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ARC-LENGTH ESTIMATIONS FOR QUADRATIC RATIONAL Bézier CURVES COINCIDING WITH ARC-LENGTH OF SPECIAL SHAPES

SEON-HONG KIM[1] AND YOUNG JOON AHN[2†]

[1]DEPARTMENT OF MATHEMATICS, SOOKMYUNG WOMEN'S UNIVERSITY, SEOUL, 140-742, KOREA

[2]DEPARTMENT OF MATHEMATICS EDUCATION, CHOSUN UNIVERSITY, GWANGJU, 501–759, KOREA

ABSTRACT. In this paper, we present arc-length estimations for quadratic rational Bézier curves using the length of polygon and distance between both end points. Our arc-length estimations coincide with the arc-length of the quadratic rational Bézier curve exactly when the weight w is 0, 1 and ∞. We show that for all $w > 0$ our estimations are strictly increasing with respect to w. Moreover, we find the parameter μ^* which makes our estimation coincide with the arc-length of the quadratic rational Bézier curve when it is a circular arc too. We also show that μ^* has a special limit, which is used for optimal estimation. We present some numerical examples, and the numerical results illustrates that the estimation with the limit value of μ^* is an optimal estimation.

1. INTRODUCTION

Quadratic rational Bézier curve has been widely used in CAD/CAM and Solid Modeling [8, 16, 19]. It is also called by *conic section*. Its arc-length can not be obtained algebraically, while the arc-length of quadratic Bézier curve can be obtained[6, 17]. To find the arc-length of quadratic rational Bézier curve is an important problem in CAGD(Computer Aided Geometric Design), Geometric Modeling[20, 21, 22, 23, 26] and Geodesy on spheroid[7, 18, 24, 25].

Gravesen[14] presented a good estimation for the arc-length of Bézier curve of degree n as

$$L_G = \frac{2}{n+1}L_b + \frac{n-1}{n+1}L_p$$

where L_p is the length of control polygon and L_b is the distance between both end control points of Bézier curve. He showed that his approximate has the approximation order five and extended his result to the rational case[15] and triangular surface case. Roulier and Piper[21, 22, 23] presented theorems and algorithms which produce polynomial/rational Bézier or parametric curves having specified arc-length subject to certain constraints. Floater et al.[11, 12, 13] presented estimations of arc-length for a parametric curve having high approximation order using only samples of points.

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† Corresponding author. ahn@chosun.ac.kr.

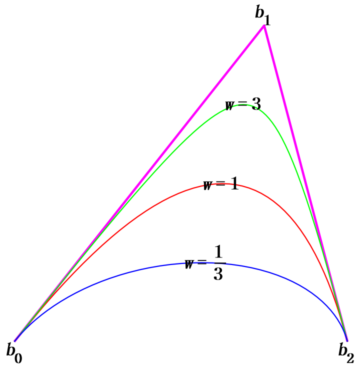

FIGURE 1. The quadratic rational Bézier curves(blue, red or green) with weight w and its control polygon(magenta) $\mathbf{b}_i$, $i = 0, 1, 2$.

The estimation method for arc-length of the quadratic rational Bézier curves is well-known, but no previous method reproduces the arc-length of special shapes of quadratic rational Bézier curves, e.g. quadratic Bézier curves or circular arcs.

We present the arc-length estimations for the quadratic rational Bézier curves reproducing the arc-length of special shapes. Our estimations are greatly inspired by Gravesen's method. The arc-length of quadratic rational Bézier curve for given control polygon is strictly increasing as the weight $w > 0$ is increasing. We show that our arc-length estimations have the same property. Also, the estimations have a parameter μ which is used to reproduce the arc-length of special shape of the quadratic rational Bézier curve, for example, line segment($w = 0$), parabola($w = 1$), or control polygon($w = \infty$). Moreover, we present the parameter μ^* which makes our estimation coincides with the arc-length of the quadratic rational Bézier curve when it is a circular arc too. We find the limit of μ^* under symmetry of quadratic rational Bézier curve. Using some numerical examples, we analyze the quality of our arc-length estimations with respect to the parameter μ. It illustrates the limit of μ^* is the optimal estimation of our estimations.

The remainder of this paper is structured as follows. In Section 2, we present our arc-length estimations which coincide with the arc-length of quadratic rational Bézier curve for $w = 0, 1, \infty$. In section 3, The arc-length estimations coinciding with the length of circular arc simultaneously are investigated. We give some numerical examples and state optimal estimation of our estimations in Section 4.

2. ARC-LENGTH ESTIMATIONS FOR QUADRATIC RATIONAL BÉZIER CURVE

In this section we present arc-length estimation for quadratic rational Bézier curve using the polygonal length, the distance between both end points, and weight. The Bézier curve of degree n with control points $\mathbf{b}_i$, $i = 0, \cdots, n$ is defined by

$$\mathbf{b}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(t), \qquad t \in [0,1]$$

where $B_i^n(t)$, $(i = 0, \cdots, n)$ is the Bernstein polynomial $B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}$. The arc-length estimation of Gravsen[14] for the Bézier curve of degree n is

$$\frac{2}{n+1} L_b + \frac{n-1}{n+1} L_p \tag{2.1}$$

where L_p is the length of the polygon of the Bézier curve and L_b is the distance between both end control points, i.e.,

$$L_p = \sum_{i=0}^{n-1} |\Delta \mathbf{b}_i| \qquad \text{and} \qquad L_b = |\Delta \mathbf{b}_n|.$$

where $\Delta \mathbf{b}_i = \mathbf{b}_{i+1} - \mathbf{b}_i$, $(i = 0, \cdots, n-1)$, $\Delta \mathbf{b}_n = \mathbf{b}_0 - \mathbf{b}_n$, and $|\mathbf{x}|$ is the length of the vector $\mathbf{x}$. Quadratic rational Bézier curve with the control points $\mathbf{b}_i$, $(i = 0, 1, 2)$ can be defined in standard form[1, 3, 9, 10]

$$\mathbf{r}(t) = \frac{\mathbf{b}_0 B_0^2(t) + w \mathbf{b}_1 B_1^2(t) + \mathbf{b}_2 B_2^2(t)}{B_0^2(t) + w B_1^2(t) + B_2^2(t)},$$

where $w > 0$ is the weight associated with $\mathbf{b}_1$. We assume that the control points are not collinear in this paper. Let $L(w)$ be the arc-length of quadratic rational Bézier curve of the weight w for given control polygon $\mathbf{b}_i$, $(i = 0, 1, 2)$. The closer w is to zero, the closer $\mathbf{r}(t)$ is to the line segment $\overline{\mathbf{b}_0 \mathbf{b}_2}$, and the closer w is to infinity, the closer $\mathbf{r}(t)$ is to the control polygon $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$, as shown in Figure 1. Thus by the limits of $L(w)$ as $w \to 0$ and $w \to \infty$, we have $L(0) = |\Delta \mathbf{b}_2| = L_b$ and $L(\infty) = |\Delta \mathbf{b}_0| + |\Delta \mathbf{b}_1| = L_p$.

Now, we present the arc-length estimation λ_μ for the quadratic Bézier curve $\mathbf{r}(t)$, $t \in [0, 1]$ with the parameter $\mu \in \mathbb{R}$ as follows, which is an extension of the Gravsen's estimation for quadratic Bézier curve($n = 2$):

$$\lambda_\mu(\mathbf{r}) = \frac{\mu}{\mu + w} L_b + \frac{w}{\mu + w} L_p. \tag{2.2}$$

Since the quadratic rational Bézier curve $\mathbf{r}$ depends on the control polygon $\mathbf{b}_i$, $(i = 0, 1, 2)$ and the weight w, we can write $\lambda_\mu(\mathbf{r}) = \lambda_\mu(\mathbf{b}_i, w)$. The estimation $\lambda_2(\mathbf{b}_i, 1)$ is equal to that of Gravsen for quadratic case, $n = 2$, in Equation (2.1).

PROPOSITION 2.1. *The arc-length estimation $\lambda_\mu(\mathbf{b}_i, w)$ coincides with the arc-length of the quadratic rational Bézier curve $\mathbf{r}(t)$ for $w = 0, \infty$, and it is strictly increasing with respect to w for any control polygon $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$ and for all positive $\mu > 0$.*

Proof. Equation (2.2) yields $\lambda_\mu(\mathbf{b}_i, 0) = L_b$ and $\lambda_\mu(\mathbf{b}_i, \infty) = L_p$, so λ_μ coincides with the arc-length $L(w)$ for $w = 0, \infty$. By the derivative

$$\frac{\partial \lambda_\mu(\mathbf{b}_i, w)}{\partial w} = \frac{\mu(L_p - L_b)}{(\mu + w)^2}$$

and by the inequality $L_p > L_b$, $\lambda_\mu(\mathbf{b}_i, w)$ is strictly increasing with respect to w. □

The arc-length of quadratic Bézier curve with control points $\mathbf{b}_i$, $i = 0, 1, 2$ is well-known [6, 17] as

$$L_q = \frac{\beta_1|\Delta\mathbf{b}_1| + \beta_3|\Delta\mathbf{b}_0|}{2\beta_2} + \frac{|\Delta\mathbf{b}_0 \times \Delta\mathbf{b}_1|^2}{8(\beta_2)^{3/2}} \cdot \ln\left(\frac{|\Delta\mathbf{b}_1|\sqrt{\beta_2} + \beta_1}{|\Delta\mathbf{b}_0|\sqrt{\beta_2} - \beta_3}\right)$$

where

$$\beta_i = \frac{1}{4}(i|\Delta\mathbf{b}_0|^2 + (4-i)|\Delta\mathbf{b}_1|^2 - |\Delta\mathbf{b}_2|^2)$$

for $i = 1, 2, 3$. If we take $\mu = \hat{\mu}$ where

$$\hat{\mu} = \frac{L_p - L_q}{L_q - L_b}, \tag{2.3}$$

then the estimation $\lambda_\mu(\mathbf{b}_i, w)$ coincide with the arc-length $L(w)$ for $w = 1$, and Equation (2.2) yields

$$\lambda_{\hat{\mu}}(\mathbf{b}_i, w) = \frac{wL_p(L_q - L_b) + L_b(L_p - L_q)}{w(L_q - L_b) + L_p - L_q}. \tag{2.4}$$

It follows from $L_b < L_q < L_p$ that $\hat{\mu} > 0$, for all $w > 0$. For the symmetric quadratic rational Bézier curve $\mathbf{r}(t)$, i.e., $|\Delta\mathbf{b}_0| = |\Delta\mathbf{b}_1|$, the value $\hat{\mu}$ has the special limit as follows.

PROPOSITION 2.2. *If the quadratic rational Bézier curve $\mathbf{r}(t)$ is symmetric, then*

$$\lim_{\mathbf{b}_1 \to (\mathbf{b}_0 + \mathbf{b}_2)/2} \hat{\mu} = 2 \qquad \textit{and} \qquad \lim_{\mathbf{b}_1 \to \infty} \hat{\mu} = 1$$

independently of the length $|\Delta\mathbf{b}_2|$.

Proof. Let $h = |\mathbf{b}_1 - (\mathbf{b}_0 + \mathbf{b}_2)/2|$. For sufficiently small $h > 0$, the following series expansions hold.

$$\begin{aligned}
L_b &= |\Delta\mathbf{b}_2| \\
L_p &= 2|\Delta\mathbf{b}_0| = 2\sqrt{h^2 + (|\Delta\mathbf{b}_2|/2)^2} \approx |\Delta\mathbf{b}_2| + \frac{2h^2}{|\Delta\mathbf{b}_2|} + \mathcal{O}(h^4) \\
L_q &\approx |\Delta\mathbf{b}_2| + \frac{2h^2}{3|\Delta\mathbf{b}_2|} + \mathcal{O}(h^4) \\
\frac{L_p - L_q}{L_q - L_b} &\approx 2 - \frac{6}{5}\frac{h^2}{|\Delta\mathbf{b}_2|^2} + \mathcal{O}(h^4).
\end{aligned} \tag{2.5}$$

Thus we have the limit

$$\lim_{\mathbf{b}_1 \to (\mathbf{b}_0 + \mathbf{b}_2)/2} \hat{\mu} = 2.$$

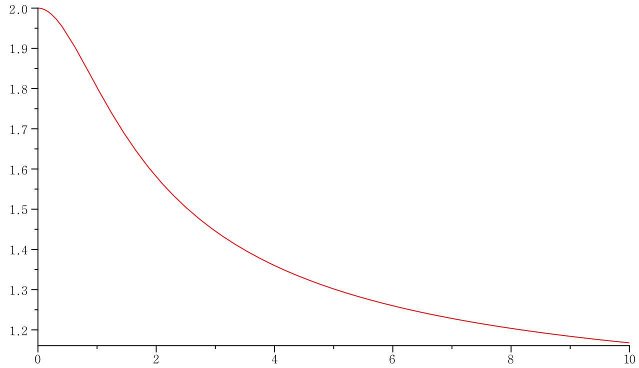

FIGURE 2. Graph of $\hat{\mu}$ for the symmetric quadratic Bézier curve, i.e. $|\Delta \mathbf{b}_0| = |\Delta \mathbf{b}_1|$: The horizontal axis is the distance between $\mathbf{b}_1$ and $(\mathbf{b}_0 + \mathbf{b}_2)/2$.

Also, for sufficiently large $h > 0$, we have following series expansions.

$$\begin{aligned} L_p &= 2|\Delta \mathbf{b}_0| = 2h + \frac{|\Delta \mathbf{b}_2|^2}{4h} + \mathcal{O}\left(\frac{1}{h^3}\right) \\ L_q &\approx h + \frac{1}{8}|\Delta \mathbf{b}_2|^2(1 + 4\ln 2 - 2\ln|\Delta \mathbf{b}_2| - 2\ln h)\frac{1}{h} + \mathcal{O}\left(\frac{1}{h^3}\right) \\ \frac{L_p - L_q}{L_q - L_b} &\approx 1 + \frac{|\Delta \mathbf{b}_2|}{h} + \mathcal{O}\left(\frac{1}{h^2}\right) \end{aligned} \tag{2.6}$$

which yield

$$\lim_{\mathbf{b}_1 \to \infty} \hat{\mu} = 1.$$

□

Figure 2 illustrates Proposition 2.2.

REMARK 2.3. *In general the finer the quadratic rational Bézier curve is subdivide, the closer the middle control point* $\mathbf{b}_1$ *is to the point* $(\mathbf{b}_0 + \mathbf{b}_1)/2$ *and the closer the weight is to one*[3, 10]. *Thus Proposition 2.2 means that* $\mu = 2$ *is a good candidate for optimal estimation of* λ_μ. *In Section 4, Figure 6 illustrates this assertion.*

3. ARC-LENGTH ESTIMATIONS FOR QUADRATIC RATIONAL BÉZIER CURVE COINCIDING WITH LENGTH OF CIRCULAR ARC

In this section we present another arc-length estimation $\mathcal{L}_\mu$ with parameter μ which coincides with the arc-length of the quadratic rational Bézier curve $\mathbf{r}(t)$ for $w = 0, 1, \infty$, simultaneously. The arc-length estimation $\mathcal{L}_\mu$ is a natural extension of $\lambda_{\hat{\mu}}(\mathbf{b}_i, w)$ in Equation (2.4) as follows:

$$\mathcal{L}_\mu(\mathbf{r}) = \frac{w^\mu L_p(L_q - L_b) + L_b(L_p - L_q)}{w^\mu(L_q - L_b) + L_p - L_q}, \tag{3.1}$$

which satisfies $\mathcal{L}_1(\mathbf{b}_i, w) = \lambda_{\hat{\mu}}(\mathbf{b}_i, w)$.

PROPOSITION 3.1. *The estimation $\mathcal{L}_\mu(\mathbf{b}_i, w)$ coincides with the arc-length of $\mathbf{r}(t)$ for $w = 0, 1, \infty$, and it is strictly increasing with respect to w for any $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$ and for all $\mu > 0$.*

Proof. Since Equation (3.1) yields $\mathcal{L}_\mu(\mathbf{b}_i, 0) = L_b$, $\mathcal{L}_\mu(\mathbf{b}_i, 1) = L_q$ and $\mathcal{L}_\mu(\mathbf{b}_i, \infty) = L_p$, it satisfies $\mathcal{L}_\mu(\mathbf{b}_i, w) = L(w)$ for $w = 0, 1, \infty$. By the derivative

$$\frac{\partial \mathcal{L}_\mu(\mathbf{b}_i, w)}{\partial w} = \frac{\mu w^{\mu-1}(L_q - L_b)(L_p - L_q)(L_p - L_b)}{(w^\mu(L_q - L_b) + L_p - L_q)^2}$$

and by the inequalities $L_b < L_q < L_p$, we have $\partial\mathcal{L}(\mathbf{b}_i, w)/\partial w > 0$ for all positive $\mu > 0$, and the assertion follows. □

Figure 3 shows the monotonicity of the estimations $\lambda_\mu(\mathbf{b}_i, w)$ and $\mathcal{L}_\mu(\mathbf{b}_i, w)$ with respect to w.

The length of circular arc is well-known. We make our estimation $\mathcal{L}_\mu(\mathbf{r})$ coincide with the length of circular arc by special choice of μ. The quadratic rational Bézier curve is a circular arc if and only if it is symmetric and the weight satisfies $w = L_b/L_p$ [2, 4, 5, 8, 16]. Moreover, the angle of the circular arc is 2α, where $\alpha = \arccos w = \arccos(L_b/L_p)$ and the radius is $L_b/2\sin\alpha$, so its length is

$$L_c = \frac{\alpha \cdot L_b}{\sin\alpha} = \frac{\arccos(L_b/L_p) \cdot L_b \cdot L_p}{\sqrt{L_p^2 - L_b^2}}.$$

Solving the equation $\mathcal{L}_\mu(\mathbf{b}_i, w) = L_c$ we have $\mu = \mu^*$, where

$$\mu^* = \frac{1}{\ln(L_b/L_p)} \ln\frac{(L_p - L_q)(L_c - L_b)}{(L_p - L_c)(L_q - L_b)}. \tag{3.2}$$

Since $L_b < L_q < L_p$ and $L_b < L_c < L_p$, the definition of μ^* in Equation (3.2) is well defined.

The value μ^* is equal to zero if and only if $(L_p - L_q)(L_c - L_b) = (L_p - L_c)(L_q - L_b)$ which is equivalent to $L_q = L_c$. Let the plane curve γ be set of all points $\mathbf{b}_1 = [x_1, y_1]$ satisfying $L_q = L_c$ for fixed $\mathbf{b}_0$ and $\mathbf{b}_2$. Figure 4 shows that the curve γ is passing through the mid-point of $\mathbf{b}_0$ and $\mathbf{b}_2$, and the slope of γ with respect to the line $\overline{\mathbf{b}_0\mathbf{b}_2}$ at the point is $\pm\pi/4$. Inside of the curve γ, $\mu^* < 0$, and Outside $\mu^* > 0$.

In the following proposition we confirm that $\mathcal{L}_{|\mu^*|}(\mathbf{b}_i, w)$ reproduces the arc-length of $\mathbf{r}(t)$ when it is a circular arc, i.e., $w = L_b/L_p$ and $|\Delta\mathbf{b}_0| = |\Delta\mathbf{b}_1|$.

PROPOSITION 3.2. *If $\mathbf{b}_1$ does not lie on the curve γ, then the estimation $L_{|\mu^*|}(\mathbf{b}_i, w)$ coincides with the arc-length of the quadratic Bézier curve $\mathbf{r}(t)$ for $w = 0, 1, \infty$ and for it to be a circular arc.*

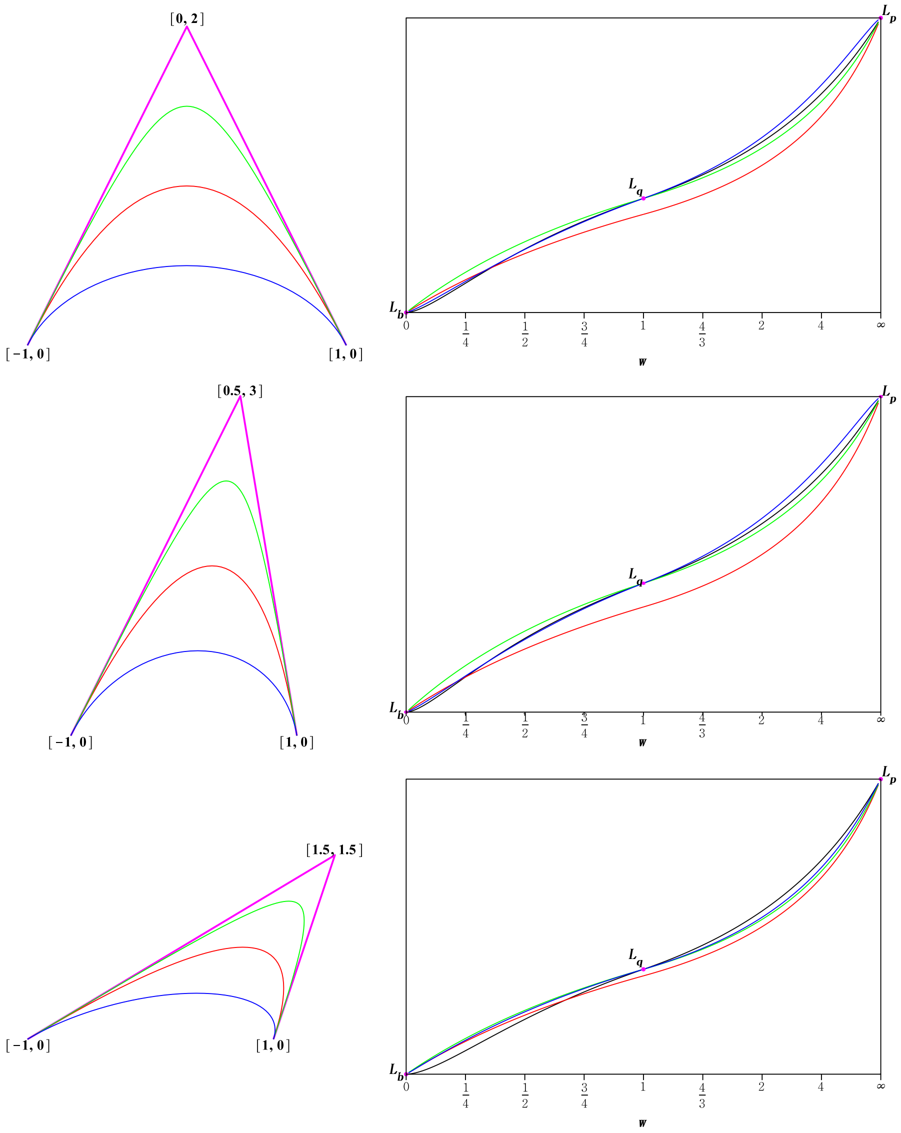

FIGURE 3. Left column : The conic section having the control points(magenta) $\mathbf{b}_0 = [-1.0]$, $\mathbf{b}_2 = [1, 0]$ and $\mathbf{b}_1 = [0, 2]$, $[0.5, 3]$, or $[1.5, 1.5]$ (from top to bottom) with the weights $w = 3$(green), 1(red), or 1/3(blue). Right column : The real arc-length $L(w)$ of the conic(black), its estimations $\lambda_2(\mathbf{b}_i, w)$(red), $\lambda_{\hat{\mu}}(\mathbf{b}_i, w) = \mathcal{L}_1(\mathbf{b}_i, w)$(green) and $\mathcal{L}_{\mu^*}(\mathbf{b}_i, w)$(blue), where $(\hat{\mu}, \mu^*) = (1.581, 1.213)$, (1.448,1.208), or (1.812,1.043)(from top to bottom). The estimation $\mathcal{L}_\mu(\mathbf{b}_i, w)$ always coincides with $L(w)$ at L_b, L_q and L_p.

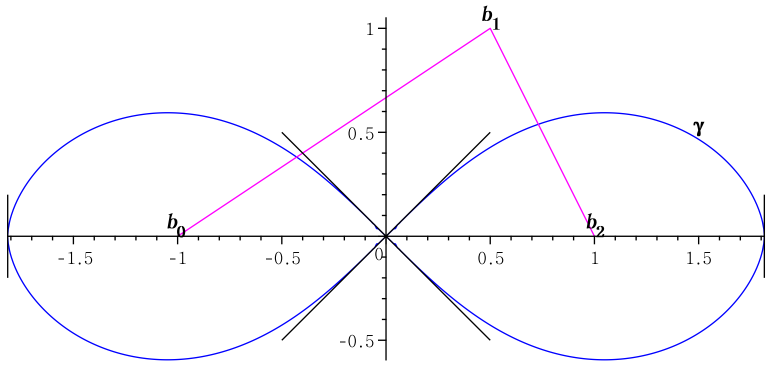

FIGURE 4. The curve γ(blue) is the set of all points $\mathbf{b}_1 = [x_1, y_1]$ satisfying $\mu^* = 0$ when $\mathbf{b}_0$ and $\mathbf{b}_2$ are fixed.

Proof. Proposition 3.1 shows that $\mathcal{L}_{|\mu^*|}(\mathbf{b}_i, w)$ coincides with the arc-length $L(w)$ for $w = 0, 1, \infty$ if $\mathbf{b}_1 \notin \gamma$ or equivalently $\mu^* \neq 0$. Thus it suffices to show that the estimation reproduces the arc-length for $\mathbf{r}(t)$ to be a circular arc. If $\mathbf{r}(t)$ is a circular arc, then it is symmetric, i.e., $|\Delta \mathbf{b}_0| = |\Delta \mathbf{b}_1|$, and the weight is $w = L_b/L_p$. The symmetry of $\mathbf{r}(t)$ implies $L_q > L_c$ and so $\mu^* > 0$ or equivalently $|\mu^*| = \mu^*$. Also the weight $w = L_b/L_p$ yields

$$w^{\mu^*} = \left(\frac{L_b}{L_p}\right)^{\frac{1}{\ln(L_b/L_p)} \ln \frac{(L_p - L_q)(L_c - L_b)}{(L_p - L_c)(L_q - L_b)}} = \frac{(L_p - L_q)(L_c - L_b)}{(L_p - L_c)(L_q - L_b)},$$

and thus we have

$$\mathcal{L}_{\mu^*}(\mathbf{b}_i, L_b/L_p) = \frac{\frac{(L_c - L_b)}{(L_p - L_c)} L_p + L_b}{\frac{(L_c - L_b)}{(L_p - L_c)} + 1} = L_c.$$

□

Although the estimation $\mathcal{L}_{|\mu^*|}(\mathbf{r})$ represents the arc-length of the quadratic rational Bézier curve $\mathbf{r}(t)$ when it is circular arc or its weight is $0, 1, \infty$, simultaneously, the estimation doesn't works well when $\mathbf{b}_1$ lie inside of the curve γ. Thus we present the limit of μ^* for symmetric quadratic rational Bézier curve to use it as an optimal estimation of $\mathcal{L}_\mu$.

PROPOSITION 3.3. *If the quadratic Bézier curve is symmetric, i.e.,* $|\Delta \mathbf{b}_0| = |\Delta \mathbf{b}_1|$, *then* μ^* *has the limit values*

$$\lim_{\mathbf{b}_1 \to (\mathbf{b}_0 + \mathbf{b}_2)/2} \mu^* = \frac{6}{5} \quad \textit{and} \quad \lim_{\mathbf{b}_1 \to \infty} \mu^* = 1$$

independently of scale of $L_b = |\Delta \mathbf{b}_2|$.

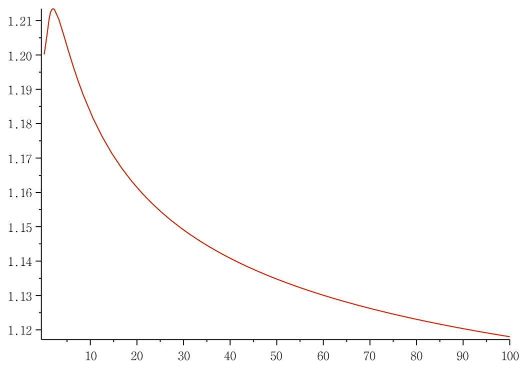

FIGURE 5. Graph of μ^* for the symmetric quadratic Bézier curve, i.e. $|\Delta \mathbf{b}_0| = |\Delta \mathbf{b}_1|$: The horizontal axis is the distance between $\mathbf{b}_1$ and $(\mathbf{b}_0 + \mathbf{b}_2)/2$.

Proof. Let $h = |\mathbf{b}_1 - (\mathbf{b}_0 + \mathbf{b}_2)/2|$. For sufficiently small $h > 0$, we have Equation (2.5) and the series expansions

$$\begin{aligned} L_c &\approx |\Delta \mathbf{b}_2| + \frac{2h^2}{3|\Delta \mathbf{b}_2|} + \mathcal{O}(h^4) \\ \frac{(L_c - L_b)}{(L_p - L_c)} &\approx \frac{1}{2} - \frac{9}{10}\frac{h^2}{|\Delta \mathbf{b}_2|^2} + \mathcal{O}(h^4) \end{aligned}$$

Thus it follows from Equation (3.2) and

$$\begin{aligned} \ln \frac{(L_p - L_q)(L_c - L_b)}{(L_q - L_b)(L_p - L_c)} &\approx -\frac{12}{5}\frac{h^2}{|\Delta \mathbf{b}_2|^2} + \mathcal{O}(h^4) \\ \ln \frac{L_b}{L_p} &\approx -2\frac{h^2}{|\Delta \mathbf{b}_2|^2} + \mathcal{O}(h^4) \end{aligned}$$

that

$$\lim_{\mathbf{b}_1 \to (\mathbf{b}_0 + \mathbf{b}_2)/2} \mu^* = \frac{6}{5}.$$

For sufficiently large $h > 0$, we have Equation (2.6) and the following series expansions.

$$\begin{aligned} L_c &\approx \frac{1}{2}|\Delta \mathbf{b}_2|\pi - \frac{1}{2}|\Delta \mathbf{b}_2|\frac{1}{h} + \mathcal{O}\left(\frac{1}{h^2}\right) \\ \frac{(L_c - L_b)}{(L_p - L_c)} &\approx \left(\frac{1}{4}|\Delta \mathbf{b}_2|\pi - \frac{1}{2}|\Delta \mathbf{b}_2|\right)\frac{1}{h} + \mathcal{O}\left(\frac{1}{h^2}\right) \end{aligned}$$

Equation (3.2) and

$$\ln\frac{(L_p-L_q)(L_c-L_b)}{(L_q-L_b)(L_p-L_c)} \approx -\ln(h)+\ln(\frac{1}{4}|\Delta\mathbf{b}_2|\pi-\frac{1}{2}|\Delta\mathbf{b}_2|)+\mathcal{O}\left(\frac{1}{h^2}\right)$$

$$\ln\frac{L_b}{L_p} \approx -\ln(h)+\ln(\frac{1}{2}|\Delta\mathbf{b}_2|)+\mathcal{O}\left(\frac{1}{h^2}\right)$$

yield

$$\lim_{\mathbf{b}_1\to\infty}\mu^*=1.$$

□

Figure 5 illustrates Proposition 3.3.

REMARK 3.4. *By the same reason of Remark 2.3, Proposition 3.3 means that $\mu=\frac{6}{5}$ is a good candidate for optimal estimation of $\mathcal{L}_\mu$.*

4. EXAMPLES AND COMMENTS

In this section we apply our estimation method to quadratic rational Bézier curves.

The examples are the unit circle $x^2+y^2=1$ and the ellipse $(x/2)^2+y^2=1$. These can be expressed by four segments of the quadratic rational Bézier curve, $k=4$. By Remarks 2.3 and 3.4, we investigate the arc-length estimations $\lambda\mu(\mathbf{r})$ and $\mathcal{L}_\mu(\mathbf{r})$ for $\mu=2,\hat{\mu},\mu^*$ and $\frac{6}{5}$. The second rows in Tables 1 and 2 show the errors $|\lambda_\mu(\mathbf{r})-L|$ and $|\mathcal{L}_\mu(\mathbf{r})-L|$. By subdivision of the quadratic rational Bézier curve at each shoulder point[1, 8, 9], the unit circle and the ellipse can be expressed by k segments of quadratic rational Bézier curves, $k=8,\cdots,64$. Tables 1 and 2 list the errors $|\lambda_\mu(\mathbf{r})-L|$ and $|\mathcal{L}_\mu(\mathbf{r})-L|$, and show $|\mathcal{L}_{\frac{6}{5}}(\mathbf{r})-L|$ has the smallest error.

As we expected in Remarks 2.3 and 3.4, Figure 6 illustrates that $\lambda_2(\mathbf{r})$ is optimal estimation of $\lambda_\mu(\mathbf{r})$ and $\mathcal{L}_{\frac{6}{5}}(\mathbf{r})$ is optimal estimation of $\mathcal{L}_\mu(\mathbf{r})$. Also we can see the decreasing ratio of the errors for each one subdivision, as shown in Tables 1 and 2. The decreasing ratios of $|\mathcal{L}_{\frac{6}{5}}(\mathbf{r})-L|$ is closer to $\frac{1}{64}$ as k increases, while those of others are closer to $\frac{1}{16}$. This means $|\mathcal{L}_{\frac{6}{5}}(\mathbf{r})-L|$ converges to zero most rapidly in our estimations.

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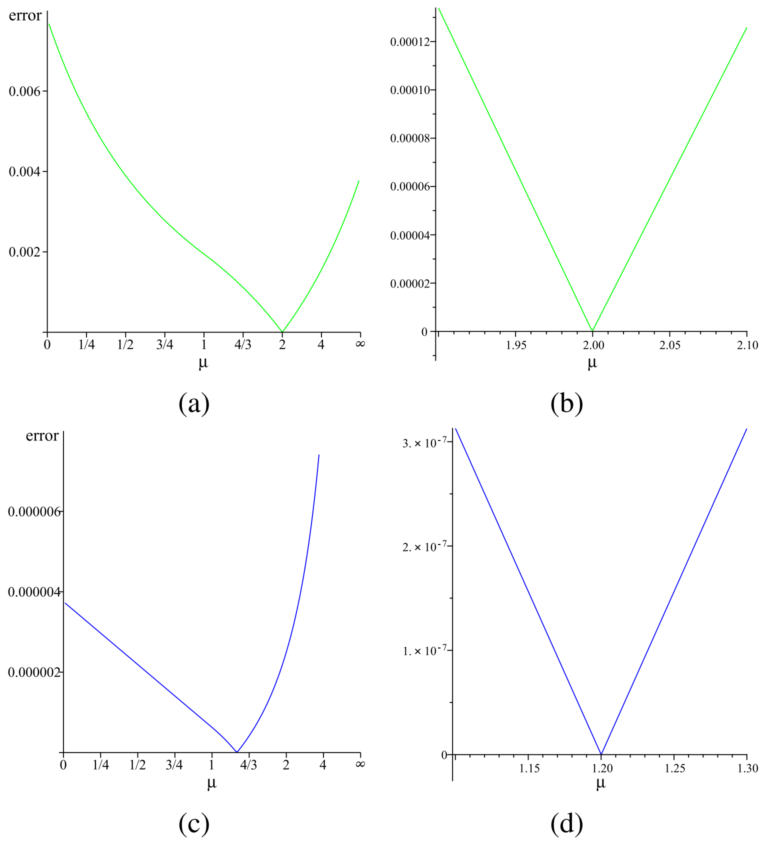

FIGURE 6. (a) The error function $|\lambda_\mu - L|$(green) for all $\mu \in (0, \infty)$. (b) $|\lambda_\mu - L|$(green) near $\mu = 2$. (c) The error function $|\mathcal{L}_\mu - L|$(blue) for all $\mu \in (0, \infty)$. (d) $|\mathcal{L}_\mu - L|$(blue) near $\mu = \frac{6}{5}$. Here the quadratic rational Bézier curve is the ellipse $(x/2)^2 + y^2 = 1$ and the number of segment is $k = 64$.

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no. segments	$\lvert\lambda_2 - 2\pi\rvert$	$\lvert\lambda_{\hat{\mu}} - 2\pi\rvert = \lvert\mathcal{L}_1 - 2\pi\rvert$	$\lvert\mathcal{L}_{\mu^*} - 2\pi\rvert$	$\lvert\mathcal{L}_{\frac{6}{5}} - 2\pi\rvert$
$k = 4$	1.43×10^{-2}	3.38×10^{-2}	0	1.47×10^{-3}
$k = 8$	8.46×10^{-4} $(\frac{1}{16.9})$	1.76×10^{-3} $(\frac{1}{19.2})$	0	2.23×10^{-5} $(\frac{1}{65.9})$
$k = 16$	5.21×10^{-5} $(\frac{1}{16.2})$	1.05×10^{-4} $(\frac{1}{16.7})$	0	3.44×10^{-7} $(\frac{1}{64.8})$
$k = 32$	3.25×10^{-6} $(\frac{1}{16.1})$	6.51×10^{-6} $(\frac{1}{16.2})$	0	5.36×10^{-9} $(\frac{1}{64.2})$
$k = 64$	2.03×10^{-7} $(\frac{1}{16.0})$	4.06×10^{-7} $(\frac{1}{16.0})$	0	8.37×10^{-11} $(\frac{1}{64.1})$

TABLE 1. Errors $|\lambda_\mu - 2\pi|$ and $|\mathcal{L}_\mu - 2\pi|$, after subdivision of the unit circle into k segments. The fractional number in bracket is decreasing ratio after each one subdivision. The value of $(\hat{\mu}, \mu^*)$ is (1.8026,1.2093), (1.9530,1.2026), (1.9884,1.2007), (1.9971,1.2002), (1.9993,1.2000) for $k = 4, \cdots, 64$, respectively.

no. segments	$\lvert\lambda_2 - L\rvert$	$\lvert\lambda_{\hat{\mu}} - L\rvert = \lvert\mathcal{L}_1 - 2\pi\rvert$	$\lvert\mathcal{L}_{\mu^*} - L\rvert$	$\lvert\mathcal{L}_{\frac{6}{5}} - L\rvert$
$k = 4$	5.40×10^{-2}	4.69×10^{-2}	7.42×10^{-2}	6.90×10^{-3}
$k = 8$	1.57×10^{-3} $(\frac{1}{34.4})$	2.68×10^{-3} $(\frac{1}{17.5})$	3.40×10^{-3} $(\frac{1}{21.8})$	7.51×10^{-5} $(\frac{1}{91.9})$
$k = 16$	6.84×10^{-5} $(\frac{1}{22.9})$	1.62×10^{-4} $(\frac{1}{16.5})$	2.10×10^{-4} $(\frac{1}{16.2})$	5.87×10^{-7} $(\frac{1}{128.0})$
$k = 32$	5.01×10^{-6} $(\frac{1}{13.7})$	1.00×10^{-5} $(\frac{1}{16.2})$	1.34×10^{-5} $(\frac{1}{15.6})$	8.36×10^{-9} $(\frac{1}{70.2})$
$k = 64$	3.13×10^{-7} $(\frac{1}{16.0})$	6.26×10^{-7} $(\frac{1}{16.0})$	8.45×10^{-7} $(\frac{1}{15.9})$	1.29×10^{-10} $(\frac{1}{64.5})$

TABLE 2. Errors $|\lambda_\mu - L|$ and $|\mathcal{L}_\mu - L|$, after subdivision of the ellipse $(x/2)^2 + y^2 = 1$ into k segments.

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