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Interval-valued Fuzzy Ideals and Bi-ideals of a Semigroup

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Abstract

We apply the concept of interval-valued fuzzy sets to theory of semigroups. We give some properties of interval-valued fuzzy ideals and interval-valued fuzzy bi-ideals, and characterize which is left [right] simple, left [right] duo and a semilattice of left [right] simple semigroups or another type of semigroups in terms of interval-valued fuzzy ideals and interval-valued fuzzy bi-ideals.

Key Words: interval-valued fuzzy set, interval-valued fuzzy semigroup, interval-valued fuzzy ideal, interval-valued fuzzy bi-ideal, interval-valued fuzzy duo.

1. Introduction

As a generalization of fuzzy sets introduced by Zadeh[14], he[15] introduced the concept of interval-valued fuzzy sets. After that time, Gorzalczany[4] applied it to a method of inference in approximate resoning, Biswass[1] to group theory and Montal and Samanta[12] to topology. Recently, Hur et al.[5] introduced the notion of interval-valued fuzzy relations and obtained some of it's properties. Moreover, Choi et al.[3] introduced the concept of interval-valued smooth topological spaces and studied it. Kang and Hur[6] investigated interval-valued fuzzy subgroups and rings.

In this paper, we apply the notion of interval-valued fuzzy sets to theory of semigroups. We give some properties of interval-valued fuzzy ideals and interval-valued fuzzy bi-ideals, and characterize which is left [right] simple, left [right] duo and a semilattice of left [right] simple semigroups or another type of semigroups in terms of interval-valued fuzzy ideals and interval-valued fuzzy bi-ideals.

2. Preliminaries

We will list some concepts needed in the later sections.

Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0,1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denoted , $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$, and $\boldsymbol{a} = [a,a]$ for every $a \in (0,1)$, We also note that

(i) $(\forall\, M, N \in D(I))$
$(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,

(ii) $(\forall\, M, N \in D(I))$
$(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the *complement* of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$ (See [12]).

Definition 2.1. [4, 12, 15] A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set* (in short, *IVFS*) in X, denoted by $A = [A^L, A^U]$, if $A^L, A^U \in I^X$ such that $A^L \leq A^U$, i.e., $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)$[resp. $A^U(x)$] is called the *lower*[resp. *upper*] *end point of x to A*. For any $[a,b] \in D(I)$, the interval-valued fuzzy set A in X defined by $A(x) = [A^L(x), A^U(x)] = [a,b]$ for each $x \in X$ is denoted by $\widetilde{[a,b]}$ and if $a = b$, then the IVFS$\widetilde{[a,b]}$ is denoted by simply $\widetilde{a}$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively.

We will denote the set of all IVFSs in X as $D(I)^X$. It is clear that set $A = [A, A] \in D(I)^X$ for each $A \in I^X$.

Definition 2.2. [12] Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then:

(i) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

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(iii) $A^c = [1 - A^U, 1 - A^L]$.

(iv) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.

$(iv)'$ $\bigcup_{\alpha \in \Gamma} A_\alpha = [\bigvee_{\alpha \in \Gamma} A^L_\alpha, \bigvee_{\alpha \in \Gamma} A^U_\alpha]$.

(v) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.

$(v)'$ $\bigcap_{\alpha \in \Gamma} A_\alpha = [\bigwedge_{\alpha \in \Gamma} A^L_\alpha, \bigwedge_{\alpha \in \Gamma} A^U_\alpha]$.

Definition 2.3. [6] An interval-valued fuzzy set A in G is called an *interval-valued fuzzy subgroupoid*(in short, *IVGP*) in G if

$$A^L(xy) \geq A^L(x) \wedge A^L(y),$$

and

$$A^U(xy) \geq A^U(x) \wedge A^U(y),\ \forall\, x, y \in G.$$

It is clear that $\widetilde{0}, \widetilde{1} \in$ IVGP(G). We will denote the IVGPs in G as IVGP(G).

Definition 2.4. [6] Let A be an IVFS of a group G and $[\lambda, \mu] \in D(I)$. Then the subgroup $A^{[\lambda,\mu]}$ is called a $[\lambda, \mu]$-*level subset* of A.

3. Interval-valued fuzzy ideals and bi-ideals of a semigroup

Let S be a semigroup. By a *subsemigroup* of S we mean a non-empty subset A of S such that $A^2 \subset A$ and by a *left* [resp. *right*] ideal of S we mean a non-empty subset A of S such that

$$SA \subset A \text{ [resp. } AS \subset A].$$

By *two-sided ideal* or simply *ideal* we mean a subset A of S which is both a left and a right ideal of S. A semigroup S is said to be *left*[resp. *right*] *simple* if S itself is the only left [resp. right] ideal of S. S is said to be *simple* if it contains no proper ideal.

Definition 3.1. Let S be a semigroup and let $A \in D(I)^S$. Then A is called an :

(1) *interval-valued fuzzy subsemigroup* (in short, *IVSG*) of S if

$$A^L(xy) \geq A^L(x) \wedge A^L(y),$$

and

$$A^U(xy) \geq A^U(x) \wedge A^U(y)$$

for any $x, y \in S$.

(2) *interval-valued fuzzy left ideal* (in short, *IVLI*) of S if

$$A^L(xy) \geq A^L(y), \text{ and } A^U(xy) \geq A^U(y)$$

for any $x, y \in S$.

(3) *interval-valued fuzzy right ideal* (in short, *IVRI*) of S if

$$A^L(xy) \geq A^L(x), \text{ and } A^U(xy) \geq A^U(x)$$

for any $x, y \in S$.

(4) *interval-valued fuzzy (two-sided) ideal* (in short, *IVI*) of S if it is both an interval-valued fuzzy left and an interval-valued fuzzy right ideal of S.

We will denote the set or all IVSGs [resp. IVLIs, IVRIs and IVIs] of S as IVSG(S) [resp. IVLI(S), IVRI(S) and IVI(S)].

It is clear that $A \in$ IVI(S) if and only if

$$A^L(xy) \geq A^L(x) \wedge A^L(y),$$

and

$$A^U(xy) \geq A^U(x) \wedge A^U(y)$$

for any $x, y \in S$, and if $A \in$ IVLI(S)[resp. IVRI(S) and IVI(S)], then $A \in$ IVSG(S).

Remark 3.2. *Let S be a semigroup.*

(a) If A is a fuzzy subsemigroup of S, then

$$A = [A, A] \in \text{IVSG}(S).$$

(b) If $A \in$ IVSG(S) [resp. IVI(S), IVLI(S) and IVRI(S)], then A^L and A^U are fuzzy subsemigroup [resp. ideal, left ideal and right ideal] of S.

Result 3.A. [6, Proposition 3.7] *Let A be a non-empty subset of a groupoid S. A is a subgroupoid of S if and only if $[\chi_A, \chi_A] \in$ IVGP(S).*

The following is the immediate result of Definition 3.1 and Result 3.A.

Theorem 3.3. Let A be a non-empty subset of a semigroup S. Then A is a subsemigroup of S if and only if $[\chi_A, \chi_A] \in$ IVSG(S).

Result 3.B. [6, Proposition 6.6] *Let R be a ring. Then A is an ideal [resp. a left ideal and a right ideal] of R if and only if $[\chi_A, \chi_A] \in$ IVI(R) [resp. IVLI(R) and IVRI(R)].*

The following is the immediate result of Definition 3.1 and Result 3.B.

Theorem 3.4. Let A be a nonempty subset of a semigroup S. Then A is an ideal [resp. a left ideal and a right ideal] of S if and only if $[\chi_A, \chi_A] \in$ IVI(S) [resp. IVLI(S) and IVRI(S)].

Proposition 3.5. Let S be a semigroup. If $A \in$ IVSG(S)[resp. IVI(S), IVLI(S) and IVRI(S)], then $A^{[\lambda,\mu]}$ is a subsemigroup [resp. ideal, left ideal and right ideal] of S.

The following result is the converse of Proposition 3.5:

Proposition 3.6. Let S be a semigroup and let $A \in D(I)^S$. If $A^{[\lambda,\mu]}$ is a subsemigroup [resp. ideal, left ideal and right ideal] of S for each $[\lambda,\mu] \in D(I)$, then $A \in$ IVSG(S)[resp. IVI(S), IVLI(S) and IVRI(S)].

Proof. Suppose $A^{[\lambda,\mu]}$ is a subsemigroup of S for each $[\lambda,\mu] \in D(I)$. For any $x, y \in S$, let $A(x) = [\lambda_1, \mu_1]$ and let $A(y) = [\lambda_2, \mu_2]$. Then $A^L(x) = \lambda_1 \geq \lambda_1 \wedge \lambda_2$, $A^U(x) = \mu_1 \geq \mu_1 \wedge \mu_2$ and $A^L(y) = \lambda_2 \geq \lambda_1 \wedge \lambda_2$, $A^U(y) = \mu_2 \geq \mu_1 \wedge \mu_2$. Thus $x, y \in A^{[\lambda_1\wedge\lambda_2, \mu_1\wedge\mu_2]}$. Since $[\lambda_1 \wedge \lambda_2, \mu_1 \wedge \mu_2] \in D(I)$, by the hypothesis, $xy \in A^{[\lambda_1\wedge\lambda_2,\mu_1\wedge\mu_2]}$. Then $A^L(xy) \geq \lambda_1 \wedge \lambda_2 = A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq \mu_1 \wedge \mu_2 \geq A^U(x) \wedge A^U(y)$. Hence $A \in$ IVSG(S).

Now suppose $A^{[\lambda,\mu]}$ is a left ideal of S for each $[\lambda,\mu] \in D(I)$. For each $y \in S$, let $A(y) = [\lambda,\mu]$. Then clearly $y \in A^{[\lambda,\mu]}$. Let $x \in S$. Then, by the hypothesis, $xy \in A^{[\lambda,\mu]}$. Thus $A^L(xy) \geq \lambda = A^L(y)$ and $A^U(xy) \geq \mu = A^U(y)$. Hence $A \in$ IVLI(S).

Also, we easily see the rest. This completes the proof. □

A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subset A$. We will denote the set of all bi-ideals of S as BI(S).

Definition 3.7. Let S be a semigroup and let $A \in IVSG(S)$. Then A is called an *interval-valued fuzzy bi-ideal* (in short, *IVBI*) of S if

$$A^L(xyz) \geq A^L(x) \wedge A^L(z),$$

and

$$A^U(xyz) \geq A^U(x) \wedge A^U(z)$$

for any $x, y, z \in S$.

We will denote the set of all IVBIs of S as IVBI(S). The following result shows that the concept of an IVBI in a semigroup is an extended one of a bi-ideal.

Theorem 3.8. Let A be a non-empty subset of a semigroup S. Then A is a bi-ideal of S if and only if $[\chi_A, \chi_A] \in$ IVBI(S).

Proof. (⇒): Suppose $A \in$ BI(S) and let $x, y, z \in S$.

Case (i): Suppose $x \in A$ and $z \in A$. Then $\chi_A(x) = \chi_A(z) = 1$. Since A is a bi-ideal of S, $xyz \in ASA \subset A$. Thus $\chi_A(xyz) = 1 = \chi_A(x) \wedge \chi_A(z)$.

Case (ii): Suppose $x \notin A$ or $z \notin A$. Then $\chi_A(x) = 0$ or $\chi_A(z) = 0$. Thus $\chi_A(xyz) \geq 0 = \chi_A(x) \wedge \chi_A(z)$. So, in either cases, $\chi_A(xyz) \geq \chi_A(x) \wedge \chi_A(z)$. Moreover, by Theorem 3.2, $[\chi_A, \chi_A] \in$ IVSG(S). Hence $[\chi_A, \chi_A] \in$ IVBI(S).

(⇐): Suppose $[\chi_A, \chi_A] \in$ IVBI(S). Let $t \in ASA$. Then there exist $x, z \in A$ and $y \in S$ such that $t = xyz$. Since $x, z \in A$, $\chi_A(x) = \chi_A(z) = 1$. Since $[\chi_A, \chi_A] \in$ IVBI(S), $\chi_A(xyz) \geq \chi_A(x) \wedge \chi_A(z) = 1$. Then $\chi_A(xyz) = 1$. Thus $t = xyz \in A$. So $ASA \subset A$. Moreover, by Theorem 3.3, A is a subsemigroup of S. Hence $A \in$ BI(S). □

Theorem 3.9. Let S be a semigroup. Then S is a group if and only if every IVBI of S is a constant mapping.

Proof. (⇒): Suppose S is a group with the identity e. Let $A \in$ IVBI(S), and let $a \in S$. Then

$$\begin{aligned} A^L(a) = A^L(eae) &\geq A^L(e) \wedge A^L(e) = A^L(e) \\ &= A^L(ee) = A^L((aa^{-1})(a^{-1}a)) \\ &= A^L(a(a^{-1}a^{-1})a) \geq A^L(a) \wedge A^L(a) \\ &== A^L(a). \end{aligned}$$

By the similar arguments, we have that $A^U(a) \geq A^U(a)$. Thus $A(a) = A(e)$. Hence A is a constant mapping.

(⇐): Suppose the necessary condition holds. Assume that S is not a group. Then it follows from p.84 in [2] that S contains a proper bi-ideal A of S. Then there exists an $x \in S$ such that $x \notin A$. Let $y \in A$ with $y \neq x$. Since A is a bi-ideal of S, by Theorem 3.8, $[\chi_A, \chi_A] \in$ IVBI(S). By the hypothesis, $[\chi_A, \chi_A]$ is a constant mapping. Thus $[\chi_A, \chi_A](x) = [\chi_A, \chi_A](y)$, i.e., $\chi_A(x) = \chi_A(y)$. Since $x \notin A$ and $y \in A$, $\chi_A(x) = 0 < \chi_A(y) = 1$, i.e., $[\chi_A, \chi_A](x) = [0, 0] \neq [1, 1] = [\chi_A, \chi_A](y)$. This is a contradiction. Hence S is a group. This completes the proof. □

Proposition 3.10. Every IVLI[resp. IVRI and IVI] of S is an IVBI of S.

Proof. Suppose $A \in$ IVLI(S), and let $x, y, z \in S$. Then $A^L(xyz) = A^L((xy)z) \geq A^L(z) \geq A^L(x) \wedge A^L(z)$ and $A^U(xyz) = A^U((xy)z) \geq A^U(z) \geq A^U(x) \wedge A^U(z)$. So $A \in$ IVBI(S). Similarly, we can see that the other cases hold. □

Theorem 3.11. Let S be a semigroup and let $A \in D(I)^S$. Then $A \in$ IVBI(S) if and only if $A^{[\lambda,\mu]} \in$ BI(S) for each $[\lambda,\mu] \in D(I)$.

Proof. (⇒): Suppose $A \in IVBI(S)$, and let $[\lambda,\mu] \in D(I)$. Since $A \in$ IVSGS, by Proposition 3.5, $A^{[\lambda,\,\mu]}$ is a subsemigroup of S. Let $t \in A^{[\lambda,\,\mu]}SA^{[\lambda,\,\mu]}$. Then there exist $x, z \in A^{[\lambda,\mu]}$ and $y \in S$ such that $t = xyz$. Since $A \in$ IVBI(S), we have

$$A^L(t) \geq A^L(x) \wedge A^L(z) \geq \lambda,$$

and

$$A^U(t) \geq A^U(x) \wedge A^U(y) \geq \mu.$$

Thus $t \in A^{[\lambda,\mu]}$. So $A^{[\lambda,\mu]}SA^{[\lambda,\mu]} \subset A^{[\lambda,\mu]}$. Hence $A^{[\lambda,\mu]} \in$ BI(S).

($\Leftarrow$): Suppose the necessary condition holds. Since $A^{[\lambda,\,\mu]}$ is a subsemigroup of S, by Proposition 3.6, $A \in$ IVSG(S). For any $x, z \in S$, let $A(x) = [\lambda_1, \mu_1]$ and let $A(z) = [\lambda_2, \mu_2]$. Then, by the process of the proof of Proposition 3.6, $x, z \in A^{[\lambda_1 \wedge \lambda_2,\, \mu_1 \wedge \mu_2]}$. Let $y \in S$. Then, by the hypothesis, $xyz \in A^{[\lambda_1 \wedge \lambda_2,\, \mu_1 \wedge \mu_2]}$. Thus

$$A^L(xyz) \geq \lambda_1 \wedge \lambda_2 = A^L(x) \wedge A^L(z),$$

and

$$A^U(xyz) \geq \mu_1 \wedge \mu_2 \leq A^U(x) \wedge A^U(z).$$

Hence $A \in$ IVBI(S). This completes the proof. □

4. Interval-valued fuzzy duos, ideals and bi-ideals of a regular semigroup

A semigroup S is said to be *regular* of for each $a \in S$ there exists an $x \in S$ such that $a = axa$.

A semigroup S is said to be *left duo*[resp. *right duo*] if every left [resp. right] ideal of S is a two-sided ideal of S.

A semigroup S is said to be *duo* if it is both left and right duo.

Definition 4.1. A semigroup S is said to be :

(1) *interval-valued fuzzy left duo*(in short, *IVLD*) if every IVLI of S is an IVI of S.

(2) *interval-valued fuzzy right duo*(in short, *IVRD*) if every IVRI of S is an IVI of S.

(3) *interval-valued fuzzy duo*(in short, *IVD*) if it is both interval-valued fuzzy left and interval-valued fuzzy right duo.

Theorem 4.2. Let S be a regular semigroup. Then S is left duo if and only if S is IVLD.

Proof. ($\Rightarrow$): Suppose S is left duo. Let $A \in$ IVLI(S) and let $a, b \in S$. Then, by the process of the proof of Theorem 3.1 in [8], $ab \in (aSa)b \subset (Sa)S \subset Sa$. Thus there exists an $x \in S$ such that $ab = xa$. Since $A \in$ IVLI(S),

$$A^L(ab) = A^L(xa) \geq A^L(a),$$

and

$$A^U(ab) = A^U(xa) \geq A^U(a).$$

Then $A \in$ IVRI(S). Thus $A \in$ IVI(S). Hence S is IVLD.

($\Leftarrow$): Suppose S is IVLD, and let A be any left ideal of S. Then, by Theorem 3.4, $[\chi_A, \chi_A] \in$ IVLI(S). By the assumption, $[\chi_A, \chi_A] \in$ IVI(S). Since $A \neq \emptyset$, by Theorem 3.4, A is an ideal of S. Hence S is left duo. This completes the proof. □

Theorem 4.2.$'$ [The dual of Theorem 4.2] *Let S be a regular semigroup. Then S is right duo if and only if S is IVRD.*

The following is the immediate result of Theorem 4.2 and 4.2$'$.

Theorem 4.3. Let S be a regular semigroup. Then S is duo if and only if S is IVD.

Theorem 4.4. Let S be a regular semigroup. Then every bi-ideal of S is a right ideal of S if and only if every IVBI of S is an IVRI of S.

Proof. ($\Rightarrow$): Suppose every bi-ideal of S is a right ideal of S. Let $A \in$ IVBI(S) and let $a, b \in S$. Then, by the process of proof of Theorem 3.4 in [8], $ab \in (aSa)S \subset aSa$. Thus there exists an $x \in S$ such that $ab = axa$. Since $A \in$ IVBI(S), we have

$$A^L(ab) = A^L(axa) \geq A^L(a) \wedge A^L(a) = A^L(a),$$

and

$$A^U(ab) = A^U(axa) \geq A^U(a) \wedge A^U(a) = A^U(a).$$

Hence $A \in$ IVRI(S).

($\Leftarrow$): Suppose that every IVBI of S is an IVRI of S, and let A be any bi-ideal of S. Then, by Theorem 3.8, $[\chi_A, \chi_A] \in$ IVBI(S). By the assumption, $[\chi_A, \chi_A] \in$ IVRI(S). Since $A \neq \varnothing$, by Theorem 3.4, A is a right ideal of S. This completes the proof. □

Result 4.A. [11, Theorem 3] *Every bi-ideal of a regular left duo semigroup S is a right ideal of S.*

Corollary 4.5. Let S be a regular duo semigroup. Then every IVBI of S is a IVRI of S.

Proof. By Result 4.A, every bi-ideal of S is a right ideal of S. Hence, by Theorem 4.3, it follows that every IVBI of S is an IVRI of S. □

Theorem 4.4.$'$ [The dual of Theorem 4.4] *Let S be a regular semigroup. Then every bi-ideal of S is a left ideal of S if and only if every IVBI of S is an IVLI of S.*

The following is the immediate result of Theorem 4.4 and 4.4$'$.

Theorem 4.6. Let S be a regular duo semigroup. Then every bi-ideal of S is an ideal of S if and only if every IVBI of S is an IVI of S.

A semigroup S is called a *semilattice of groups* [2] if it is the set-theoretical union of a set of mutually disjoint subgroups $G_\alpha(\alpha \in \Gamma)$, i.e., $S = \bigcup_{\alpha\in\Gamma} G_\alpha$ such that for any α, $\beta \in \Gamma$, $G_\alpha G_\beta \subset G_\gamma$ and $G_\beta G_\alpha \subset G_\gamma$ for some $\gamma \in \Gamma$.

Result 4.B. [10, Theorem 4] *Every bi-ideal of a semigroup S which is a semilattice of groups, is an ideal of S.*

The following is the immediate result of Result 4.B and Theorem 4.6.

Corollary 4.7. Let S be a semigroup which is a semilattice of groups. Then every IVBI of S is an IVI of S.

We denote by $L[a]$ [resp. $J[a]$] the principle left [resp. two-sided] ideal of a semigroup S generated by a in S, i.e.,

$$L[a] = \{a\} \cup Sa,$$

and

$$J[a] = \{a\} \cup Sa \cup aS \cup SaS.$$

It is well-known [2, Lemma 2.13] that if S is a regular semigroup, then $L[a] = Sa$ for each $a \in S$.

A semigroup S is said to be *right zero*[resp. *left zero*] if $xy = y$[resp. $xy = x$] for any $x, y \in S$.

Theorem 4.8. Let S be a regular semigroup and let E_S the set of all idempotent elements of S. Then E_S forms a left zero subsemigroup of S if and only if for each $A \in$ IVLI(S), $A(e) = A(f)$ for any $e, f \in E_S$, where E_S denotes the set of all idempotent elements of S.

Proof. ($\Rightarrow$): Suppose E_S forms a left zero subsemigroup of S. Let $A \in$ IVLI(S), and let $e, f \in E_S$. Then, by the hypothesis, $ef = e$ and $fe = f$. Since $A \in$ IVLI(S), we have

$$A^L(e) = A^L(ef) \geq f^L = A^L(fe) \geq A^L(e),$$

and

$$A^U(e) = A^U(ef) \geq f^U = A^U(fe) \geq A^U(e).$$

Hence $A(e) = A(f)$.

($\Leftarrow$): Suppose the necessary condition holds. Since S is regular, $E_S \neq \emptyset$. Let $e, f \in E_S$. Then, by Theorem 3.4, $[\chi_{L[f]}, \chi_{L[f]}] \in$ IVLI(S). Thus $\chi_{L[f]}(e) = \chi_{L[f]}(f) = 1$. So $e \in L[f] = Sf$. Then there exists an $x \in S$ such that $e = xf = xff = ef$. Hence E_S is a left zero semigroup. This completes the proof. $\square$

Corollary 4.9. Let S be an idempotent semigroup. Then S is left zero if and only if for each $A \in$ IVLI(S), $A(e) = A(f)$ for any $e, f \in S$.

Theorem 4.8.′ [The dual of Theorem 4.8] Let S be a regular semi group. Then E_S forms a right zero subsemigroup of S if and only if for each $A \in$ IVRI(S), $A(e) = A(f)$ for any $e, f \in E_S$.

Corollary 4.9.′ [The dual of Corollary 4.9] Let S be an semigroup. Then S is right zero if and only if for each $A \in$ IVRI(S), $A(e) = A(f)$ for any $e, f \in S$.

Theorem 4.10. Let S be a regular semigroup. Then S is a group if and only if for each $A \in$ IVBI(S), $A(e) = A(f)$ for any $e, f \in E_S$.

Proof. ($\Rightarrow$): Suppose S is a group. Let $A \in$ IVBI(S). Then, by Theorem 3.8, A is a constant mapping. Hence $A(e) = A(f)$ for any $e, f \in E_S$.

($\Leftarrow$): Suppose the necessary condition holds. Let $e, f \in E_S$. Let $B[x]$ denote the principal bi-ideal of S generated by x in S, i.e., $B[x] = \{x\} \cup \{x^2\} \cup xSx$ [2, p.84]. Moreover, if S is regular, then $B[x] = xSx$ for each $x \in S$. Then, by Theorem 3.8, $[\chi_{B[f]}, \chi_{B[f]}] \in$ IVBI(S). Since $f \in B[f]$, $\chi_{B[f]}(e) = \chi_{B[f]}(f) = 1$. Then $e \in B[f] = fSf$. Thus, by the process of the proof of Theorem 3.14 in [8], $e = f$. Since S is regular, $E_s \neq \emptyset$ and S contains exactly one idempotent. So it follows from [2, p.33(Ex. 4)] that S is a group. This completes the proof. $\square$

5. Intra-regular semigroups

A semigroup S is said to be *intra-regular* if for each $a \in S$, there exist $x, y \in S$ such that $a = xa^2y$. For characterization of such a semigroup, see [2, Theorem 4.4] and [13, II.4.5 Theorem].

Theorem 5.1. Let S be a semigroup. Then S is intra-regular if and only if for each $A \in$ IVI(S), $A(a) = A(a^2)$ for each $a \in S$.

Proof. ($\Rightarrow$): Suppose S is intra-regular. Let $A \in$ IVI(S), and let $a \in S$. Then, by the hypothesis, there exist $x, y \in S$ such that $a = xa^2y$. Since $A \in$ IVI(S), we have

$$A^L(a) = A^L(xa^2y) \geq A^L(xa^2) \geq A^L(a^2) \geq A^L(a),$$

and

$$A^U(a) = A^U(xa^2y) \geq A^U(xa^2) \geq A^U(a^2) \geq A^U(a).$$

Hence $A(a) = A(a^2)$ for each $a \in S$.

($\Leftarrow$): Suppose the necessary condition holds and let $a \in S$. Then, by Theorem 3.4, $[\chi_{J[a^2]}, \chi_{J[a^2]}] \in$ IVI(S). Since $a^2 \in J[a^2]$, $\chi_{J[a^2]}(a) = \chi_{J[a^2]}(a^2) = 1$. Thus $a \in J[a^2] = \{a\} \cup Sa^2 \cup a^2S \cup Sa^2S$. So we can easily see that S is intra-regular. This completes the proof. $\square$

Proposition 5.2. Let S be an intra-regular semigroup. Then for each $A \in$ IVI(S), $A(ab) = A(ba)$ for any $a, b \in S$.

Proof. Let $A \in$ IVI(S), and let $a, b \in S$. Then, by Theorem 5.1, $A^L(ab) = A^L((ab)^2) = A^L(a(ba)b) \geq A^L(ba)$ $= A^L((ba)^2) = A^L(b(ab)a) \geq A^L(ab)$. By the similar arguments, we have that $A^U(ab) \geq A^U(ab)$. Thus $A(ab) = A(ba)$. This completes the proof. □

6. Completely regular semigroups

A semigroup S is said to be *completely regular* if for each $a \in S$, there exists an $x \in S$ such that

$$a = axa \text{ and } ax = xa.$$

A semigroup S is said to be *left regular*[resp. *right regular*] if for each $a \in S$, there exists an $x \in S$ such that

$$a = xa^2 \text{ [resp. } a = a^2x\text{]}.$$

For characterizations of such a semigroup, see [2, Theorem 4.2.]. It is well-known[2, Theorem 4.3.] that S is completely regular if and only if it is left and right regular.

Result 6.A. [13, p. 105] *Let S be a semigroup. Then the followings are equivalent:*

(1) S is completely regular.

(2) S is a union of groups.

(3) $a \in a^2Sa^2$ for each $a \in S$.

Theorem 6.1. Let S be a semigroup. Then S is left regular if and only if, for each $A \in$ IVLI(S), $A(a) = A(a^2)$ for each $a \in S$.

Proof. ($\Rightarrow$): Suppose S is left regular. Let $A \in$ IVLI(S), and let $a \in S$. Then, by the hypothesis, there exists an $x \in S$ such that $a = xa^2$. Since $A \in$ IVLI(S), $A^L(a) = A^L(xa^2) \geq A^L(a^2) \geq A^L(a)$ and $A^U(a) = A^U(xa^2) \geq A^U(a^2) \geq A^U(a)$. Hence $A(a) = A(a^2)$, for each $a \in S$.

($\Leftarrow$): Suppose the necessary condition holds. Let $a \in S$. Then, by Theorem 3.4, $(\chi_{L[a^2]}, \chi_{L[a^2]^c}) \in$ IVLI(S). Since $a^2 \in L[a^2]$, $(\chi_{L[a^2]}(a) = \chi_{L[a^2]}(a^2) = 1$. Then $a \in L[a^2] = \{a^2\} \cup Sa^2$. Hence S is left regular. This completes the proof. □

Theorem 6.1,′ [The dual of Theorem 6.1] Let S be a semigroup. Then S is right regular if and only if for each $A \in$ IVRI(S), $A(a) = A(a^2)$ for each $a \in S$.

Now we give another characterization of a completely regular semigroup by interval-valued fuzzy bi-ideals.

Theorem 6.2. Let S be a semigroup. Then the followings are equivalent:

(1) S is completely regular.

(2) For each $A \in$ IVBI(S), $A(a) = A(a^2)$ for each $a \in S$.

(3) For each $B \in$ IVLI(S) and each $C \in$ IVRI(S), $B(a) = B(a^2)$ and $C(a) = C(a^2)$ for each $a \in S$.

Proof. It is clear that (1)$\Leftrightarrow$(3) by Theorem 6.1 and 6.1′. Thus it is sufficient to show that(1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2): Suppose the condition (1) holds. Let $A \in$ IVBI(S),and let $a \in S$. Then, by Result 6.A(3), there exists an $x \in S$ such that $a = a^2xa^2$. Since $A \in$ IVBI(S), $A^L(a) = A^L(a^2xa^2) \geq A^L(a^2) \wedge A^L(a^2) = A^L(a^2) \geq A^L(a) \wedge A^L(a) = A^L(a)$. By the similar arguments, we have that $A^U(a) \geq A^U(a)$. Hence $A(a) = A(a^2)$.

(2) $\Rightarrow$ (1): Suppose the condition (2) holds. For each $x \in S$, let $B[x]$ denote the principal bi-ideal of S generated by x, i.e., $B[x] = \{x\} \cup \{x^2\} \cup xSx$. Let $a \in S$. Then, by Theorem 3.8, $[\chi_{B[a^2]}, \chi_{B[a^2]}] \in$ IVBI(S). Since $a^2 \in B[a^2]$, $\chi_{B[a^2]}(a) = \chi_{B[a^2]}(a^2) = 1$. Thus $a \in B[a^2] = \{a^2\} \cup \{a^4\} \cup a^2Sa^2$. Hence S is completely regular. This completes the proof. □

Result 6.B. [9, Theorem 1] *Let S be a semigroup. Then S is a semilattice of groups if and only if* BI(S) *is a semilattice under the multiplication of subsets.*

Theorem 6.3. Let S be a semigroup. Then S is a semilattice of groups if and only if for each $A \in$ IVBI(S), $A(a) = A(a^2)$ and $A(ab) = A(ba)$ for any $a, b \in S$.

Proof. ($\Rightarrow$): Suppose S is a semilattice of groups. Then S is a union of groups. By Result 6.A, S is completely regular. Let $A \in$ IVBI(S), and let $a \in S$. Then, by Theorem 6.2, $A(a) = A(a^2)$. Now let $a, b \in S$. Then, by the process of the proof of Theorem 6 in [7], there exists an $x \in S$ such that $(ab)^3 = (ba)x(ba)$. Thus $A^L(ab) = A^L((ab)^3) = A^L((ba)x(ba)) \geq A^L(ba) \wedge A^L(ba) = A^L(ba)$. By the similar arguments, we have that $A^U(ab) \geq A^U(ba)$. Similarly, we can see that $A^L(ba) \geq A^L(ab)$ and $A^U(ba) \geq A^U(ab)$. So $A(ab) = A(ba)$. Hence the necessary conditions hold.

($\Leftarrow$): Suppose the necessary conditions hold. Then, by the first condition and Theorem 6.2, S is completely regular. Thus it is easily shown that A is idempotent for each $A \in$ BI(S). Let $A, B \in$ BI(S), and let $t \in BA$. Then there exist $a \in A$ and $b \in B$ such that $t = ab$. Moreover $B[t] = B[ab] \in$ BI(S). By Theorem 3.8, $[\chi_{B[ab]}, \chi_{B[ab]}] \in$ IVBI(S). By the hypothesis, $[\chi_{B[ab]}, \chi_{B[ab]}](ab) = [\chi_{B[ab]}, \chi_{B[ab]}](ba)$. Since $ab \in B[ab]$, $\chi_{B[ab]}(ab) = \chi_{B[ab]}(ba) = 1$. Then $ba \in B[ab] = \{ab\} \cup \{abab\} \cup abSab$. It follows from the process of the proof of Theorem 6 in [7] that $BA = AB$.

So $(\mathrm{BI}(S), \cdot)$ is a commutative idempotent semigroup. Hence, by Result 6.B, S is a semilattice of groups. This completes the proof. □

Corollary 6.4. Let S be an idempotent semigroup. Then S is commutative if and only if for each $A \in \mathrm{IVBI}(S)$, $A(ab) = A(ba)$ for any $a, b \in S$.

7. Semigroups that are semilattices of left [resp. right] simple semigroups

Result 7.A. [9, Theorem 7 and 18, Theorem] *Let S be a semigroup. Then the followings are equivalent :*

(1) S is a semilattice of left simple semigroups.

(2) S is left regular and $AB = BA$ for any two left ideals A and B of S.

(3) S is left regular and every left ideal of it is an ideal of S.

The following result can be proved in a similar way as in the proof of Theorem 4.2 and 4.2′.

Theorem 7.1. Let S be a left [resp. right] regular semigroup. Then S is left [resp. right] duo if and only if S is IVLD [resp. IVRD].

The characterization of a semigroup that is a semilattice of left simple semigroups can be founded in [13, Theorem II.4.9].

Theorem 7.2. Let S be a semigroup. Then S is a semilattice of left simple semigroups if and only if for each $A \in \mathrm{IVLI}(S)$, $A(a) = A(a^2)$ and $A(ab) = A(ba)$ for any $a, b \in S$.

Proof. $(\Rightarrow)$: Suppose S is a semilattice of left simple semigroups. Let $A \in \mathrm{IVLI}(S)$, and let $a, b \in S$. Then, by Result 7.A, S is left regular. By Theorem 6.1, $A(a) = A(a^2)$. By the hypothesis and Result 6.A, S is left duo. By Theorem 7.1, S is IVLD. Then $A \in \mathrm{IVI}(S)$. Thus $A^L(ab) = A^L((ab^2)) = A^L(a(ba)b) \geq A^L(ba)$ and $A^U(ab) = A^U((ab^2)) = A^U(a(ba)b) \geq A^U(ba)$. By the similar arguments, we have $A^L(ba) \geq A^L(ab)$ and $A^U(ba) \geq A^U(ab)$. Hence $A(ab) = A(ba)$ for any $a, b \in S$.

$(\Leftarrow)$: Suppose the necessary conditions hold. Then, by the first condition and Theorem 7.1, S is left regular. Let A and B be any left ideals of S and let $x \in AB$. Then there exist $a \in A$ and $b \in B$ such that $x = ab$. By Theorem 3.4, $[\chi_{L[ba]}, \chi_{L[ba]}] \in \mathrm{IVLI}(S)$. Since $ba \in L[ba]$, $\chi_{L[ba]}(ab) = \chi_{L[ba]}(ba) = 1$. Thus $ab \in L[ba] = \{ba\} \cup Sba \subset BA \cup SBA \subset BA$. So ,by the process of the proof of Theorem 6.3 in [8], we have $AB = BA$. Hence, by Result 6.A, S is a semilattice of left simple semigroups. This completes the proof. □

Theorem 7.2.′ [The dual of Theorem 7.2] Let S be a semigroup. Then the S is a semilattice of right simple semigroups if and only if for each $A \in \mathrm{IVRI}(S)$, $A(a) = A(a^2)$ and $A(ab) = A(b)$ for any $a, b \in S$.

8. Left [resp. right] simple semigroups

Definition 8.1. A semigroup S is said to be *interval-valued fuzzy left simple*[resp. *interval-valued fuzzy right simple*] if every IVLI [resp. IVRI] of S is a constant mapping and is said to be *interval-valued fuzzy simple* if every IVI of S is a constant mapping.

Theorem 8.2. Let S be a semigroup. Then S is left simple if and only if S is interval-valued fuzzy left simple.

Proof. $(\Rightarrow)$: Suppose S is left simple. Let $A \in \mathrm{IVLI}(S)$, and let $a, b \in S$. Since S is left simple, from [2, p.6], there exist $x, y \in S$ such that $b = xa$ and $a = yb$. Since $A \in \mathrm{IVLI}(S)$, $A^L(a) = A^L(yb) \geq A^L(b) = A^L(xa) \geq A^L(a)$ and $A^U(a) = A^U(yb) \geq A^U(b) = A^U(xa) \geq A^U(a)$. Thus $A(a) = A(b)$. So A is a constant mapping. Hence S is interval-valued fuzzy left simple.

$(\Leftarrow)$: Suppose the necessary condition holds. Let A be any left ideal of S. By Theorem 3.4, $[\chi_A, \chi_A] \in \mathrm{IVLI}(S)$. By the hypothesis, $[\chi_A, \chi_A]$ is a constant mapping. Since $A \neq \emptyset$, $[\chi_A, \chi_A] = \tilde{1}$. Then $\chi_A(a) = 1$ for each $a \in S$. Thus $a \in A$ for each $a \in S$, i.e., $S \subset A$. Hence S is left simple. This completes the proof. □

The following two results can be seen in a similar way as in the proof of Theorem 8.2.

Theorem 8.2.′ [The dual of Theorem 8.2] Let S be a semigroups. Then S is right simple if and only if S is interval-valued fuzzy simple.

Theorem 8.3. Let S be a semigroup. Then S is simple if and only if S is interval-valued fuzzy simple.

It is well-known that a semigroup S is a group if and only if it is left and right simple. Thus from this and Theorem 8.2 and 8.2′, we obtain the following result :

Theorem 8.4. Let S be a semigroup. Then S is a group if and only if S is both interval-valued fuzzy left and interval-valued fuzzy right simple.

Proposition 8.5. Let S be a left simple semigroup. Then every IVBI of S is an IVRI of S.

Proof. Let $A \in \mathrm{IVBI}(S)$, and let $a, b \in S$. Since S is left simple, there exists an $x \in S$ such that $b = xa$. Since $A \in \mathrm{IVBI}(S)$, $A^L(ab) = A^L(axa) \geq A^L(a) \wedge A^L(a) = A^L(a)$ and $A^U(ab) = A^U(axa) \geq A^U(a) \wedge A^U(a) = A^U(a)$. Hence $A \in \mathrm{IVRI}(S)$. This completes the proof. □

Corollary 8.6. Let S be a left simple semigroup. Then every bi-ideal of S is a right ideal of S.

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