

$$\begin{aligned}
&\|(\Phi x)(t) - (\Phi y)(t)\| \\
&\leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|A(s)(x(s) - y(s))\| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|A(s)(x(s) - y(s))\| ds \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \eta)^{q-1} \|F(\eta, x) - F(\eta, y)\| d\eta \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-\eta)^{q-1} \|F(\eta, x) - F(\eta, y)\| d\eta \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|f(s, x(s)) - f(s, y(s))\| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \big[\|f(s, x(s)) - f(s, y(s))\|\big] ds \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s \|g(s, \tau, x(\tau)) - g(s, \tau, y(\tau))\| d\tau ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \int_0^s \|g(s, \tau, x(\tau)) - g(s, \tau, y(\tau))\| d\tau ds \\
&\quad + \sum_{0<t_k<t} \|I_k(x(t_k^-)) - I_k(y(t_k^-))\| \\
&\leq \Big[(m+1)\gamma \Big[K\big[(m+1)\gamma(M + L_1 + TL_2) + mL_3\big] + [M + L_1 + TL_2]\Big] \\
&\quad + mL_3\Big] \|x - y\| \\
&= p\|x - y\|.
\end{aligned}$$

Since $0 \leq p < 1$, then $\Phi$ is a contraction and so by Banach fixed point theorem there exists a unique fixed point $x \in PC(J, B_r)$ such that $(\Phi x)(t) = x(t)$. This fixed point is then a solution of the system (1.1)-(1.3) and clearly, $x(T) = (\Phi x)(T) = x_1$, which implies that the system is controllable on $J$. □

## 4. Nonlocal Controllability Result

In this section, we discuss the controllability system (1.1)-(1.2) with a nonlocal condition of the form

$$x(0) + h(x) = x_0. \tag{4.1}$$

where $h : PC(J, \mathbb{X}) \to \mathbb{X}$ is a given function.

The nonlocal condition can be applied in physics with better effect than the classical initial condition $x(0) = x_0$. For example, $h(x)$ may be given by

$$h(x) = \sum_{i=1}^{m} c_i x(t_i),$$

where $c_i (i = 1, 2, \ldots, m)$ are given constants and $0 < t_1 < t_2 < \cdots < t_m < T$. Nonlocal conditions were initiated by Byszewski [20] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski and Lakshmikantham [21], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For more details on fractional order with nonlocal condition, we refer [11, 12, 13, 26, 43, 44]

For the study of the controllability system (1.1)-(1.2) with (4.1), we need the following hypotheses:

(H7) $h : PC(J, \mathbb{X}) \to \mathbb{X}$ is continuous and there exist constants $L_4 > 0, \widetilde{L}_4 > 0$, such that

$$\|h(x) - h(y)\| \leq L_4 \|x - y\|_{PC}, \quad \text{for } x, y \in PC(J, \mathbb{X}),$$

and $\widetilde{L}_4 = \|h(0)\|$.

(H8) There exists a constant $r > 0$ such that

$$\|x_0\| + (mL_3 + L_4)r + (m\widetilde{L}_3 + \widetilde{L}_4) + (m+1)\gamma\Big[(M + L_1 + TL_2)r + (\widetilde{K} + \widetilde{L}_1 + T\widetilde{L}_2)\Big] \leq r,$$

where

$$\widetilde{K} = K\Big[\|x_1\| + \|x_0\| + (mL_3 + L_4)r + (m\widetilde{L}_3 + \widetilde{L}_4) + (m+1)\gamma\big[(M + L_1 + TL_2)r + (\widetilde{L}_1 + T\widetilde{L}_2)\big]\Big]$$

**Definition 5.** *The impulsive integrodifferential system (1.1)-(1.2) with the condition (4.1) is said to be controllable on the interval $J$ if for every $x_0, x_1 \in \mathbb{X}$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1.1)-(1.2) with (4.1) satisfies $x(0) + h(x) = x_0$ and $x(T) = x_1$.*

**Theorem 4.1.** *If the hypotheses (H1)-(H5) and (H7)-(H8) are satisfied, then the impulsive fractional integrodifferential system (1.1)-(1.2) with the condition (4.1) is controllable on $J$ provided*

$$p' = \Big[(mL_3 + L_4) + (m+1)\gamma\{K\big[(m+1)\gamma(M + L_1 + TL_2) + (mL_3 + L_4)\big] + [M + L_1 + TL_2]\}\Big] < 1.$$

*Proof.* Using the hypothesis (H2) for an arbitrary function $x(\cdot)$, define the control

$$
\begin{aligned}
u(t) = W^{-1}\Big[&x_1 - (x_0 - h(x)) - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)x(s)ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T - s)^{q-1} A(s)x(s)ds - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s))ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T - s)^{q-1} f(s, x(s))ds \\
&- \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T - s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds - \sum_{0<t_k<T} I_k(x(t_k^-))\Big](t).
\end{aligned}
$$

We have to show that when using this control, the operator $\Psi : PC(J, B_r) \to PC(J, B_r)$ defined by

$$
\begin{aligned}
(\Psi x)(t) = x_0 - h(x) &+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)x(s)ds \\
&+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} A(s)x(s)ds + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \eta)^{q-1} BW^{-1}\Big[x_1 \\
&- (x_0 - h(x)) - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)x(s)ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T - s)^{q-1} A(s)x(s)ds - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s))ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T - s)^{q-1} f(s, x(s))ds \\
&- \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T - s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds - \sum_{0<t_k<T} I_k(x(t_k^-))\Big](\eta)d\eta \\
&+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - \eta)^{q-1} BW^{-1}\Big[x_1 - (x_0 - h(x))
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)x(s)ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} A(s)x(s)ds - \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s))ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} f(s, x(s))ds \\
&- \frac{1}{\Gamma(q)} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{T} (T-s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds - \sum_{0<t_k<T} I_k(x(t_k^-))\Big](\eta)d\eta \\
&+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s))ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, x(s))ds \\
&+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds \\
&+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \int_0^s g(s, \tau, x(\tau))d\tau ds + \sum_{0<t_k<t} I_k(x(t_k^-))
\end{aligned}
$$

has a fixed point. This fixed point is then a solution of the control problem (1.1)-(1.2) with (4.1). Clearly, $(\Psi x)(T) = x_1$, which means that the control $u$ steers the system (1.1)-(1.2) with (4.1) from the initial state $x_0$ to $x_1$ in time $T$ provided we can obtain a fixed point of the operator $\Psi$. The rest of the proof is similar to Theorem 3.1, hence omitted. □

## 5. EXAMPLE

Consider the following nonlinear impulsive partial integrodifferential system of the form [11]

$$
\frac{\partial^\alpha}{\partial t^\alpha} z(t,y) = a(t,y) \frac{\partial^2}{\partial y^2} z(t,y) + \mu(t,y) + k_o(y) \sin z(t,y) + k_1 \int_0^t e^{-z(s,y)} ds, \qquad (5.1)
$$

$$
z(0,y) + \sum_{i=1}^{m} c_i z(t_i, y) = z_0(y), \quad 0 \le y \le \pi, \qquad (5.2)
$$

$$
z(t,0) = z(t,\pi) = 0, \quad t \in J = [0,T], \qquad (5.3)
$$

$$
\Delta z|_{t=t_i} = I_i(z(y)) = \frac{1}{(\alpha_i |z(y)| + t_i)}, \quad 1 \le i \le m, \qquad (5.4)
$$

where $0 < \alpha < 1$, and $c_i$ and $\alpha_i$ are small and $k_0(y)$ is continuous on $[0,\pi]$. We define $X = U = L^2[0,\pi]$, $B_r = \{y \in L^2[0,\pi] : \|y\| \le r\}$.

Let $x(t) = z(t, \cdot)$ and $u(t) = \mu(t, \cdot)$, where $\mu : J \times [0, \pi] \to [0, \pi]$ is continuous, $h(x(t)) = \sum_{i=1}^{m} c_i z(t_i, y)$ and $f(t, x) = k_0(\cdot) \sin z(t, \cdot),\ g(t, s, x) = k_1 e^{-z(s,\cdot)}$.

The operator $A(t)$ is defined by

$$A(t)z = -a(t, y)z'' \quad \text{with the domain}$$

$$D(A) = \{z(\cdot) \in \mathbb{X} : z, z' \text{ are absolutely continuous}, \quad z'' \in \mathbb{X},\ z(0) = z(\pi) = 0\}$$

generates an evolution system and satisfies (H1).

With this choice of $A(t), I_i, f, g, h$ and $B = I$, the identity operator, we see that the system (5.1)-(5.4) are an abstract formulation of the system (1.1)-(1.2) with (4.1). Assume that the operator $W : L^2(J, U)/KerW \to \mathbb{X}$ defined by

$$Wu = \frac{1}{\Gamma(\alpha)} \int_0^T (T - s)^{q-1} \mu(s, \cdot) ds$$

has an inverse operator and satisfies the hypothesis (H2) for every $x \in B_r$.

Further other conditions (H3)-(H5) and (H7)-(H8) are satisfied and it is possible to choose $c_i, \alpha_i, k_0, k_1$ in such a way that the constant $p' < 1$. Hence, by Theorem (4.1), the system (5.1)-(5.4) is controllable on $J$.

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