

ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS IN \mathbb{R}^n

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ABSTRACT. We investigate the asymptotic behavior of positive solutions to the elliptic equation

$$(0.1) \quad \Delta u + |x|^{l_1} u^p + |x|^{l_2} u^q = 0 \text{ in } \mathbb{R}^n.$$

We obtain a conclusion that, for $n \geq 3$, $-2 < l_2 < l_1 \leq 0$ and $q > p > 1$, any positive radial solution to (0.1) has the following properties: $\lim_{r \rightarrow \infty} r^{\frac{2+l_1}{p-1}} u$ and $\lim_{r \rightarrow 0} r^{\frac{2+l_2}{q-1}} u$ always exist if $\frac{n+l_1}{n-2} < p < q$, $p \neq \frac{n+2+2l_1}{n-2}$, $q \neq \frac{n+2+2l_2}{n-2}$. In addition, we prove that the singular positive solution of (0.1) is unique under some conditions.

1. Introduction

In this paper, we study the asymptotic behavior of positive solutions to the following equation

$$(1.1) \quad \Delta u + K_1(|x|)u^p + K_2(|x|)u^q = 0 \text{ in } \mathbb{R}^n, \quad n \geq 3,$$

and in particular, that of positive radial solutions to the equation

$$(1.2) \quad \Delta u + |x|^{l_1} u^p + |x|^{l_2} u^q = 0 \text{ in } \mathbb{R}^n, \quad n \geq 3,$$

where $-2 < l_2 < l_1 \leq 0$, $1 < p < q$ and $\Delta = \sum_1^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, and K_1, K_2 are locally Hölder continuous in $\mathbb{R}^n \setminus \{0\}$. We say a positive function $u(x)$ is an entire solution to (1.1), if $u(x)$ satisfies (1.1) pointwise in $\mathbb{R}^n \setminus \{0\}$.

When $K_1(|x|) = K_2(|x|)$, $p = q$, rescaling, (1.1) reduces to

$$(1.3) \quad \Delta u + K(|x|)u^p = 0 \text{ in } \mathbb{R}^n, \quad n \geq 3.$$

(1.3) originates from many mathematical and physical fields, e.g., the well-known scalar curvature equation in the study of Riemannian geometry, the scalar field equation for the standing wave of nonlinear Schrödinger and the

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Klein-Görder equations, the Matukuma equation describing the dynamics of globular cluster of star, etc. Please refer to [16, 18, 20, 21, 23] and the references therein. There have been many works devoted to studying the existence of positive solutions of (1.3) in \mathbb{R}^n after the first contribution by Ni [20] in 1982, see [2, 6, 7, 8, 11, 12, 23] and the references therein. One of the features of the equation is that (1.3) can possess infinitely many solutions as long as the exponent p and the dimension n are large enough. Recent studies in [1, 3, 4, 9, 10] have paid special attention to this phenomenon. The purpose of this paper is to study the asymptotic behavior of positive entire solutions and to present the uniqueness result. In such perspective, we review related works as follows. In the fast decay case, i.e., $|K| \leq Cr^l, l < -2$, Ni [20] shows that (1.3) possess infinitely many positive solutions which are bounded from below by positive constants. Li and Ni [18] shows that, for positive solution to (1.3), the limit $u_\infty = \lim_{x \rightarrow \infty} u(x)$ always exists. Furthermore, if $u_\infty = 0$ for any $\varepsilon > 0$, we have

$$u(x) \leq \begin{cases} C|x|^{2-n} & \text{if } p > \frac{n+l}{n-2}, \\ C_\varepsilon|x|^{[(1-\varepsilon)(l+2)]/1-p} & \text{if } p \leq \frac{n+l}{n-2}, \end{cases}$$

where C_ε is a constant depending on ε , and if $u_\infty > 0$, we have

$$|u(x) - u_\infty| \leq \begin{cases} C|x|^{2-n} & \text{if } l < -n, \\ C|x|^{2-n} \log |x| & \text{if } l = -n, \\ C|x|^{2+l} & \text{if } -n < l < -2, \end{cases}$$

at ∞ .

For the slow decay case, i.e., $K(r) \geq Cr^l$ for $l > -2$ and r being large enough, and additionally, $K(r)$ satisfying:

- (K.1) $K(r) > 0$ in $r > 0$ and $\lim_{r \rightarrow \infty} r^{-l}K(r) = k_\infty > 0$,
- (K.2) $K(r)$ is differentiable and $[\frac{d}{dr}(r^{-l}K(r))]^+ \in L^1, r > 0$,
- (K.3) $K(r)$ is differentiable and $[\frac{d}{dr}(r^{-l}K(r))]^- \in L^1, r > 0$.

Li [17] gave an accurate description on the asymptotic behavior of positive solutions of (1.3), which can be stated as follows.

Theorem A ([17, Theorem 1]). *Let u be a positive radial solution of (1.3). Assume that K satisfies:*

- (i) (K.1) and (K.2), if $0 < \frac{2+l}{p-1} < \frac{n-2}{2}$ or
- (ii) (K.1) and (K.3), if $\frac{n-2}{2} < \frac{2+l}{p-1} < n-2$.

Then,

$$\lim_{r \rightarrow \infty} r^m u(r) = u_\infty \equiv \begin{cases} [\frac{2+l}{p-1}(n-2 - \frac{2+l}{p-1})]^{1/p-1} / k_\infty^{1/p-1} & \text{or} \\ 0. \end{cases}$$

Furthermore, if $u_\infty = 0$, then $\lim r^{n-2}u(r)$ exists and is finite and positive.

Remark 1.1. When $l = -2$, a result similar to Theorem A holds (see [16, 17]).

If $l_1 = l_2 = 0$, the equation (1.2) reduces to the following famous Lane-Emden equation

$$(1.4) \quad \Delta u + u^p + u^q = 0 \quad \text{in } \mathbb{R}^n, \quad n \geq 3.$$

The equation (1.4) has been paid much attention recently. When p and q are in the range $\frac{n}{n-2} < p < \frac{n+2}{n-2} < q$, the mixed growth structure (supercritical for u large and subcritical for u small) has a profound impact on the existence and non-existence theory, and changes the outcome for the Lane-Emden equation, the analysis is surprisingly difficult. Recently Bamon et al. ([5]) proved that if q is fixed and let p approach $\frac{n+2}{n-2}$ from below, then (1.4) has a large number of radial solutions. A similar result holds ([5]) for $p > \frac{n}{n-2}$ and p is fixed, while letting q approach $\frac{n+2}{n-2}$. In addition, they proved that (1.4) doesn't possess any solution if q is fixed and then let p be close enough to $\frac{n}{n-2}$.

For physical reasons, we consider the positive radial solutions to the equation (1.2), in which case it can be rewritten as

$$(1.5) \quad u'' + \frac{n-1}{r}u' + r^{l_1}u^p + r^{l_2}u^q = 0, \quad n \geq 3,$$

where $r = |x|$.

In order to state our main results, we need some notations, which will be used throughout this paper:

$$\begin{aligned} \alpha_1 &= \frac{2+l_1}{p-1}, & \alpha_2 &= \frac{2+l_2}{q-1}, \\ \lambda_1^{p-1} &= \alpha_1(n-2-\alpha_1), & \lambda_2^{q-1} &= \alpha_2(n-2-\alpha_2), \\ v_1(r) &= r^{\alpha_1}u(r), & v_2(r) &= r^{\alpha_2}u(r), \end{aligned}$$

(note that $\alpha_1 > \alpha_2, n-2-\alpha_1 > 0$).

We sometimes suppose

$$(1.6) \quad p \neq \frac{n+2+2l_1}{n-2}, \quad q \neq \frac{n+2+2l_2}{n-2}.$$

Now we give the following definitions:

$u(r)$ is said to be a singular solution of (1.5) (then we say $u(r)$ is singular at infinity) at infinity if $\limsup_{r \rightarrow \infty} r^{\alpha_1}u(r) > 0$, and is said to be regular at infinity if $\lim_{r \rightarrow \infty} r^{n-2}u$ exists. Similarly, $u(r)$ is said to be singular at 0 if $\limsup_{r \rightarrow 0} r^{\alpha_2}u > 0$, is said to be regular at 0 if $\lim_{r \rightarrow 0} u$ exists.

One of our main results is as follows.

Theorem 1.2. *Suppose $-2 < l_2 < l_1 \leq 0$, $\frac{n+l_1}{n-2} < p < q$, and p, q satisfy (1.6). Then, for any solution of (1.5), we have:*

$$(1.7) \quad \lim_{r \rightarrow \infty} r^{\alpha_1} u = \lambda_1, \text{ or } \lim_{r \rightarrow \infty} r^{n-2} u = c_1$$

for some constant $c_1 > 0$. Moreover,

$$(1.8) \quad \lim_{r \rightarrow 0} r^{\alpha_2} u = \lambda_2, \text{ or } \lim_{r \rightarrow 0} u = c_2$$

for some constant $c_2 > 0$.

Remark 1.3. If, say, the origin is a singularity of u , the term $|x|^{l_2} u^q$ is dominant and the term $|x|^{l_1} u^p$ is thus a small perturbation. At infinity, the situation is just reversed, i.e., the term $|x|^{l_1} u^p$ is dominant and the term $|x|^{l_2} u^q$ is a small perturbation.

The device in the proof of Theorem 1.2 is an energy function which plays a central role in the convergence of (1.7) and (1.8). If $p = \frac{n+2+2l_1}{n-2}$, then the coefficient of the energy function is 0 and $r^{\frac{2+l_1}{p-1}} u$ could oscillate endlessly near the ∞ . As the same reason, if $q = \frac{n+2+2l_2}{n-2}$, then $r^{\frac{2+l_2}{q-1}} u$ could oscillate endlessly near the origin.

Theorem 1.4. *If $-2 < l_2 < l_1 \leq 0$, $\frac{n+l_1}{n-2} < p < q$, let u be a positive radial solution of (1.2), then*

(i) *if $q = \frac{n+2+2l_2}{n-2}$, then either (1.8) holds, or $v_2(r)$ oscillates endlessly near the origin between two sequences $\mu_{1,i}$ and $\mu_{2,i}$ satisfying $0 < \mu_{1,i} < \mu_{2,i}$ and*

$$\begin{aligned} \lim_{i \rightarrow \infty} \mu_{1,i} &= \mu_1, & \lim_{i \rightarrow \infty} \mu_{2,i} &= \mu_2, \\ \mu_1 &= \liminf_{r \rightarrow 0} r^{\frac{n-2}{2}} u(r) < \limsup_{r \rightarrow 0} r^{\frac{n-2}{2}} u(r) = \mu_2, \end{aligned}$$

where μ_1 and μ_2 are fixed values satisfying

$$(1.9) \quad 0 < \mu_1 \leq \lambda_2 \leq \mu_2, \quad b(\mu_1) = b(\mu_2) < 0 \text{ with } b(v) = \frac{1}{q+1} v^{q+1} - \frac{\lambda_2^{q-1}}{2} v^2;$$

(ii) *if $p = \frac{n+2+2l_1}{n-2}$, then either (1.7) holds or $v_1(r)$ oscillates endlessly near ∞ between two sequences $\mu'_{1,i}$ and $\mu'_{2,i}$ satisfying $0 < \mu'_{1,i} < \mu'_{2,i}$ and*

$$\begin{aligned} \lim_{i \rightarrow \infty} \mu'_{1,i} &= \mu'_1, & \lim_{i \rightarrow \infty} \mu'_{2,i} &= \mu'_2, \\ \mu'_1 &= \liminf_{r \rightarrow \infty} r^{\frac{n-2}{2}} u(r) < \limsup_{r \rightarrow \infty} r^{\frac{n-2}{2}} u(r) = \mu'_2, \end{aligned}$$

where μ'_1 and μ'_2 are fixed values satisfying

$$(1.10) \quad 0 < \mu'_1 \leq \lambda_1 \leq \mu'_2, \quad b_1(\mu'_1) = b_1(\mu'_2) \text{ with } b_1(v) = \frac{1}{p+1} v^{p+1} - \frac{\lambda_1^{p-1}}{2} v^2.$$

For the equation (1.3), when $K \equiv 1$, it becomes the simplest model which is called generalized Lane-Emden equation or Emden-Fowler equation in astrophysics

$$(1.11) \quad \Delta u + u^p = 0 \quad \text{in } \mathbb{R}^n, \quad n \geq 3.$$

For (1.11), Serrin and Zhou [22] obtained the following uniqueness result.

Theorem B ([22]). *If $\frac{n}{n-2} < p \leq \frac{n+2}{n-2}$, then (1.11) admits exactly one solution $\frac{2}{p-1}(n-2-\frac{2}{p-1})r^{-\frac{2}{p-1}}$, singular at the infinity. If $p > \frac{n+2}{n-2}$, then (1.11) admits exactly one solution $\frac{2}{p-1}(n-2-\frac{2}{p-1})r^{-\frac{2}{p-1}}$, singular at the origin.*

Remark 1.5. From Theorem B, we know that (1.11) has only one solution which is singular at infinity for $\frac{n}{n-2} < p \leq \frac{n+2}{n-2}$ and singular at the origin for $p > \frac{n+2}{n-2}$. Inspired by Theorem B, we derive the following result.

Theorem 1.6. (i) *If $-2 < l_2 < l_1 \leq 0$, $\frac{n+l_1}{n-2} < p < q < \frac{n+2+2l_2}{n-2}$, then (1.5) admits exactly one positive solution singular at infinity. This solution has the following exact limits:*

$$\lim_{r \rightarrow \infty} r^{\alpha_1} u = \lambda_1, \quad \lim_{r \rightarrow 0} r^{\alpha_2} u = \lambda_2;$$

(ii) *If $-2 < l_2 < l_1 \leq 0$, $\frac{n+2+2l_1}{n-2} < p < q$, then (1.5) admits exactly one positive solution singular at the origin. This solution has the exact limits:*

$$\lim_{r \rightarrow \infty} r^{\alpha_1} u = \lambda_1, \quad \lim_{r \rightarrow 0} r^{\alpha_2} u = \lambda_2.$$

This paper is organized as follows. In Section 2, we make some basic observations and fundamental estimates of positive solutions. In Section 3, the asymptotic behaviors at ∞ and 0 of positive solutions are studied. Finally, the uniqueness result is established.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constant which is independent of u and may change its value from line to line.

2. Preliminaries

In this section we present some preliminary results for radial solution $u(r)$ of (1.5), where $r = |x|$ is the radius.

Now we prove the following priori estimates, which are inspired by the work of Ni [20].

Lemma 2.1. *If $-2 < l_2 < l_1 \leq 0$, $\frac{n+l_1}{n-2} < p < q$, let $u(r)$ be a positive radial solution of (1.5) for $r \in (0, \infty)$, then we have, for some positive constant C ,*

- (i) *If u tends to ∞ as $r \rightarrow 0$, then $u(r) \leq Cr^{-\frac{2+l_2}{q-1}}$;*
- (ii) *If u tends to 0 as $r \rightarrow \infty$, then $u(r) \leq Cr^{-\frac{2+l_1}{p-1}}$.*

Proof. (i) For a radial solution $u = u(r)$, we rewrite (1.5) in the following form:

$$(2.1) \quad (r^{n-1}u')' + r^{n-1}(r^{l_1}u^p + r^{l_2}u^q) = 0.$$

Since $u \rightarrow \infty$ as $r \rightarrow 0$, there exists small $r_0 > 0$ such that $u' < 0$ in $(0, r_0)$.

Integrating (2.1) from \bar{r} to r ($\bar{r} < r < r_0$), we obtain

$$r^{n-1}u'(r) = \bar{r}^{n-1}u'(\bar{r}) - \int_{\bar{r}}^r s^{n-1}(s^{l_1}u^p + s^{l_2}u^q)ds.$$

Therefore, $r^{n-1}u'(r) \leq -\int_{\bar{r}}^r s^{n-1}(s^{l_1}u^p + s^{l_2}u^q)ds$ for all $0 < \bar{r} < r$. Then letting $\bar{r} \rightarrow 0$, we obtain

$$\begin{aligned} r^{n-1}u'(r) &\leq -\int_0^r s^{n-1}(s^{l_1}u^p + s^{l_2}u^q)ds \\ &\leq -\int_0^r s^{n-1}s^{l_2}u^q ds. \end{aligned}$$

Since u is decreasing near $r = 0$, we find that

$$r^{n-1}u'(r) < -u^q(r) \int_0^r s^{n-1}s^{l_2} ds = -\frac{1}{n+l_2}r^{n+l_2}u^q(r),$$

which in turn leads to

$$(2.2) \quad \frac{u'(r)}{u^q(r)} \leq -\frac{C}{n+l_2}r^{l_2+1}.$$

Integrating (2.2) over (\bar{r}, r) , we have

$$(2.3) \quad \int_{\bar{r}}^r \frac{u'(s)}{u^q} ds \leq -C \int_{\bar{r}}^r s^{l_2+1} ds.$$

It follows from (2.3) that

$$u^{1-q}(r) \geq u(\bar{r})^{1-q} + \frac{q-1}{l_2+2}(r^{l_2+2} - \bar{r}^{l_2+2}).$$

Letting $\bar{r} \rightarrow 0$, we have

$$u^{1-q}(r) \geq Cr^{2+l_2}.$$

So we have $u(r) \leq Cr^{-\frac{2+l_2}{q-1}}$ for $0 < r < r_0$, and the proof is completed.

Part (ii) of Lemma 2.1 may be handled in a similar way. Similar to the proof of (i) there exists a large number $R > 0$ such that, for all $r > R$,

$$r^{n-1}u'(r) < -\int_R^r s^{n-1}s^{l_1}u^p ds.$$

By a similar computation, we have

$$(2.4) \quad \frac{u'(r)}{u^p} \leq \frac{1}{n+l_1}(R^{n+l_1}r^{1-n} - r^{1+l_1}).$$

Integrating (2.4) over (R, r) , we have

$$u^{1-p}(r) \geq Cr^{2+l_1} \text{ at } r = \infty.$$

So we have $u(r) \leq Cr^{-\frac{2+l_1}{p-1}}$ as $r \rightarrow \infty$, and the proof of part (ii) is completed. \square

Lemma 2.2. *Let u be a positive solution of (1.5). Then there exist two positive numbers \bar{r} and $\bar{\bar{r}}$ such that*

- (i) $|u'(r)| \leq Cr^{-(\frac{2+l_2}{q-1}+1)}$; $|u''(r)| \leq Cr^{-(\frac{2+l_2}{q-1}+2)}$ for $0 < r < \bar{r}$.
(ii) $|u'(r)| \leq Cr^{-(\frac{2+l_1}{p-1}+1)}$; $|u''(r)| \leq Cr^{-(\frac{2+l_1}{p-1}+2)}$ for $r > \bar{\bar{r}}$.

Proof. (i) Integrate (1.2) in a small ball B_r with radius r centered at 0. From Lemma 2.1 and Green's identity, we obtain

$$\begin{aligned} -\omega_n r^{n-1} u'(r) &= - \int_{B_r} \Delta u = \int_{B_r} (|x|^{l_1} u^p + |x|^{l_2} u^q) dx \\ &\leq C \int_0^r s^{l_2} u^q s^{n-1} ds \leq C \int_0^r s^{-\frac{2+l_2}{q-1} q + l_2 + n - 1} ds \\ &= Cr^{-\frac{2+l_2}{q-1} q + l_2 + n}. \end{aligned}$$

So as $r \rightarrow 0$, we have

$$\begin{aligned} -u'(r) = |u'(r)| &\leq Cr^{-\frac{2+l_2}{q-1} q + l_2 + 1} = Cr^{-(\frac{2+l_2}{q-1} + 1)}, \\ |u''| &\leq \frac{n-1}{r} |u'| + r^{l_1} u^p + r^{l_2} u^q \\ &\leq C[r^{-(\frac{2+l_2}{q-1} + 2)} + r^{l_2 - \frac{(2+l_2)q}{q-1}}] \leq Cr^{-(\frac{2+l_2}{q-1} + 2)}. \end{aligned}$$

And the proof of (i) is completed.

Part (ii) of Lemma 2.2 may be handled in a similar fashion. As in the proof of (i), for large r we have

$$\begin{aligned} \omega_n r^{n-1} (-u'(r)) &= - \int_{B_r} \Delta u = \int_{B_r} s^{l_1} u^p + s^{l_2} u^q dx \\ &\leq C + C \int_{R_0}^r s^{-\frac{2+l_1}{p-1} p + l_1 + n - 1} ds, \end{aligned}$$

where R_0 is a large positive number. By a simple computation, we obtain

$$|u'(r)| \leq Cr^{-(\frac{2+l_1}{p-1} + 1)}, \quad |u''(r)| \leq Cr^{-(\frac{2+l_1}{p-1} + 2)} \quad \text{at } r = \infty,$$

and the proof is over. \square

Lemma 2.3. *Suppose that u is a positive solution of (1.5). Let $v(r) = r^\alpha u(r)$. Then v satisfies*

$$v'' + \frac{n-1-2\alpha}{r} v' - \frac{(n-2-\alpha)\alpha}{r^2} v + r^{l_1 - (p-1)\alpha} v^p + r^{l_2 - (q-1)\alpha} v^q = 0.$$

Let $\alpha = \alpha_1$. Then we have $(v_1 = r^{\alpha_1} u)$

$$(2.5) \quad v_1'' + \frac{n-1-2\alpha_1}{r} v_1' - \frac{(n-2-\alpha_1)\alpha_1}{r^2} v_1 + \frac{v_1^p}{r^2} + r^{l_2 - (q-1)\alpha_1} v_1^q = 0.$$

Let $\alpha = \alpha_2$. Then we have ($v_2 = r^{\alpha_2} u$)

$$(2.6) \quad v_2'' + \frac{n-1-2\alpha_2}{r} v_2' - \frac{(n-2-\alpha_2)\alpha_2}{r^2} v_2 + \frac{v_2^q}{r^2} + r^{l_1-(p-1)\alpha_2} v_2^p = 0.$$

This lemma can be proved by straight forward calculations, thus we omit it here.

Lemma 2.4. *We have*

$$(2.7) \quad v_1'^2 r \in L^1(R, \infty), \quad v_2'^2 r \in L^1(0, R),$$

where R is a large positive number.

Proof. Multiplying (2.5) by $v_1' r^2$ and integrating from R to $r > R$, we obtain

$$(2.8) \quad \frac{v_1'^2 s^2}{2} \Big|_R^r + c^1 \int_R^r v_1'^2 s ds - \frac{\lambda_1^{p-1}}{2} v_1^2 \Big|_R^r + \frac{1}{p+1} v_1^{p+1} \Big|_R^r + \int_R^r s^{l_2-\alpha_1(q-1)+2} v_1' v_1^q ds = 0,$$

where $c^1 = n - 2 - 2\alpha_1$.

From (ii) of Lemma 2.2, we obtain

$$(2.9) \quad |v_1'| \leq \frac{C}{r}; \quad |v_1''| \leq \frac{C}{r^2} \quad \text{at } r = \infty,$$

so $v_1^{p+1} \Big|_R^r, \frac{v_1'^2 s^2}{2} \Big|_R^r$ and $\int_R^r s^{l_2-\alpha_1(q-1)+2} v_1' v_1^q ds$ are bounded at $r = \infty$, which imply

$$\int_R^r v_1'^2 s ds \leq C \quad \text{for all } r > R$$

since $c^1 \neq 0$ by (1.6), and the first one (named as $(2.7)_1$) of (2.7) follows. $(2.7)_2$ can be handled by a similar way, we omit it here. \square

Lemma 2.5. *We have*

$$(2.10) \quad \lim_{r \rightarrow \infty} r v_1' = 0; \quad \lim_{r \rightarrow 0} r v_2' = 0.$$

Proof. Now we prove $(2.10)_1$. Suppose for contradiction that it is not true, then there exists a sequence $r_k \rightarrow +\infty$ such that

$$|v_1'(r_k) r_k| \geq C.$$

From (2.9), one obviously has, near ∞ ,

$$|(v_1'^2 r^2)'| \leq \frac{M}{r}$$

for some $M > 0$. Combining the above two inequalities yields

$$|v_1'^2(r) r^2 - v_1'(r_k)^2 r_k^2| \leq M|r - r_k| \max(\frac{1}{r}, \frac{1}{r_k}),$$

and so

$$v_1'^2(r) r^2 \geq \frac{C^2}{2}, \quad r \in [(1 + \varepsilon)^{-1} r_k, (1 + \varepsilon) r_k], \quad \varepsilon(1 + \varepsilon) = \frac{C^2}{2M}.$$

This contradicts (2.7)₁.

The proof of (2.10)₂ may be handled similarly. As in the previous proof, we have, near 0,

$$|(v_2'^2 r^2)'| \leq \frac{M}{r}$$

for some $M > 0$, and by a similar calculation, we will obtain a contradiction and the proof is completed. \square

Lemma 2.6. *Let u be a positive superharmonic function near ∞ and \bar{u} be its spherical mean. Then, $r^{n-2}\bar{u}$ is increasing as $r \rightarrow \infty$.*

Proof. Put $f(t) := r^{n-2}\bar{u}(r), t = \log r$. Then f satisfies

$$f'' - (n - 2)f' \leq 0$$

and $f'(t) \leq e^{(n-2)(t-T)} f'(T)$ on $[T, t]$ for T large. Because f is positive, f must be increasing near ∞ . It implies that $(r^{n-2}\bar{u}(r))_r > 0$ near ∞ . \square

3. Asymptotic behavior

In this section we investigate the asymptotic behavior at ∞ and 0 of positive solutions of (1.5). We prove that any positive solution of (1.5) must behave either like $r^{-\alpha_1}$ or r^{2-n} at ∞ . In addition, if any positive radial solution of (1.2) is singular at the origin, then it must behave like $r^{-\alpha_2}$. Now we give the proof of Theorem 1.2.

The proof of Theorem 1.2. Consider the function $a(r) = \frac{v_1^{p+1}}{p+1} - \frac{\lambda_1^{p-1} v_1^2}{2}$. By Lemma 2.4 and Lemma 2.5, we have, for fixed $R > 0$,

$$\frac{v_1'^2 r^2}{2} \rightarrow 0, \int_R^r v_1'^2 s \rightarrow c_2, \int_R^r r^{l_2 - \frac{(2+l_1)(q-1)}{p-1} + 2} v_1' v_1^p \rightarrow c_3 \text{ as } r \rightarrow \infty$$

for some constants c_2 and c_3 . This implies, by (2.8), that $a(r)$ must tend to a finite constant c_4 as $t \rightarrow \infty$. We claim v_1 approaches a finite limit as $r \rightarrow \infty$. If not, we may choose two sequences $\{\eta_i\}$ and $\{\xi_i\}$ going to ∞ as $i \rightarrow \infty$ such that

$$\begin{cases} \{\eta_i\} \text{ are local minima of } v_1, \{\xi_i\} \text{ are local maxima.} \\ \eta_i < \xi_i < \eta_{i+1}, i = 1, 2, \dots \end{cases}$$

And we have

$$v_1(\eta_i) \rightarrow m_1, \quad v_1(\xi_i) \rightarrow m_2 \quad \text{as } i \rightarrow \infty$$

for some positive constants m_1, m_2 with $m_1 < m_2$. Since $a(r)$ tend to a finite constant as $r \rightarrow \infty$, we have

$$\frac{m_1^{p+1}}{p+1} - \frac{\lambda_1^{p-1} m_1^2}{2} = \frac{m_2^{p+1}}{p+1} - \frac{\lambda_1^{p-1} m_2^2}{2} = c_4,$$

where c_4 is a constant. However, the intermediate value theorem shows that there exists $r_i \in (\eta_i, \xi_i)$ such that

$$v(r_i) = m_0, \quad m_1 < m_0 < m_2 \text{ and } \frac{da(v)}{dv}(r_i) = 0.$$

Furthermore, we have $\frac{m_0^{p+1}}{p+1} - \frac{\lambda_1^{p-1}m_0^2}{2} \neq c_4$, since $\frac{v_1^{p+1}}{p+1} - \frac{\lambda_1^{p-1}v_1^2}{2}$ has only one minima for $v_1 \in [0, +\infty]$. A contradiction is obtained. Similarly, we conclude that $r^{\alpha_2}u(r)$ approaches a finite limit as $r \rightarrow 0$.

- Claim. (1) $\lim_{r \rightarrow \infty} v_1(r)$ must be either 0 or λ_1 ;
 (2) $\lim_{r \rightarrow 0} v_2(r)$ must be either 0 or λ_2 .

We only prove (1), the demonstration of (2) being the same. We denote the $\lim_{r \rightarrow \infty} v_1(r)$ by v_∞ . Now, if $v_\infty \neq 0$, we want to show that $v_\infty = \lambda_1$. From (2.5), we have

$$\frac{d^2v_1}{dt^2} + (n - 2 - 2\alpha_1)\frac{dv_1}{dt} - (n - 2 - \alpha_1)\alpha_1v_1 + v_1^p + e^{(l_2 - (q-1)\alpha_1 + 2)t}v_1^q = 0,$$

where $t = \log r$. From Lemma 2.5, we have $v'(t) \rightarrow 0$ as $t \rightarrow \infty$. So $\lim_{t \rightarrow \infty} v''(t)$ exists and must be 0. Immediately, we have $v_\infty = \lambda_1$ or 0.

If $\lim_{r \rightarrow \infty} v_1(r) = 0$ or $\lim_{r \rightarrow 0} v_2(r) = 0$, (2.5) and (2.6) suggest that $v_i (i = 1, 2)$ tend to zero at an algebraic rate. Indeed, since $p, q > 1$ and $v_i (i = 1, 2)$ is expected to satisfy asymptotically the following equations

$$v_1'' + \frac{n-1-2\alpha_1}{r}v_1' - \frac{\lambda_1^{p-1}}{r^2}v_1 = 0 \quad \text{as } r \rightarrow \infty,$$

$$v_2'' + \frac{n-1-2\alpha_2}{r}v_2' - \frac{\lambda_2^{q-1}}{r^2}v_2 = 0 \quad \text{as } r \rightarrow 0.$$

Therefore v_i should satisfy the following asymptotical behaviors

$$v_1 \approx r^{-(n-2-\alpha_1)} \text{ at } \infty, \quad v_2 \approx r^{\alpha_2} \text{ at } 0.$$

- Now we claim: (i) If $\lim_{r \rightarrow \infty} v_1 = 0$, then $\lim_{r \rightarrow \infty} r^{n-2}u = c > 0$;
 (ii) If $\lim_{r \rightarrow 0} v_2 = 0$, then $\lim_{r \rightarrow 0} u = c_1 > 0$.

First, we prove (i), by assumption, for any $\varepsilon > 0$ there exists a positive number r_ε such that v_1 satisfies

$$(3.11) \quad v_1'' + \frac{n-1-2\alpha_1}{r}v_1' - \frac{(\lambda_1^{p-1} - \varepsilon)}{r^2}v_1 \geq 0, \quad r > r_\varepsilon.$$

The characteristic equation of (3.11) has the two characteristic values

$$a_1 = \alpha_1 - \frac{n-2-\sqrt{(n-2)^2-4\varepsilon}}{2} = \alpha_1 + O(\varepsilon),$$

$$a_2 = \alpha_1 - \frac{n-2+\sqrt{(n-2)^2-4\varepsilon}}{2} = \alpha_1 + 2 - n + O(\varepsilon).$$

Rewrite (3.11) as

$$\left(D - \frac{a_1 - 1}{r}\right)\left(D - \frac{a_2}{r}\right)v_1 \geq 0,$$

where $D := \frac{d}{dr}$, $D^2 := \frac{d^2}{dr^2}$.

Let $(D - \frac{a_2}{r}v_1) = U_1$, so we have

$$U_1' + \frac{1 - a_1}{r}U_1 \geq 0,$$

from which, we have

$$(3.12) \quad [r^{1-\alpha_1+O(\varepsilon)}(D - \frac{a_2}{r})v_1]' \geq 0.$$

Observe that, for ε small enough,

$$\lim_{r \rightarrow \infty} r^{1-\alpha_1+O(\varepsilon)}(D - \frac{a_2}{r})v_1 = 0 \quad \text{by (2.9),}$$

since $1 - \alpha_1 + O(\varepsilon) < 1$. It follows from (3.12) that

$$(D - \frac{a_2}{r})v_1 \leq 0, \quad r > r_\varepsilon.$$

Integrating once from r_ε to r yields

$$v_1 \leq c_\varepsilon r^{a_2} = c_\varepsilon r^{\alpha_1+2-n+O(\varepsilon)}, \quad r > r_\varepsilon.$$

Thus for ε sufficiently small,

$$(3.13) \quad v_1'' + \frac{n-1-2\alpha_1}{r}v_1' - \frac{\lambda_1^{p-1}}{r^2}v_1 = g(r)$$

with

$$(3.14) \quad g(r) = \frac{v_1^p}{r^2} + r^{l_2-(q-1)\alpha_1}v_1^q = O(r^{-2-\delta}) \quad \text{near } \infty,$$

where δ is a positive constant.

Applying the method of variation of parameters to (3.13), v_1 is represented by

$$\begin{aligned} v_1(r) &= C_1(R)r^{\alpha_1} + C_2(R)r^{\alpha_1+2-n} \\ &\quad + \frac{r^{\alpha_1+2-n}}{2-n} \int_R^r s^{n-1-\alpha_1}g(s)ds - \frac{r^{\alpha_1}}{2-n} \int_R^r s^{1-\alpha_1}g(s)ds. \end{aligned}$$

By a similar computation, we have, from (3.14) and the boundedness of $v_1(r)$,

$$v_1(r) = Cr^{\alpha_1+2-n} + o(r^{\alpha_1+2-n}),$$

hence,

$$r^{n-2}u(r) \leq C \quad \text{at } r = \infty.$$

By Lemma 2.6, there exists a constant c such that $r^{n-2}u(r) \rightarrow c$ as $r \rightarrow \infty$.

The proof of (ii) is handled by the same way. As in the proof of (i), for any $\varepsilon' > 0$, there exists a positive number $r_{\varepsilon'}$ such that

$$\lim_{r \rightarrow 0} r^{n-1-\alpha_2+O(\varepsilon')}(D - \frac{\alpha_2}{r})v_2 = 0,$$

and

$$(D - \frac{\alpha_2}{r})v_2 \geq 0, \quad 0 < r < r_{\varepsilon'}.$$

Integrating once from r to $r_{\varepsilon'}$ yields

$$v_2 \leq c_{\varepsilon'} r^{\alpha_2 + O(\varepsilon')}.$$

Applying the method of variation of parameters to (2.6), we immediately obtain that v_2 is bounded by r^{α_2} , and in turn u is bounded, by standard theory, we have $\lim_{r \rightarrow 0} u(r) = c_1$.

When $p = \frac{n+2+2l_1}{n-2}$ or $q = \frac{n+2+2l_2}{n-2}$, there is another possibility: $r^{\frac{n-2}{2}}u$ could oscillate endlessly near ∞ or oscillate endlessly near the origin. \square

The proof of Theorem 1.4. (i) The function $v_2(t) := r^{\alpha_2}u, t = \log r$, satisfies

$$v_2'' - \lambda_2^{q-1}v_2 + v_2^q + e^{\delta_1 t}v_2^p = 0,$$

where $\delta_1 = (\alpha_1 - \alpha_2)(p - 1) > 0$. Note that by Lemma 2.1(i), $v_2(t)$ is bounded near $-\infty$. Define an energy function

$$E(t) := \frac{1}{2}v_2'^2 - \frac{\lambda_2^{q-1}}{2}v_2^2 + \frac{1}{q+1}v_2^{q+1}.$$

As in (2.8), we have

$$E(t) = C(T) + \int_t^T e^{\delta_1 s}v_2^p v_2' ds,$$

where T is a fixed number.

In case that v_2 oscillates near $-\infty$, then we may suppose that

$$0 \leq \mu_1 = \liminf_{t \rightarrow -\infty} v_2(t) < \limsup_{t \rightarrow -\infty} v_2(t) = \mu_2 < \infty.$$

Then, there exist two sequences η_i and ε_i going to $-\infty$ as $i \rightarrow \infty$ such that η_i and ε_i are local minima and local maxima of v_2 , respectively, satisfying $\eta_i < \varepsilon_i < \eta_{i+1}, i = 1, 2, \dots$. From (2.8), we know $\lim_{t \rightarrow -\infty} \int_t^T e^{\delta_1 s}v_2^p v_2' ds$ exists, so $E = \lim_{t \rightarrow -\infty} E(t)$ exists, and from which we have

$$\lim_{i \rightarrow \infty} E(\eta_i) = -\frac{\lambda_2^{q-1}}{2}\mu_1^2 + \frac{1}{q+1}\mu_1^{q+1} = -\frac{\lambda_2^{q-1}}{2}\mu_2^2 + \frac{1}{q+1}\mu_2^{q+1} = \lim_{i \rightarrow \infty} E(\varepsilon_i),$$

which implies that $b(\mu_1) = b(\mu_2)$.

Observe that for each $i > 1$,

$$(3.16) \quad 0 \leq v_2''(\eta_i) = \lambda_2^{q-1}v_2(\eta_i) - v_2^q(\eta_i) - v_2^p(\eta_i)e^{\delta_1 \eta_i},$$

while

$$(3.17) \quad 0 \geq v_2''(\varepsilon_i) = \lambda_2^{q-1}v_2(\varepsilon_i) - v_2^q(\varepsilon_i) - v_2^p(\varepsilon_i)e^{\delta_1 \varepsilon_i}.$$

Putting $v_2(\eta_i) = \mu_{1,i}, v_2(\varepsilon_i) = \mu_{2,i}$, we have

$$\lim_{i \rightarrow \infty} \mu_{1,i} = \mu_1, \quad \lim_{i \rightarrow \infty} \mu_{2,i} = \mu_2.$$

From (3.16) and (3.17), we obtain

$$0 < \mu_1 \leq \lambda_2 \leq \mu_2.$$

The proof of (ii) can be handled in a similar way, we omit it here. \square

4. A uniqueness result

In this section, we shall prove a uniqueness result for positive radial singular solutions of (1.5) when $p > \frac{n+2+2l_1}{n-2}$ and $q < \frac{n+2+2l_2}{n-2}$. More precisely, we require that solutions be singular at infinity for $q < \frac{n+2+2l_2}{n-2}$, and be singular at the origin for $p > \frac{n+2+2l_1}{n-2}$. In order to prove Theorem 1.6, we need a finer asymptotic behavior of solution to (1.5) near 0 and ∞ .

Let $w(t) = v_1 - \lambda_1$, $t = \log r$. If $\lim_{t \rightarrow +\infty} v(t) = \lambda_1$, then we have $\lim_{t \rightarrow +\infty} w(t) = 0$ and

$$(4.1) \quad w'' + (n - 2 - 2\alpha_1)w' + (2 + l_1)(n - 2 - \alpha_1)w + f(w) + (w + \lambda_1)^q e^{\delta t} = 0,$$

where $\delta = \frac{(2+l_1)(1-q)}{p-1} + 2 + l_2 < 0$, and

$$\begin{aligned} f(w) &= (\lambda_1 + w)^p - \lambda_1^p - p\lambda_1^{p-1}w = \lambda_1^p \sum_{k=2}^{\infty} \frac{(p-k+1)}{k!} \left(\frac{w}{\lambda_1}\right)^k \\ &= \frac{(p-1)\lambda_1^{p-2}}{2} w^2 + o(w^2) \quad \text{for } w \text{ near } 0. \end{aligned}$$

Theorem 4.1. *Let $u(r)$ be a singular solution of (1.5) at infinity and $\frac{n+l_1}{n-2} < p < \frac{n+2+2l_1}{n-2}$. Then for any $\varepsilon \in (0, -\delta)$ we have*

$$(4.2) \quad w(t) = O(e^{-\varepsilon t}), \quad w'(t) = O(e^{-\varepsilon t}) \quad \text{as } t \rightarrow +\infty.$$

Proof. For $T > 0$ and $t \in (0, T)$, multiplying (4.1) by $2w'(t)$ and integrating from t to T , we obtain

$$(4.3) \quad \begin{aligned} & [w'^2 + (2 + l_1)(n - 2 - \alpha_1)w^2] \Big|_t^T + 2(n - 2 - 2\alpha_1) \int_t^T w'^2 ds \\ & + 2 \int_t^T f(s)w' ds + 2 \int_t^T (w + \lambda_1)^q e^{\delta t} w' ds = 0. \end{aligned}$$

By (2.10), it is easy to see that for large t ,

$$\begin{aligned} w'^2(T) &\rightarrow 0, \quad w^2(T) \rightarrow 0, \\ \int_t^T f(s)w' ds &= -\frac{(p-1)\lambda_1^{p-2}}{6} w^3(t) + o(w^3(t)) \quad \text{as } T \rightarrow +\infty. \end{aligned}$$

It follows from Lemma 2.4 and (4.3) (letting $T \rightarrow \infty$) that, for large t ,

$$w'^2(t) + (2 + l_1)(n - 2 - \alpha_1)w^2(t) \leq C|w|^3 + \int_t^\infty (w + \lambda_1)^q e^{\delta s} w' ds,$$

since $n - 2 - 2\alpha_1 < 0$. It follows that for large t

$$(4.4) \quad w'^2(t) + (2 + l_1)(n - 2 - \alpha_1)w^2(t) \leq C \int_t^\infty (w + \lambda_1)^q e^{\delta s} w' ds \leq C e^{\delta t}.$$

Hence by (4.4) and the fact $(2 + l_1)(n - 2 - 2\alpha_1) > 0$, we have

$$|w'(t)| + |w(t)| \leq Ce^{\frac{\delta}{2}t}.$$

Therefore

$$\left| \int_t^T (w + \lambda_1)^q e^{\delta t} w' ds \right| \leq C \int_t^T e^{\frac{3\delta}{2}s} \leq Ce^{\frac{3\delta}{2}t},$$

which, together with (4.4), imply, that

$$|w'(t)| + |w(t)| \leq Ce^{\frac{3\delta}{4}t}.$$

Thus for any $m > 0$, using a simple iteration of m -step in (4.4) yields

$$|w'(t)| + |w(t)| \leq Ce^{\frac{(2^m - 1)\delta}{2^m}t},$$

and (4.2) follows by taking m large. □

Remark 4.2. As a matter of fact, applying the method of variation of parameters to (4.1), we have that ε can reach at $-\delta$. Because of (4.2), we have $f(t) = O(w^2(t)) = O(e^{\delta t})$ and

$$w(t) = -\frac{1}{\omega} \int_t^\infty e^{\frac{n-2-2\alpha_1}{2}(s-t)} \sin \omega(s-t) [f(s) + e^{\delta t}(v_1 + \lambda_1)^p] ds = O(e^{\delta t}),$$

where $\omega = ((2 + l_1)(n - 2 - \alpha_1) - \frac{1}{4}(n - 2 - 2\alpha_1)^2)^{\frac{1}{2}}$.

Remark 4.3. If $u(r)$ is a singular solution of (1.5) at 0, and $\frac{n+l_1}{n-2} < q < \frac{n+2+2l_2}{n-2}$, then we have a similar result

$$v_2(r) - \lambda_2 = O(r^\epsilon), \quad (v_2(r) - \lambda_2)' = O(r^{\epsilon+1}) \quad \text{as } r \rightarrow 0,$$

where $\epsilon \in (0, \frac{(2+l_2)(1-p)}{q-1} + 2 + l_1)$.

Now we are ready to prove Theorem 1.6. Let $u_1(r)$ and $u_2(r)$ be two different singular solutions of (1.5) at infinity. We introduce the function

$$\bar{w}(t) = r^{\alpha_1} u_1(r) - r^{\alpha_1} u_2(r) = w_1(t) - w_2(t), \quad t = \log r,$$

and show that $\bar{w}(t)$ is identical to zero.

Proof of Theorem 1.6. Clearly $\bar{w}(t)$ satisfies

$$(4.5) \quad \bar{w}''(t) + (n - 2 - 2\alpha_1)\bar{w}'(t) + (2 + l_1)(n - 2 - \alpha_1)\bar{w} + (f(w_1) - f(w_2)) + [(w_1(t) + \lambda_1)^q - (w_2(t) + \lambda_1)^q]e^{\delta t} = 0.$$

For $T > 0$ and $t \in (0, T)$, multiplying (4.5) by $2\bar{w}'$ and integrating from t to T , we obtain

$$(4.6) \quad [\bar{w}'^2 + (2 + l_1)(n - 2 - \alpha_1)\bar{w}^2] \Big|_t^T + (n - 2 - 2\alpha_1) \int_t^T \bar{w}'^2 + 2 \int_t^T [f(w_1) - f(w_2)]\bar{w}' + 2 \int_t^T [(w_1(t) + \lambda_1)^q - (w_2(t) + \lambda_1)^q]e^{\delta s}\bar{w}' = 0.$$

Thus we have the following estimates:

$$\left| \int_t^T [f(w_1) - f(w_2)]\bar{w}' \right| \leq C \int_t^T |\bar{w}'\bar{w}|e^{-\varepsilon s} \leq C \int_t^T e^{-\varepsilon s}(\bar{w}^2 + \bar{w}'^2)$$

and

$$\int_t^T [(w_1(t) + \lambda_1)^q - (w_2(t) + \lambda_1)^q]e^{\delta s}\bar{w}' \leq C \int_t^T e^{\delta s}(\bar{w}^2 + \bar{w}'^2).$$

For large t , it follows, by letting $T \rightarrow +\infty$ and using Theorem 4.1, that

$$\bar{w}'^2(t) + (2 + l_1)(n - 2 - \alpha_1)\bar{w}^2(t) \leq C \int_t^\infty e^{-\varepsilon s}(\bar{w}^2 + \bar{w}'^2),$$

since $n - 2 - 2\alpha_1 \leq 0$. From which, we have

$$\bar{w}'^2(t) + \bar{w}^2(t) \leq C \int_t^\infty e^{-\varepsilon s}(\bar{w}^2 + \bar{w}'^2),$$

since $(2 + l_1)(n - 2 - \alpha_1) > 0$. Hence $\bar{w}'^2(t) + \bar{w}^2(t) \equiv 0$ for all sufficiently large t by the Gronwall inequality. According to the uniqueness of ordinary differential equation, w is identically zero for all t .

When $p > \frac{n+2+2l_1}{n-2}$, we use the transformation

$$\bar{w}_1(t) = r^{\alpha_2}u_1(r) - r^{\alpha_2}u_2(r) = z_1(t) - z_2(t), \quad t = \log r,$$

where $z_i(t) = r^{\alpha_2}u_i(r) - \lambda_2 \rightarrow 0$ ($i, = 1, 2$) as $t \rightarrow -\infty$. Then $\bar{w}_1(t)$ satisfies

$$(4.7) \quad \begin{aligned} &\bar{w}_1''(t) + (n - 2 - 2\alpha_2)\bar{w}_1'(t) + (2 + l_2)(n - 2 - \alpha_2)\bar{w}_1 \\ &+ (\bar{f}(z_1) - \bar{f}(z_2)) + [(z_1(t) + \lambda_2)^p - (z_2(t) + \lambda_2)^p]e^{\delta_2 t} = 0, \end{aligned}$$

where $\delta_2 = (p - 1)(\alpha_1 - \alpha_2) > 0$, and

$$\begin{aligned} \bar{f}(z) &= (\lambda_2 + z)^q - \lambda_2^q - q\lambda_2^{q-1}z = \lambda_2^q \sum_{k=2}^\infty \frac{q - k + 1}{k!} \left(\frac{z}{\lambda_2}\right)^k \\ &= \frac{(q - 1)\lambda_2^{q-2}}{2}z^2 + o(z^2) \quad \text{for } t \text{ near } -\infty. \end{aligned}$$

The equation (4.7) is the same as (4.5). On the other hand, one can easily see from previous proof that the key ingredient of the proof for the case $p > \frac{n+2+2l_1}{n-2}$ is that the coefficient of the term $\bar{w}_1'(t)$ is not less than 0. Fortunately, we have $n - 2 - 2\alpha_2 > 0$ when $p > \frac{n+2+2l_1}{n-2}$. Hence the proof above carries over immediately. □

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