J. KSIAM Vol.15, No.1, 43–55, 2011

LIOUVILLE THEOREMS OF SLOW DIFFUSION DIFFERENTIAL INEQUALITIES WITH VARIABLE COEFFICIENTS IN CONE

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ABSTRACT. We here investigate the Liouville type theorems of slow diffusion differential inequality and its coupled system with variable coefficients in cone. First, we give the definition of global weak solution, and then we establish the universal estimate (does not depend on the initial value) of solution by constructing test function. At last, we obtain the nonexistence of non-negative non-trivial global weak solution within the appropriate critical exponent. The main feature of this method is that we need not use comparison theorem or the maximum principle.

1. INTRODUCTION

In this paper, we consider non-negative global weak solutions of the slow diffusion differential inequality with variable coefficients

$$\frac{\partial u}{\partial t} - div(|x|^{\alpha} Du^m) \geq |x|^{\beta} u^q \quad in \;\; K \times (0, \infty)$$

and its coupled system in cone, where $2 > \alpha > 1 - N$, $2 - \alpha + m\beta > 0$, $N \geq 3$, $q > m \geq 1$.

The inequality appears in the diffusion theory, biological theory and biological population dynamics, etc(see [2, 3, 4, 8]). Recently some authors have studied Liouville type theorems for parabolic differential inequality and its coupled system in $R^N \times (0, \infty)$. For example, Kartsatos et al. studied the nonexistence of the global solution within suitable critical range when $\alpha = \beta = 0$, $m = 1$, see [11]. In paper [18], Piccirillo considered entire solutions for a class of general evolution of inequality by using test function method which was introduced by Mitidieri and Pohozaev [16, 17]. For the studies of elliptic problems, we refer the readers to see [9, 10, 16]. In the case of the cone domain, Kondrat'ev studied boundary problems of linear elliptic equations by using comparison principle, see [5, 12, 15, 19, 21]. The research of the corresponding parabolic problems was initiated by Bandle and Levine [1]. For the developed theories of the linear and semi-linear parabolic problems in cone, we refer to the classical

Received by the editors March 16, 2011; Accepted March 18, 2011.

2000 *Mathematics Subject Classification.* 35R45, 35K65, 34A12.

Key words and phrases. slow diffusion parabolic differential inequalities, test function, Liouville theorem.

The Project-sponsored by the Fundamental Research Funds for the Central Universities(NO.201013043), NSFC(NO.10971204) and SRF for ROCS, SEM..

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papers of Laptev [13, 14] and the references therein. G.Caristi studied semi-linear parabolic inequalities by test function method, see [6]. In paper [13], G.G. Laptev studied the semilinear parabolic inequality and its coupled system with variable coefficients by using test function method. Here we use the nonlinear capacity in the form of the test function method to develop the Liouville type theorems for more generalized parabolic inequality in cone.

Our goal is to study the Liouville type theorems of slow diffusion parabolic differential inequality and its coupled system with variable coefficients in the diffusion and source terms. The difficult is to find how the variable coefficient exponents α, β and nonlinear exponents m, q impact on the global weak solution and its critical value closely links with the first eigenvalue corresponding to *Laplace-Beltrami* operator of the boundary problem with homogeneous boundary conditions in cone K_w. Moreover, we do not give any regularity assumptions on initial value, and then it does not lead to a good ›trace‹ on hyperplane t=0. Besides, we do not use comparison theorems or the maximum principle.

The plan of this paper is as follows. In Section 1, we introduce some basic knowledge, symbols and the process of constructing test function. In Section 2, we prove the Liouville type theorems of global weak solution of slow diffusion inequality. In Section 3, we prove the Liouville type theorems of its coupled system.

2. Basic knowledge review

In this section, we simply introduce some relevant basic knowledge and marks and then we detail the process of constructing test function.

Let K_ω be a subdomain of the unit sphere S^{N-1} $(N \geq 3)$ with piecewise smooth boundary ∂K_ω. A cone K is a set that can be described as follows in the spherical system of coordinates (r, ω), $0 \leq r < \infty$, $\omega \in S^{N-1}$,

$$K = \{x = (r, \omega) : \ 0 < r < \infty, \ \omega \in K_\omega\}$$

We denote the lateral surface of a cone by ∂K.

Let Ω be an unbounded subdomain of R^{N+1} with piecewise smooth boundary. We will use the well-known *Sobolev* anisotropic spaces $W_q^{2,1}(\Omega)$ and the local space $L^q_{loc}(\Omega)$ whose elements belong to $L^q(\Omega^{'})$ for each compact subset $\Omega^{'}(\bar{\Omega}^{'} \subset \Omega)$. Denote the space of continuous function by $C(\bar{\Omega})$ and the space of smooth functions by $C^m(\bar{\Omega})$ on the closed domain $\bar{\Omega}$. The expression $Du = (\frac{\partial u}{\partial x_i}))$ denotes the vector of partial derivatives and $Du \times D\varphi = \sum_{i=1}^N \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i}$ denotes product of the vector of partial derivatives for two differentiable functions $u(x)$ and $\varphi(x)$. Symbol $\frac{\partial u}{\partial n}$ denotes the derivative of the function u in the direction of the outward normal n to the boundary ∂K of the cone.

Recall that the Laplace operator Δ has the following form

$$\Delta = \frac{1}{r^{N-1}}\frac{\partial(r^{N-1}\frac{\partial}{\partial r})}{\partial r} + \frac{1}{r^2\Delta_\omega} = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega$$

where Δ_ω is the *Laplace-Beltrami* operator on the unit sphere $S^{N-1} \subset R^N$. Moreover, throughout this paper we use the first (the least) eigenvalue $\lambda_\omega \equiv \lambda_1(K_\omega) > 0$ and the corresponding eigenfunction $\phi(\omega)$ of the operator Δ_ω which solve the problem

$$\begin{cases} \Delta_\omega\phi + \lambda\phi = 0 \quad in\ K_\omega, \\ \phi = 0 \qquad on\ \partial K_\omega. \end{cases} \tag{2.1}$$

It is well known that $\phi(\omega) > 0$ for $\omega \in K_\omega$ and assume that $\phi(\omega) \le 1$.

In the following, we will detail the process of constructing test function. In fact, we construct test functions by using the separation of variables. Therefore, we consider the corresponding t-function firstly and then x-function.

We begin with a standard cut-off function $\zeta(y) \in C^\infty(R^+)$ such that $0 \le \zeta(y) \le 1$ when $0 \le y \le 1$, $\zeta(y) = 1$ and when $y \ge 2$, $\zeta(y) = 0$.

Next, we construct a function $\eta(y)$ such that the pointwise inequalities

$$|\eta'(y)|^p \le c_\eta \eta^{p-1}(y), \quad |\eta''(y)|^p \le c_\eta \eta^{p-1}(y)$$

hold with some constant c_η for all $1 < p \le p_0$, where p_0 is fixed. It is sufficient to take

$$\eta(y) = (\zeta(y))^{2p_0}.$$

Then

$$|\eta'(y)|^p = (2p_0)^p \zeta^{2p_0(p-1)}\zeta^{2p_0-p}|\zeta'|^p = (2p_0)^p\zeta^{2p_0-p}|\zeta'|^p\eta^{p-1} \le c_\eta\eta^{p-1}(y),$$
$$|\eta''(y)|^p \le (2p_0)^p\zeta^{2p_0(p-1)}\zeta^{2p_0-2p}((2p_0-1)|\zeta'|^2 + \zeta|\zeta''|)^p \le c_\eta\eta^{p-1}(y).$$

Introduce a parameter ρ which will be increasing without limit in our subsequent constructions and a positive exponent θ. And we also consider the function $\eta(\frac{t}{\rho^\theta})$ of the variable t$(t \ge 0)$. By directly calculating, we obtain

$$\eta'(\frac{t}{\rho^\theta}) = 2p_0\zeta^{2p_0-1}\zeta'\frac{1}{\rho^\theta},$$

so

$$|\eta'(\frac{t}{\rho^\theta})|^p = (2p_0)^p\zeta^{2p_0(p-1)}\zeta^{2p_0-p}|\zeta'|^p\rho^{-\theta p} \le c_\eta\rho^{-\theta p}\eta^{p-1}(\frac{t}{\rho^\theta}),$$

and then

$$\int_{supp|\frac{d\eta(\frac{t}{\rho^\theta})}{dt}|} \frac{|\frac{d\eta(\frac{t}{\rho^\theta})}{dt}|^p}{\eta^{p-1}(\frac{t}{\rho^\theta})}dt \le \int_{supp|\frac{d\eta(\frac{t}{\rho^\theta})}{dt}|} c_\eta\rho^{-\theta p}dt = c_\eta\rho^{-\theta(p-1)}, \tag{2.2}$$

here

$$supp|\frac{d\eta(\frac{t}{\rho^\theta})}{dt}| = \{\rho^\theta < t < 2\rho^\theta\}.$$

Now we perform similar constructions for the test function of the variable x. However, since we deal with a conical domain in this case, we consider a product of two functions

$$r^s\eta(r/\rho)$$

where $r = |x|$ and $s > 0$. Hence, we have

$$\frac{\partial(r^s\eta(\frac{r}{\rho}))}{\partial r} = sr^{s-1}\eta(\frac{r}{\rho}) + r^s\eta'(\frac{r}{\rho})\frac{1}{\rho},$$

and

$$
\begin{aligned}
|\frac{\partial(r^s\eta(\frac{r}{\rho}))}{\partial r}|^p &= |sr^{s-1}\eta(\frac{r}{\rho}) + r^s\eta'(\frac{r}{\rho})\frac{1}{\rho}|^p \\
&\le c_p s^p r^{p(s-1)}\eta^p(\frac{r}{\rho}) + c_p r^{ps}|\eta'(\frac{r}{\rho})|^p\ \frac{1}{\rho} \\
&\le c_p s^p r^{p(s-1)}\eta^{p-1}(\frac{r}{\rho}) + c_p r^{ps} c_\eta \eta^{p-1}(\frac{r}{\rho})\frac{1}{\rho} \\
&\le c_{ps\eta} r^{p(s-1)}\eta^{p-1}(\frac{r}{\rho})(1+\frac{r^p}{\rho^p}),
\end{aligned}
$$

where constant $c_{\rho s\eta}$ is independent of ρ and r. A similar estimate can also be established for the second derivative

$$
\begin{aligned}
\frac{\partial^2(r^s\eta(\frac{r}{\rho}))}{\partial r^2} &= \frac{\partial}{\partial r}(sr^{s-1}\eta(\frac{r}{\rho}) + r^s\eta'(\frac{r}{\rho})\frac{1}{\rho}) \\
= s(s-1)r^{s-2}\eta(\frac{r}{\rho}) &+ sr^{s-1}\eta'(\frac{r}{\rho})\frac{1}{\rho} + sr^{s-1}\eta'(\frac{r}{\rho})\frac{1}{\rho} + r^s\eta''(\frac{r}{\rho})\frac{1}{\rho^2} \\
&= r^{s-2}[s(s-1)\eta(\frac{r}{\rho}) + 2s\eta'(\frac{r}{\rho})\frac{r}{\rho} + \eta''(\frac{r}{\rho})\frac{r^2}{\rho^2}],
\end{aligned}
$$

and

$$
\begin{aligned}
|\frac{\partial^2(r^s\eta(\frac{r}{\rho}))}{\partial r^2}|^p &= r^{p(s-2)}[(s(s-1)\eta(\frac{r}{\rho}) + 2s\eta'(\frac{r}{\rho})\frac{r}{\rho} + \eta''(\frac{r}{\rho})\frac{r^2}{\rho^2}]^p \\
&\le r^{p(s-2)}[c_p s^p(s-1)^p\eta^p(\frac{r}{\rho}) + 2c_p s^p|\eta'(\frac{r}{\rho}|^p\frac{r^p}{\rho^p} + c_p|\eta''(\frac{r}{\rho})|^p\frac{r^{2p}}{\rho^{2p}}] \\
&\le r^{p(s-2)}[c_p s^p(s-1)^p\eta^{p-1}(\frac{r}{\rho}) + 2c_p s^p c_\eta\eta^{p-1}(\frac{r}{\rho})\frac{r^p}{\rho^p} + c_p c_\eta\eta^{p-1}(\frac{r}{\rho})\frac{r^{2p}}{\rho^{2p}}] \\
&\le r^{p(s-2)}c_{ps\eta}\eta^{p-1}(\frac{r}{\rho})(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}}).
\end{aligned}
$$

Then we clarify the meaning of the above estimates. First, we introduce the Laplace operator $\triangle$. We consider the function$r^s\Phi(\omega)$. Taking account of the equality $\triangle_\omega\Phi = -\lambda_\omega\Phi$, we obtain

$$
\begin{aligned}
\triangle(r^s\Phi(\omega)) &= \frac{\partial^2}{\partial r^2}(r^s\Phi(\omega) + \frac{N-1}{r}\frac{\partial(r^s\Phi(\omega) + \frac{1}{r^2}\triangle_\omega(r^s\Phi(\omega)))}{\partial r}) \\
&= s(s-1)r^{s-2}\Phi(\omega) + \frac{N-1}{r}sr^{s-1}\Phi(\omega) + \frac{1}{r^2}(-\lambda_\omega r^s\Phi(\omega)) \\
&= s(s-1)r^{s-2}\Phi(\omega) + s(N-1)r^{s-2}\Phi(\omega) - \lambda_\omega r^{s-2}\Phi(\omega)) \\
&= r^{s-2}\Phi(\omega)\{s(s-1)+s(N-1)-\lambda_\omega\}.
\end{aligned}
$$

Let us introduce the parameters

$$\begin{cases} s^* = \frac{N-2}{2} + \sqrt{(\frac{N-2}{2})^2 + \lambda_\omega} \\ s_* = -\frac{N-2}{2} + \sqrt{(\frac{N-2}{2})^2 + \lambda_\omega} \end{cases} \tag{2.3}$$

The quantities s^*, s_* are the roots of the polynomial in the curly brackets in the above expression for $\triangle(r^s\Phi(\omega))$. We consider now the function $\xi(x) \equiv \xi(r,\omega) = r^{s_*}\Phi(\omega)$, so there is the identity $\triangle\xi = 0$ in K. It is also obvious that it vanishes at ∂K.

We now establish an estimate which is similar to (2.2).

Set $\psi(x) = \xi(x)\eta(\frac{|x|}{\rho}) = r^{s_*}(\frac{r}{\rho})\Phi(\omega)$, we observe that

$$\triangle\psi_\rho = \frac{\partial^2\psi_\rho}{\partial r^2} + \frac{N-1}{r}\frac{\partial\psi_\rho}{\partial r} + \frac{1}{r^2}\triangle_\omega\psi_\rho,$$

where $\triangle_\omega\psi_\rho = r^{s_*}\eta(\frac{|x|}{\rho})(-\lambda_\omega\Phi) = -\lambda_\omega\psi_\rho$.

Taking account of the above estimates for the derivatives of the product $r^s\eta(\frac{r}{\rho})$, we obtain

$$|\triangle\psi(x)|^p = |(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} - \frac{\lambda_\omega}{r^2})(r^s\eta(\frac{r}{\rho}))|^p\Phi^p(\omega)$$

$$\leq c_{ps\eta}\eta^{p-1}(\frac{r}{\rho})r^{p(s_*-2)}(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}})+(\frac{N-1}{r})^p c_{ps\eta}\eta^{p-1}(\frac{r}{\rho})r^{p(s_*-1)}(1+\frac{r^p}{\rho^p})+\frac{\lambda_\omega^p}{r^{2p}}r^{ps_*}\eta^p(\frac{r}{\rho})$$

$$\leq c\eta^{p-1}(\frac{r}{\rho})r^{p(s_*-2)}(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}}) = c\psi_\rho^{p-1}(x)\frac{1}{r^{2p-s_*}}(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}}).$$

By the construction of $\eta(\frac{|x|}{\rho}) \equiv 1$ for $|x| \leq p$, we obtain $\triangle\psi_\rho(x) = \triangle\xi(x) = 0$ and $supp|\triangle\psi_\rho(x)| = \{K\bigcap\{\rho < |x| < 2\rho\}\}$, hence $(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}})$ is bounded on $supp|\triangle\psi_\rho(x)|$, what is more

$$r^{s^*-2p} \leq c\rho^{s^*-2p},$$

therefore

$$|\triangle\psi_\rho(x)|^p \leq c\psi_\rho(x)^{p-1}\rho^{s^*-2p},$$

so

$$\int_{supp|\triangle\psi_\rho|}\frac{|\triangle\psi_\rho(x)|^p}{\psi_\rho^{p-1}(x)}dx \leq c\int_\rho^{2\rho}\int_{K_\omega}\frac{\psi_\rho^{p-1}(x)r^{N-1}}{\psi_\rho^{p-1}(x)\rho^{2p-s_*}}drd\omega \leq c_\psi\rho^{s_*-2p+N}.$$

In this way, we take the composite test function in the following form

$$\varphi_\rho(x,t) = \eta(\frac{t}{\rho^\theta})\psi_\rho(x).$$

At the same time, we can also get the accurate estimate about test function

$$\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}|}\frac{|\frac{\partial\varphi_\rho}{\partial t}|^p}{\varphi_\rho^{p-1}}dxdt \leq \int_{K\bigcap\{r<2\rho\}}\psi_\rho(x)dx \times \int_{supp|\frac{d\eta(\frac{t}{\rho^\theta})}{dt}|}\frac{|\frac{d\eta(\frac{t}{\rho^\theta})}{dt}|^p}{\eta^{p-1}(\frac{t}{\rho^\theta})}dt$$

$$\leq \int_0^{2\rho} r^{s_*} dx \times c_\eta \rho^{-\theta(p-1)}$$
$$= c_\eta \rho^{-\theta(p-1)} \int_0^{2\rho} r^{N-1+s_*} dr$$
$$= c_\varphi \rho^{s_*+N-\theta(p-1)}, \tag{2.4}$$

$$\int\int_{supp|\triangle\varphi_\rho|} \frac{|\triangle\varphi_\rho(x,t)|^p}{\varphi_\rho^{p-1}} dxdt \leq \int_0^{2\rho^\theta} \eta(\frac{t}{\rho^\theta})dt \int_{supp|\triangle\varphi_\rho|} \frac{|\triangle\varphi_\rho|^p}{\varphi_\rho^{p-1}} dx$$
$$\leq \int_0^{2\rho^\theta} 1dt \int_{supp|\triangle\varphi_\rho|} \frac{|\triangle\varphi_\rho|^p}{\varphi_\rho^{p-1}} dx \leq c_\varphi \rho^{\theta-2p+s_*+N}. \tag{2.5}$$

When $\theta = 2$, the powers on the right-hand sides of two above-mentioned inequalities are the same, hence we get the estimate

$$\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+\triangle\varphi_\rho|} \frac{|\frac{\partial\varphi_\rho}{\partial t} + \triangle\varphi_\rho|^p}{\varphi_\rho^{p-1}} dxdt \leq c_0 \rho^{-2p+s_*+N+2}.$$

3. LIOUVILLE THEOREM OF SLOW DIFFUSION INEQUALITY

In this section, we obtain the Liouville theorem of the slow diffusion inequality. Now we consider the parabolic type inequality as following

$$\frac{\partial u}{\partial t} - div(|x|^\alpha D(u^m)) \geq |x|^\beta u^q, \tag{3.1}$$

where

$$2 > \alpha > 1-N, 2-\alpha+m\beta > 0, N \geq 3, u \geq 0, q > m > 1.$$

Then we interpret the definition of a nonnegative weak solution.

Definition 3.1. *Let $u(x,t) \in C(\bar{K} \times [0,\infty))$. A non-negative function $u(x,t)$ is called a weak solution of (3.1) if for each non-negative test function $\varphi(x,t) \in W_\infty^{2,1}(K \times (0,\infty))$ with compact support with respect to $r = |x|$ and t such that $\varphi|_{\partial(K\times(0,\infty))} = 0$, we have the following integral inequality holding*

$$\int_0^\infty \int_{\partial K} u^m |x|^\alpha \frac{\partial\varphi}{\partial n} dxdt - \int_0^\infty \int_K u\frac{\partial\varphi}{\partial t} dxdt - \int_0^\infty \int_K u^m div(|x|^\alpha D\varphi) dxdt$$
$$\geq \int_0^\infty \int_K |x|^\beta u^q \varphi dxdt + \int_K u(x,0)\varphi(x,0)dx. \tag{3.2}$$

Theorem 3.1. *For*

$$1 < m < q \leq q^* = m + \frac{2-\alpha+m\beta}{s^{*\alpha}+2-\alpha},$$

the solution of (3.1) is identical to zero, where

$$s^{*\alpha} = \frac{\alpha+N-2}{2} + \sqrt{(\frac{\alpha+N-2}{2})^2 + \lambda_\omega}.$$

Proof. We proceed by contradiction. Assume that $u(x,t)$ is not identical to zero and $1 < m < q \leq q^*$. We choose a test function in a form similar to the introduction. Draw into an exponent $\theta > 0$, and set $\psi_\rho(x) = \xi_\alpha \eta(\frac{|x|}{\rho})$, $\varphi_\rho(x,t) = \eta(\frac{t}{\rho^\theta})\psi_\rho(x)$, where

$$\xi_\alpha(x) \equiv \xi_\alpha(r,\omega) = r^{s_{*\alpha}}\Phi(\omega).$$

In this case, taking place of the Laplace operator we have the operator

$$A_\alpha \equiv div(|x|^\alpha D) = r^\alpha\{\frac{\partial^2}{\partial r^2} + \frac{\alpha+N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_\omega\}.$$

By calculation in K, so

$$A_\alpha \xi_\alpha \equiv div(|x|^\alpha D\xi_\alpha) = r^\alpha\{\frac{\partial^2 \xi_\alpha}{\partial r^2} + \frac{\alpha+N-1}{r}\frac{\partial \xi_\alpha}{\partial r} + \frac{1}{r^2}\triangle_\omega \xi_\alpha\}$$

$$= r^\alpha\{s_{*\alpha}(s_{*\alpha}-1)r^{s_{*\alpha}-2}\Phi(\omega) + \frac{\alpha+N-1}{r}s_{*\alpha}r^{s_{*\alpha}-1}\Phi(\omega) + \frac{1}{r^2}(-\lambda_\omega r^{s_{*\alpha}}\Phi(\omega))\}$$

$$= r^\alpha\{s_{*\alpha}(s_{*\alpha}-1)r^{s_{*\alpha}-2}\Phi(\omega) + s_{*\alpha}(\alpha+N-1)r^{s_{*\alpha}-2}\Phi(\omega) - \lambda_\omega r^{s_{*\alpha}-2}\Phi(\omega))\}$$

$$= r^\alpha r^{s_{*\alpha}-2}\{s_{*\alpha}(s_{*\alpha}-1) + s_{*\alpha}(\alpha+N-1) - \lambda_\omega\}\Phi(\omega) \equiv 0.$$

The first integral on the left-hand side of (3.2) is non-positive. Indeed, we have

$$\frac{\partial \xi}{\partial n} = r^{s_{*\alpha}}\frac{\partial \phi(\omega)}{\partial n_\omega} \leq 0,$$

where n_ω is the outward normal to the boundary of K_ω and the non-positiveness of the derivative $\frac{\partial \phi(\omega)}{\partial n_\omega}$ is a consequence of Hopf's Lemma, because the function $\phi(\omega)$ is positive in K_ω and vanishes at the boundary. By applying Hölder inequality to the second and third integrals on the left-hand side of (3.2), we obtain

$$\int_K u(x,0)\varphi(x,0)dx + \int_0^\infty \int_K |x|^\beta u^q \varphi dxdt$$

$$\leq \int\int_{supp|\frac{\partial \varphi_\rho}{\partial t}|} u|\frac{\partial \varphi_\rho}{\partial t}|dxdt + \int\int_{supp|A_\alpha \varphi_\rho|} u^m |A_\alpha \varphi_\rho| dxdt$$

$$\leq (\int\int_{supp|\frac{\partial \varphi_\rho}{\partial t}|} u^q \varphi_\rho dxdt)^{\frac{1}{q}} \times (\int\int_{supp|\frac{\partial \varphi_\rho}{\partial t}|} \frac{|\frac{\partial \varphi_\rho}{\partial t}|^{q'}}{\varphi_\rho^{q'-1}} dxdt)^{\frac{1}{q'}}$$

$$+(\int\int_{supp|A_\alpha \varphi_\rho|} u^q \varphi_\rho dxdt)^{\frac{m}{q}} \times (\int\int_{supp|A_\alpha \varphi_\rho|} \frac{|A_\alpha \varphi_\rho|^{\frac{q}{q-m}}}{\varphi_\rho^{\frac{m}{q-m}}} dxdt)^{\frac{q-m}{q}}, \qquad (3.3)$$

where

$$\frac{1}{q} + \frac{1}{q'} = 1, \frac{m}{q} + \frac{q-m}{q} = 1.$$

Similar to the derivation in the introduction, we can also obtain integral estimate as follows

$$\int\int_{supp|\frac{\partial \varphi_\rho}{\partial t}|} \frac{|\frac{\partial \varphi_\rho}{\partial t}|^{q'}}{\varphi_\rho^{q'-1}} dxdt \leq c\rho^{-\theta q' + s_{*\alpha} + \theta + N}, \qquad (3.4)$$

$$\int\int_{supp|A_\alpha\varphi_\rho|}\frac{|A_\alpha\varphi_\rho|^{\frac{q}{q-m}}}{\varphi_\rho^{\frac{m}{q-m}}}dxdt\le C\rho^{\frac{-(2-\alpha)q}{q-m}+\theta+s_{*\alpha}+N},\tag{3.5}$$

Set $-\theta q'+s_{*\alpha}+\theta+N=\frac{-(2-\alpha)q}{q-m}+\theta+s_{*\alpha}+N$, we obtain

$$\theta=\frac{(2-\alpha)(q-1)}{q-m}>0.$$

Hence the powers on the right-hand sides of estimates (3.4) and (3.5) are the same and we get

$$\int\int_{\varphi_\rho=\xi(x)}u^q\xi(x)dxdt\le\rho^{\frac{-\beta q}{q-m}}\int\int_{supp|A_\alpha\varphi_\rho|}\frac{|A_\alpha\varphi_\rho|^{\frac{q}{q-m}}}{\varphi_\rho^{\frac{m}{q-m}}}dxdt$$
$$\le c_0\rho^{\frac{-(2-\alpha)}{q-m}+s_{*\alpha}+N-\frac{\beta q}{q-m}},$$

that is

$$\int\int_{\varphi_\rho=\xi(x)}|x|^\beta u^q\xi(x)dxdt\le\rho^\beta\rho^{\frac{-(2-\alpha)}{q-m}+s_{*\alpha}+N-\frac{\beta q}{q-m}}$$
$$=c_0\rho^{\frac{-(2-\alpha)}{q-m}+s_{*\alpha}+N-\frac{\beta m}{q-m}}.$$

By $\frac{1}{q}+\frac{1}{q'}=1$, $1<m<q\le q^*=m+\frac{2-\alpha+m\beta}{s^{*\alpha}+2-\alpha}$, we know

$$\rho\frac{-(2-\alpha)}{q-m}+s_{*\alpha}+N-\frac{\beta m}{q-m}\le 0,$$

what is more

$$\int_0^\infty\int_K|x|^\beta u^q\xi_\alpha dxdt=0.$$

Since ξ is positive in K, it follows that $u\equiv 0$ which is in contradiction with our assumption. □

4. LIOUVILLE THEOREM OF COUPLING INEQUALITIES

In this section, we shall prove the Liouville theorem of coupling inequalities with variable coefficients.

We consider slow diffusion coupling inequalities (4.1) with variable coefficients as follows.

$$\begin{cases}\frac{\partial u}{\partial t}-div(|x|^\alpha Du^m)\ge|x|^\beta v^{q_1} & in\ \ K\times(0,\infty)\\ \frac{\partial v}{\partial t}-div(|x|^\alpha Dv^m)\ge|x|^\beta u^{q_2} & in\ \ K\times(0,\infty)\\ u\ge 0,\quad v\ge 0.\end{cases}\tag{4.1}$$

where

$$2>\alpha>1-N,\ \ 2-\alpha+m\beta>0,\ \ N\ge 3,\ \ m>1,\ \ q_1>m,\ \ q_2>m.$$

Definition 4.1. *A pair of non-negative functions $u, v \in C(\bar{K}\times[0,\infty))$ is called a weak solution of (4.1) if for each non-negative test function $\varphi(x,t) \in W^{2,1}_{\infty}(K \times (0,\infty))$ with compact support with respect to $r = |x|$ and t such that $\varphi|_{\partial(K\times(0,\infty))} = 0$, we have the following integral inequalities holding*

$$\int_0^\infty \int_{\partial K} u^m |x|^\alpha \frac{\partial \varphi}{\partial n} dxdt - \int_0^\infty \int_K u \frac{\partial \varphi}{\partial t} dxdt - \int_0^\infty \int_K u^m div(|x|^\alpha D\varphi) dxdt$$
$$\geq \int_0^\infty \int_K |x|^\beta v^{q_1} \varphi dxdt + \int_K u(x,0)\varphi(x,0)dx,$$

Theorem 4.1. *For*

$$max\{\gamma_1, \gamma_2\} \geq \frac{s^*_\alpha + 2 - \alpha}{2 - \alpha + m\beta},$$

the solution of (4.1) is identical to zero, where

$$\gamma_1 = \frac{q_1 + m}{q_1 q_2 - m^2}\ , \qquad \gamma_2 = \frac{q_2 + m}{q_1 q_2 - m^2}\ .$$

Proof. Assume $u(x,t)$, $v(x,t)$ are not identical to zero. We choose the same test function as that in the theorem 3.1, that is $\varphi_\rho(x,t) = \eta(\frac{t}{\rho^\theta})\psi_\rho(x)$. By using Hölder inequality, we obtain the integral estimates

$$\int_K u(x,0)\varphi_\rho(x,0)dx + \int_0^\infty \int_K |x|^\beta v^{q_1} \varphi_\rho dxdt$$
$$\leq \int\int_{supp|\frac{\partial \varphi_\rho}{\partial t}|} u|\frac{\partial \varphi_\rho}{\partial t}|dxdt + \int\int_{supp|A_\alpha \varphi_\rho|} u^m |A_\alpha \varphi_\rho| dxdt$$
$$\leq (\int\int_{supp|\frac{\partial \varphi_\rho}{\partial t}|} u^{q_2}\varphi_\rho dxdt)^{\frac{1}{q_2}} \times (\int\int_{supp|\frac{\partial \varphi_\rho}{\partial t}|} \frac{|\frac{\partial \varphi_\rho}{\partial t}|^{q'_2}}{\varphi_\rho^{q'_2-1}} dxdt)^{\frac{1}{q'_2}}$$
$$+(\int\int_{supp|A_\alpha \varphi_\rho|} u^{q_2}\varphi_\rho dxdt)^{\frac{m}{q_2}} \times (\int\int_{supp|A_\alpha \varphi_\rho|} \frac{|A_\alpha \varphi_\rho|^{\frac{q_2}{q_2-m}}}{\varphi_\rho^{\frac{m}{q_2-m}}} dxdt)^{\frac{q_2-m}{q_2}}$$
$$\leq (\int\int_{supp|\frac{\partial \varphi_\rho}{\partial t}+A_\alpha \varphi_\rho|} u^{q_2}\varphi_\rho dxdt)^{\frac{m}{q_2}} \times (\int\int_{supp|A_\alpha \varphi_\rho|} \frac{|A_\alpha \varphi_\rho|^{\frac{q_2}{q_2-m}}}{\varphi_\rho^{\frac{m}{q_2-m}}} dxdt)^{\frac{q_2-m}{q_2}}, \quad (4.2)$$

$$\int_K v(x,0)\varphi_\rho(x,0)dx + \int_0^\infty \int_K |x|^\beta u^{q_2} \varphi_\rho dxdt$$
$$\leq \int\int_{supp|\frac{\partial \varphi_\rho}{\partial t}|} v|\frac{\partial \varphi_\rho}{\partial t}|dxdt + \int\int_{supp|A_\alpha \varphi_\rho|} v^m |A_\alpha \varphi_\rho| dxdt$$
$$\leq (\int\int_{supp|\frac{\partial \varphi_\rho}{\partial t}|} v^{q_1}\varphi_\rho dxdt)^{\frac{1}{q_1}} \times (\int\int_{supp|\frac{\partial \varphi_\rho}{\partial t}|} \frac{|\frac{\partial \varphi_\rho}{\partial t}|^{q'_1}}{\varphi_\rho^{q'_1-1}} dxdt)^{\frac{1}{q'_1}}$$

$$+(\int\int_{supp|A_\alpha\varphi_\rho|} v^{q_1}\varphi_\rho dxdt)^{\frac{m}{q_1}} \times (\int\int_{supp|A_\alpha\varphi_\rho|} \frac{|A_\alpha\varphi_\rho|^{\frac{q_1}{q_1-m}}}{\varphi_\rho^{\frac{m}{q_1-m}}} dxdt)^{\frac{q_1-m}{q_1}}$$

$$\leq (\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho} |v^{q_1}\varphi_\rho dxdt)^{\frac{m}{q_1}} \times (\int\int_{supp|A_\alpha\varphi_\rho|} \frac{|A_\alpha\varphi_\rho|^{\frac{q_1}{q_1-m}}}{\varphi_\rho^{\frac{m}{q_1-m}}} dxdt)^{\frac{q_1-m}{q_1}}. \quad (4.3)$$

Here we denote

$$J_1 = \int\int_{supp|A_\alpha\varphi_\rho|} \frac{|A_\alpha\varphi_\rho|^{\frac{q_2}{q_2-m}}}{\varphi_\rho^{\frac{m}{q_2-m}}} dxdt,$$

$$J_2 = \int\int_{supp|A_\alpha\varphi_\rho|} \frac{|A_\alpha\varphi_\rho|^{\frac{q_1}{q_1-m}}}{\varphi_\rho^{\frac{m}{q_1-m}}} dxdt.$$

For

$$\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}|} \frac{|\frac{\partial\varphi_\rho}{\partial t}|^{q'}}{\varphi_\rho^{q'-1}} dxdt \leq c\rho^{-\theta q'+s_{*\alpha}+\theta+N},$$

$$\int\int_{supp|A_\alpha\varphi_\rho|} \frac{|A_\alpha\varphi_\rho|^{\frac{q}{q-m}}}{\varphi_\rho^{\frac{m}{q-m}}} dxdt \leq c\rho^{-\frac{(2-\alpha)q}{q-m}+\theta+s_{*\alpha}+N},$$

we obtain $\theta = \frac{(2-\alpha)(q-1)}{q-m} > 0$, and thus the powers on the right-hand sides of estimates are the same, furthermore we mark

$$J_1 \leq c\rho^{-\frac{2-\alpha}{q_2-m}+s_{*\alpha}+N}, J_2 \leq c\rho^{-\frac{2-\alpha}{q_1-m}+s_{*\alpha}+N}.$$

Due to (4.2) and (4.3), we have

$$\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} |x|^\beta v^{q_1}\varphi_\rho dxdt$$
$$\leq c\rho^\beta \int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} v^{q_1}\varphi_\rho dxdt$$
$$\leq (\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} u^{q_2}\varphi_\rho dxdt)^{\frac{m}{q_2}} \times J_1^{\frac{q_2-m}{q_2}},$$

$$\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} |x|^\beta u^{q_2}\varphi_\rho dxdt$$
$$\leq c\rho^\beta \int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} u^{q_2}\varphi_\rho dxdt$$
$$\leq (\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} v^{q_1}\varphi_\rho dxdt)^{\frac{m}{q_1}} \times J_2^{\frac{q_1-m}{q_1}}.$$

Then we see that

$$\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} v^{q_1}\varphi_\rho dxdt$$
$$\leq (\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} u^{q_2}\varphi_\rho dxdt)^{\frac{m}{q_2}} \times J_1^{\frac{q_2-m}{q_2}}\rho^{-\beta}$$
$$\leq ((\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} v^{q_1}\varphi_\rho dxdt)^{\frac{m}{q_1}} \times J_2^{\frac{q_1-m}{q_1}}\rho^{-\beta})^{\frac{m}{q_2}} \times J_1^{\frac{q_2-m}{q_2}}\rho^{-\beta}.$$

That is

$$(\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} v^{q_1}\varphi_\rho dxdt)^{1-\frac{m^2}{q_1q_2}} \leq J_2^{\frac{m(q_1-m)}{q_1q_2}}\rho^{-\frac{\beta m}{q_2}}J_1^{\frac{q_2-m}{q_2}}\rho^{-\beta},$$

i.e.

$$(\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} v^{q_1}\varphi_\rho dxdt)^{\frac{m^2}{q_1q_2}} \leq (J_2^{\frac{m(q_1-m)}{q_1q_2}}\rho^{-\frac{\beta(q_2+m)}{q_2}}J_1^{\frac{q_2-m}{q_2}})^{\frac{m^2}{q_1q_2-m^2}}.$$

Simplify the exponents of these estimates:

$$\int_0^\infty\int_K |x|^\beta v^{q_1}\varphi_\rho dxdt$$
$$\leq (\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} u^{q_2}\varphi_\rho dxdt)^{\frac{m}{q_2}} \times J_1^{\frac{q_2-m}{q_2}}$$
$$\leq ((\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} v^{q_1}\varphi_\rho dxdt)^{\frac{m}{q_1}} \times J_2^{\frac{q_1-m}{q_1}}\rho^{-\beta}dxdt)^{\frac{m}{q_2}} \times J_1^{\frac{q_2-m}{q_2}}$$
$$= (\int\int_{supp|\frac{\partial\varphi_\rho}{\partial t}+A_\alpha\varphi_\rho|} v^{q_1}\varphi_\rho dxdt)^{\frac{m^2}{q_1q_2}} \times J_2^{\frac{m(q_1-m)}{q_1q_2}}\rho^{-\frac{\beta m}{q_2}}J_1^{\frac{q_2-m}{q_2}}$$
$$\leq (J_2^{\frac{m(q_1-m)}{q_1q_2}}\rho^{-\frac{\beta(q_2+m)}{q_2}}J_1^{\frac{q_2-m}{q_2}})^{\frac{m^2}{q_1q_2-m^2}} \times J_2^{\frac{m(q_1-m)}{q_1q_2}}\rho^{-\frac{\beta m}{q_2}}J_1^{\frac{q_2-m}{q_2}}$$
$$= J_2^{\frac{m(q_1-m)}{q_1q_2}(\frac{m^2}{q_1q_2-m^2}+1)} \times J_1^{\frac{q_2-m}{q_2}(\frac{m^2}{q_1q_2-m^2}+1)}\rho^{-\frac{\beta(q_2+m)m^2}{q_2(q_1q_2-m^2)}-\frac{\beta m}{q_2}}$$
$$= J_2^{\frac{m(q_1-m)}{q_1q_2-m^2}} \times J_1^{\frac{q_1(q_2-m)}{q_1q_2-m^2}}\rho^{-\frac{\beta m(q_1+m)}{q_1q_2-m^2}}$$
$$\leq C\rho^{-\frac{q_1(2-\alpha)}{q_1q_2-m}-\frac{m(2-\alpha)}{q_1q_2-m}+(s_{*\alpha}+N)\frac{q_1q_2-q_1+mq_1-m}{q_1q_2-m^2}-\beta m\gamma_1}$$
$$= C\rho^{-(2-\alpha)\gamma_1+(s_{*\alpha}+N)-\beta m\gamma_1}.$$

By $max\{\gamma_1,\gamma_2\} \geq \frac{s^*_\alpha+2-\alpha}{2-\alpha+m\beta}$, if $\gamma_1 \geq \gamma_2$, we can receive

$$\gamma_1 \geq \frac{s^*_\alpha+2-\alpha}{2-\alpha+m\beta},$$

that is, when $-(2-\alpha)\gamma_1+(s_{*\alpha}+N)-\beta m\gamma_1 \leq 0$, the functions $v(x,t)$ identically vanishes which is in contradiction with our assumption.

In a similar way,

$$\int_0^\infty \int_K |x|^\beta u^{q_2} \varphi_\rho dxdt \le C\rho^{-(2-\alpha)\gamma_2+(s_{*\alpha}+N)-\beta m\gamma_2}.$$

If $\gamma_2 \ge \gamma_1$, we can see that when

$$-(2-\alpha)\gamma_2 + (s_{*\alpha}+N) - \beta m\gamma_2 \le 0,$$

the functions $u(x,t)$ identically vanishes which is in contradiction with our assumption. Obviously, if one of the functions $u(x,t)$ and $v(x,t)$ is identical to zero and then so is the second. □

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