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SEMIGROUPS OF TRANSFORMATIONS WITH INVARIANT SET

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Abstract. Let $T(X)$ denote the semigroup (under composition) of transformations from X into itself. For a fixed nonempty subset Y of X, let

$$S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

Then $S(X,Y)$ is a semigroup of total transformations of X which leave a subset Y of X invariant. In this paper, we characterize when $S(X,Y)$ is isomorphic to $T(Z)$ for some set Z and prove that every semigroup A can be embedded in $S(A^1,A)$. Then we describe Green's relations for $S(X,Y)$ and apply these results to obtain its group $\mathcal{H}$-classes and ideals.

1. Introduction

The full transformation semigroup $T(X)$ is extremely important and the Green's relations play an essential role in semigroup theory. As far back in 1952, Malcev [5] determined ideals of $T(X)$, later in 1955 Miller and Doss [2] described its Green's relations and group $\mathcal{H}$-classes. This paper is devoted to generalizations of these results.

The semigroup we consider is $S(X,Y)$ consists of all mappings in $T(X)$ which leave $Y \subseteq X$ invariant. To the extent that $S(X,X) = T(X)$, we may regard $S(X,Y)$ as a generalization of $T(X)$.

Magill [4] introduced and studied the semigroup $S(X,Y)$ in 1966. Later in 1975, Symons [7] described the automorphism group of this semigroup. In 2005 Nenthein, Youngkhong, and Kemprasit [6] showed that $S(X,Y)$ is a regular semigroup if and only if $X = Y$ or Y contains exactly one element, and $E = \{\alpha \in S(X,Y) : X\alpha \cap Y = Y\alpha\}$ is the set of all regular elements of $S(X,Y)$. Here, in Section 2, we prove that: $S(X,Y)$ is isomorphic to $T(Z)$ if and only if $X = Y$ and $|Y| = |Z|$, and we also prove that every semigroup A can be embedded in $S(A^1,A)$. In Section 3, we characterize Green's relations on $S(X,Y)$ and find that $\mathcal{D} = \mathcal{J}$ if and only if X is a finite set or $X = Y$ or $|Y| = 1$,

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and we also show that its group $\mathcal{H}$-class is isomorphic to a certain subgroup of a permutation group. In Section 4, we describe ideals of the semigroup $S(X,Y)$.

Throughout the paper, the set X we consider can be finite or infinite. The cardinality of a set A is denoted by $|A|$ and $X = A\dot{\cup}B$ means X is a disjoint union of A and B. Also, we write functions on the right; in particular, this means that for a composition $\alpha\beta$, α is applied first.

2. Isomorphisms and embeddings

Let X be any set and Y a fixed nonempty subset of X. We consider the subsemigroup of $T(X)$ defined by

$$S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\},$$

where $Y\alpha$ denotes the range of Y under α. Note that id_X, the identity map on X, belongs to $S(X,Y)$ and

$$E = \{\alpha \in S(X,Y) : X\alpha \cap Y = Y\alpha\}$$

is the set of all regular elements of $S(X,Y)$.

As in Clifford and Preston [1] vol 2, p. 241, we shall use the notation

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}$$

to mean that $\alpha \in T(X)$ and take as understood that the subscript i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$.

With the above notation, for any $\alpha \in S(X,Y)$ we can write

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix},$$

where $A_i \cap Y \neq \emptyset; B_j, C_k \subseteq X\backslash Y; \{a_i\} \subseteq Y, \{b_j\} \subseteq Y\backslash\{a_i\}$ and $\{c_k\} \subseteq X\backslash Y$. Here, I is a nonempty set, but J or K can be empty. For examples: If $\alpha \in E$, then J is an empty set. And if $\alpha \in S(X,Y)\backslash E$, then both I and J are nonempty but K can be an empty set.

The following example shows that in general E is not a subsemigroup of $S(X,Y)$.

Example 1. (a) Let $X = \{1,2,3\}$ and $Y = \{1,2\}$. Define

$$\alpha = \begin{pmatrix} \{1,2\} & 3 \\ 1 & 3 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & \{2,3\} \\ 1 & 2 \end{pmatrix}.$$

Then we have $\alpha, \beta \in E$, but

$$\alpha\beta = \begin{pmatrix} \{1,2\} & 3 \\ 1 & 2 \end{pmatrix} \notin E.$$

(b) Let $X = \mathbb{N}$ denote the set of positive integers and $Y = \{1, 2\}$. Define

$$\alpha = \begin{pmatrix} \{1,2\} & X\backslash\{1,2\} \\ 1 & 3 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & X\backslash\{1\} \\ 1 & 2 \end{pmatrix}.$$

Thus we have $\alpha, \beta \in E$, but

$$\alpha\beta = \begin{pmatrix} \{1,2\} & X\backslash\{1,2\} \\ 1 & 2 \end{pmatrix} \notin E.$$

Therefore, E in (a) and (b) are not subsemigroups of $S(X, Y)$.

To give a necessary and sufficient condition for E to be a regular subsemigroup, we first note the following.

(1) If $X = Y$, then $E = S(X, Y) = T(X)$ which is a regular semigroup.

(2) If $|Y| = 1$, say $Y = \{a\}$, then for each $\alpha \in S(X, Y)$ we have $X\alpha \cap Y = \{a\} = Y\alpha$, so $S(X, Y) = E$.

Lemma 1. *The following statements are equivalent*:

(1) *E is a regular subsemigroup of $S(X, Y)$.*

(2) *$S(X, Y)$ is a regular semigroup.*

(3) *$X = Y$ or $|Y| = 1$.*

Proof. From [6], Corollary 2.4, we have $S(X, Y)$ is regular if and only if $X = Y$ or $|Y| = 1$. Now, assume that E is a regular subsemigroup of $S(X, Y)$ and suppose that $Y \subsetneq X$ and $|Y| \geq 2$. Let $a, b \in Y$ be such that $a \neq b$ and $c \in X\backslash Y$. Define $\alpha, \beta \in E$ by

$$\alpha = \begin{pmatrix} Y & X\backslash Y \\ a & c \end{pmatrix}, \ \beta = \begin{pmatrix} a & X\backslash\{a\} \\ a & b \end{pmatrix}.$$

Then $\alpha\beta = \left(\begin{smallmatrix} Y & X\backslash Y \\ a & b \end{smallmatrix}\right) \notin E$ which is a contradiction. Therefore, $Y = X$ or $|Y| = 1$. Conversely, assume that $X = Y$ or $|Y| = 1$. If $X = Y$, then $E = T(X)$ which is a regular semigroup. If $|Y| = 1$, then $S(X, Y)$ is regular and $E = S(X, Y)$, thus E is a regular subsemigroup. □

If X is an infinite set and Y is a finite subset of X, then $X \neq Y$. Thus we have the following corollary.

Corollary 1. *If X is an infinite set and Y is a finite subset of X, then $E = S(X, Y)$ is regular if and only if $|Y| = 1$.*

Theorem 1. *$S(X, Y) \cong T(Z)$ for some set Z if and only if $X = Y$ and $|Y| = |Z|$.*

Proof. If $X = Y$ and $|Y| = |Z|$, then $S(X, Y) = S(Y, Y) = T(Y) \cong T(Z)$. Conversely, assume that $S(X, Y) \cong T(Z)$. Suppose that $Y \subsetneq X$, then $|X| > 1$. If $|Y| = 1$, say $Y = \{a\}$, then $S(X, Y)$ is a regular semigroup (by Lemma 1) with more than one element and having

$$\alpha = \begin{pmatrix} X \\ a \end{pmatrix}$$

as a zero element. That means $S(X,Y) \cong T(Z)$ is a semigroup with zero which is a contradiction. But, if $|Y| > 1$, then $S(X,Y)$ is not regular, thus $S(X,Y) \not\cong T(Z)$ which is a contradiction. Therefore $X = Y$ and hence $S(X,Y) = S(Y,Y) = T(Y)$. Thus $T(Y) \cong T(Z)$ and this gives $|Y| = |Z|$. □

Since $X \neq Y$ when X is an infinite set and Y is a finite subset of X, the following corollary is an immediate consequence of Theorem 1.

Corollary 2. *If X is an infinite set and Y is a finite subset of X, then $S(X,Y)$ is never isomorphic to $T(Z)$ for any set Z.*

From Theorem 1, we see that $S(X,Y)$ is almost never isomorphic to $T(Z)$. However, we can embedded $T(Y)$ into $S(X,Y)$ by sending $\alpha \mapsto \alpha'$ where $\alpha' \in S(X,Y)$ is defined by

$$x\alpha' = \begin{cases} x\alpha & \text{if } x \in Y, \\ x & \text{if } x \in X \backslash Y. \end{cases}$$

In 1959, M. Hall ([3], Theorem 1.1.2) showed that every semigroup A can be embedded in the full transformation semigroup by using the extended right regular representation of A. That is for each $a \in A$, define a map $\rho_a : A^1 \to A^1$ by $x\rho_a = xa \quad (x \in A^1)$. Then $\rho_a \in T(A^1)$ and $\Phi : A \to T(A^1)$ given by $a\Phi = \rho_a$ is a monomorphism. Since for each $a \in A$, we have $x\rho_a = xa \in A$ for all $x \in A$, it follows that $\rho_a \in S(A^1, A)$ and so Φ maps A into $S(A^1, A)$ is a well-defined monomorphism. That means A can be embedded in $S(A^1, A)$ which is a proper non-regular subsemigroup of $T(A^1)$ if A does not contains an identity element. Thus we have proved the following theorem.

Theorem 2. *Every semigroup A can be embedded in $S(A^1, A)$.*

3. Green's relations on $S(X,Y)$

Since $S(X,Y)$ is not a regular subsemigroup of $T(X)$ if $Y \subsetneq X$ and $|Y| > 1$, Hall's theorem ([3], Proposition 2.4.2) can not be applied to find the $\mathcal{L}$ and $\mathcal{R}$ relations on this semigroup. However, it is well-known that $\alpha\mathcal{L}\beta$ in $T(X)$ if and only if $X\alpha = X\beta$; and $\alpha\mathcal{R}\beta$ in $T(X)$ if and only if $\pi_\alpha = \pi_\beta$ (see [1] vol 1, Lemma 2.5 and Lemma 2.6).

Lemma 2. *Let $\alpha, \beta \in S(X,Y)$. Then $\alpha = \gamma\beta$ for some $\gamma \in S(X,Y)$ if and only if $X\alpha \subseteq X\beta$ and $Y\alpha \subseteq Y\beta$. Consequently, $\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$ and $Y\alpha = Y\beta$.*

Proof. We first note that if $\alpha = \gamma\beta$ for some $\gamma \in S(X,Y)$, then $X\alpha \subseteq X\beta$ and $Y\alpha = Y\gamma\beta = (Y\gamma)\beta \subseteq Y\beta$ since $\gamma \in S(X,Y)$.

To prove the converse, we suppose that $X\alpha \subseteq X\beta$ and $Y\alpha \subseteq Y\beta$. Then $Y(\alpha|_Y) \subseteq Y(\beta|_Y)$ where $\alpha|_Y, \beta|_Y \in T(Y)$. Hence, by a standard result, $\alpha|_Y = \delta(\beta|_Y)$ for some $\delta \in T(Y)$: that is, $y\alpha = (y\delta)\beta$ for each $y \in Y$. Now, for each

$x \notin Y$, there exists some $x' \in X$ such that $x\alpha = x'\beta$ since $X\alpha \subseteq X\beta$. Thus for each $x \notin Y$, choose such an x' and extend $\delta \in T(Y)$ to $\gamma \in T(X)$ by

$$x\gamma = \begin{cases} x\delta & \text{if } x \in Y, \\ x' & \text{if } x \notin Y. \end{cases}$$

Then $\gamma \in S(X, Y)$ and $\alpha = \gamma\beta$ as required. □

We note that for any $\alpha \in S(X, Y)$, the symbol π_α will denote the composition of X induced by the map α, namely

$$\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\},$$

and $\pi_\alpha(Y)$ will denote the subset of π_α defined by

$$\pi_\alpha(Y) = \{y\alpha^{-1} : y \in X\alpha \cap Y\}.$$

For $\alpha, \beta \in S(X, Y)$, $\mathcal{A} \subseteq \pi_\alpha$, and $\mathcal{B} \subseteq \pi_\beta$, we say that $\mathcal{A}$ *refines* $\mathcal{B}$ if for each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subseteq B$.

Lemma 3. *Let $\alpha, \beta \in S(X, Y)$. Then $\alpha = \beta\gamma$ for some $\gamma \in S(X, Y)$ if and only if π_β refines π_α and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$. Consequently, $\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$ and $\pi_\alpha(Y) = \pi_\beta(Y)$.*

Proof. It is clear that if $\alpha = \beta\gamma$ for some $\gamma \in S(X, Y)$, then π_β refines π_α and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$.

Conversely, assume that the conditions hold. For each $x \in X\beta$, there exists $z \in X$ such that $x = z\beta$, so we define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} z\alpha, & \text{if } x \in X\beta, \\ x\beta, & \text{if } x \in X \backslash X\beta. \end{cases}$$

Then γ is well-defined since π_β refines π_α. Now, we prove that $\gamma \in S(X, Y)$. For each $y \in Y$, we have $y \in X\backslash X\beta$ or $y \in X\beta \cap Y$. If $y \in X\backslash X\beta$, then $y\gamma = y\beta \in Y$ since $\beta \in S(X, Y)$. If $y \in X\beta \cap Y$, then there exists $x \in X$ such that $y = x\beta$. Since $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$, we have $x \in y\beta^{-1} \subseteq y'\alpha^{-1}$ for some $y' \in X\alpha \cap Y$. Thus $y\gamma = x\beta\gamma = x\alpha = y' \in Y$. Also, we have $x\beta\gamma = (x\beta)\gamma = x\alpha$ for all $x \in X$ by the definition of γ. □

Recall that each group $\mathcal{H}$-class of $T(X)$ is isomorphic to a permutation group $G(A)$ for some $A \subseteq X$ ([1] vol 1, Theorem 2.10). Here, for the semigroup $S(X, Y)$, the result depends on the group which is denoted by $G(A, B)$ and

$$G(A, B) = \{\rho \in G(A) : \rho|_B \in G(B)\},$$

where $B \subseteq A$ for some $A \subseteq X$ and $B \subseteq Y$.

Theorem 3. *Let ϵ be an idempotent in $S(X, Y)$. Then the group $\mathcal{H}$-class H_ϵ is isomorphic to $G(A, B)$. In this case, A is a cross section of π_ϵ.*

Proof. Since ϵ is an idempotent, we can write

$$\epsilon = \begin{pmatrix} C_i & D_j \\ c_i & d_j \end{pmatrix},$$

where $c_i \in C_i \cap Y$ and $d_j \in D_j \subseteq X \backslash Y$. Let $B = \{c_i\} \subseteq Y$ and $A = \{c_i\} \cup \{d_j\} \subseteq X$. Since $H_\epsilon = L_\epsilon \cap R_\epsilon$, we have by Lemma 2 and Lemma 3 that

$$H_\epsilon = \left\{ \begin{pmatrix} C_i & D_j \\ c_i\sigma & d_j\delta \end{pmatrix} : \sigma \in G(B), \delta \in G(A \backslash B) \right\}.$$

Let $\rho = \sigma \cup \delta$. Then $\rho \in G(A, B)$ and

$$H_\epsilon = \left\{ \begin{pmatrix} C_i & D_j \\ c_i\rho & d_j\rho \end{pmatrix} : \rho \in G(A, B) \right\}.$$

Therefore, H_ϵ is isomorphic to $G(A, B)$ by sending $\left(\begin{smallmatrix} C_i & D_j \\ c_i\rho & d_j\rho \end{smallmatrix}\right) \mapsto \rho$ where $G(A, B)$ is a subgroup of the permutation group $G(A)$. $\square$

We note that when $\epsilon = id_X$, then H_ϵ the group of units of $S(X, Y)$ is isomorphic to $G(X, Y)$ and this group was shown to isomorphic to the automorphism group of $S(X, Y)$ when $|Y| > 2$ (see [7], Theorem 4.2).

Clifford and Preston in [1] vol 1, Lemma 2.8, proved that two elements of $T(X)$ are $\mathcal{D}$-related if and only if they have the same rank, that is, the ranges of the two elements have the same cardinality. But, for $S(X, Y)$ we have the following theorem.

Theorem 4. *Let $\alpha, \beta \in S(X, Y)$. Then $\alpha\mathcal{D}\beta$ if and only if $|Y\alpha| = |Y\beta|$, $|X\alpha \backslash Y| = |X\beta \backslash Y|$ and $|(X\alpha \cap Y) \backslash Y\alpha| = |(X\beta \cap Y) \backslash Y\beta|$.*

Proof. First assume that $\alpha\mathcal{L}\gamma\mathcal{R}\beta$ for some $\gamma \in S(X, Y)$. Then by Lemma 3, we have $\pi_\beta = \pi_\gamma$ and $\pi_\beta(Y) = \pi_\gamma(Y)$. Thus we can write

$$\beta = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix}, \ \gamma = \begin{pmatrix} A_i & B_j & C_k \\ x_i & y_j & z_k \end{pmatrix},$$

where $A_i \cap Y \neq \emptyset; B_j, C_k \subseteq X \backslash Y; \{a_i\}, \{x_i\} \subseteq Y; \{b_j\} \subseteq Y \backslash \{a_i\}, \{y_j\} \subseteq Y \backslash \{x_i\}$ and $\{c_k\}, \{z_k\} \subseteq X \backslash Y$. Since $X\alpha = X\gamma$, $Y\alpha = Y\gamma$ by Lemma 2, we must have

$$\alpha = \begin{pmatrix} L_i & M_j & N_k \\ x_i & y_j & z_k \end{pmatrix},$$

where $L_i \cap Y \neq \emptyset$ and $M_j, N_k \subseteq X \backslash Y$. Then $|Y\alpha| = |\{x_i\}| = |I| = |\{a_i\}| = |Y\beta|$, $|X\alpha \backslash Y| = |\{z_k\}| = |K| = |\{c_k\}| = |X\beta \backslash Y|$ and $|(X\alpha \cap Y) \backslash Y\alpha| = |\{y_j\}| = |J| = |\{b_j\}| = |(X\beta \cap Y) \backslash Y\beta|$.

Conversely, assume that the conditions hold. We can write

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix}, \ \beta = \begin{pmatrix} U_i & V_j & W_k \\ u_i & v_j & w_k \end{pmatrix},$$

where $A_i \cap Y \neq \emptyset \neq U_i \cap Y; B_j, C_k, V_j, W_k \subseteq X\backslash Y; \{a_i\}, \{u_i\} \subseteq Y; \{b_j\} \subseteq Y\backslash\{a_i\}, \{v_j\} \subseteq Y\backslash\{u_i\}$ and $\{c_k\}, \{w_k\} \subseteq X\backslash Y$. Then we define

$$\mu = \begin{pmatrix} U_i & V_j & W_k \\ a_i & b_j & c_k \end{pmatrix},$$

thus $\mu \in S(X,Y)$ and $Y\mu = \{a_i\} = Y\alpha$, $X\mu = \{a_i\} \cup \{b_j\} \cup \{c_k\} = X\alpha$. So $\alpha\mathcal{L}\mu$ by Lemma 2. Also we have $\pi_\mu = \pi_\beta$ and $\pi_\mu(Y) = \{U_i\} \cup \{V_j\} = \pi_\beta(Y)$. Hence $\mu\mathcal{R}\beta$ by Lemma 3 and therefore $\alpha\mathcal{D}\beta$. □

Corollary 3. *Let $\alpha, \beta \in S(X,Y)$. If Y is a finite subset of X, then $\alpha\mathcal{D}\beta$ if and only if $|X\alpha| = |X\beta|, |Y\alpha| = |Y\beta|$ and $|X\alpha \cap Y| = |X\beta \cap Y|$.*

Proof. Suppose that Y is a finite subset of X. If $\alpha\mathcal{D}\beta$, then by Theorem 4, we have $|Y\alpha| = |Y\beta|, |X\alpha\backslash Y| = |X\beta\backslash Y|$ and $|(X\alpha \cap Y)\backslash Y\alpha| = |(X\beta \cap Y)\backslash Y\beta|$. Since $X\alpha \cap Y = Y\alpha\dot{\cup}[(X\alpha \cap Y)\backslash Y\alpha]$, it follows that $|X\alpha \cap Y| = |Y\alpha| + |(X\alpha \cap Y)\backslash Y\alpha| = |Y\beta| + |(X\beta \cap Y)\backslash Y\beta| = |Y\beta\dot{\cup}[(X\beta \cap Y)\backslash Y\beta]| = |X\beta \cap Y|$. Since $X\alpha = (X\alpha \cap Y)\dot{\cup}(X\alpha\backslash Y)$, we get $|X\alpha| = |X\alpha \cap Y| + |X\alpha\backslash Y| = |X\beta \cap Y| + |X\beta\backslash Y| = |(X\beta \cap Y)\dot{\cup}(X\beta\backslash Y)| = |X\beta|$.

Conversely, assume that the conditions hold. Since Y is a finite set, we have $Y\alpha, Y\beta, X\alpha \cap Y$ and $X\beta \cap Y$ are finite. Hence $|Y\alpha| + |(X\alpha \cap Y)\backslash Y\alpha| = |X\alpha \cap Y| = |X\beta \cap Y| = |Y\beta| + |(X\beta \cap Y)\backslash Y\beta|$ which implies that $|(X\alpha \cap Y)\backslash Y\alpha| = |(X\beta \cap Y)\backslash Y\beta|$ since $|Y\alpha| = |Y\beta|$ is finite. Since $|X\alpha \cap Y| = |X\beta \cap Y|$ is finite and $|X\alpha \cap Y| + |X\alpha\backslash Y| = |X\alpha| = |X\beta| = |X\beta \cap Y| + |X\beta\backslash Y|$, we have $|X\alpha\backslash Y| = |X\beta\backslash Y|$. Therefore, $\alpha\mathcal{D}\beta$ as required. □

Theorem 5. *Let $\alpha, \beta \in S(X,Y)$. Then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in S(X,Y)$ if and only if $|X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta|$ and $|X\alpha\backslash Y| \leq |X\beta\backslash Y|$. Consequently, $\alpha\mathcal{J}\beta$ if and only if $|X\alpha| = |X\beta|, |Y\alpha| = |Y\beta|$ and $|X\alpha\backslash Y| = |X\beta\backslash Y|$.*

Proof. Assume that $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in S(X,Y)$. Then

$$\begin{aligned}
|X\alpha| &= |X\lambda\beta\mu| = |(X\lambda\beta)\mu| \leq |X\lambda\beta| = |(X\lambda)\beta| \leq |X\beta|, \\
|Y\alpha| &= |Y\lambda\beta\mu| = |(Y\lambda\beta)\mu| \leq |Y\lambda\beta| = |(Y\lambda)\beta| \leq |Y\beta|, \text{ and} \\
|X\alpha\backslash Y| &= |X\lambda\beta\mu\backslash Y| = |(X\lambda)\beta\mu\backslash Y| \leq |X\beta\mu\backslash Y|, \\
&= |(X\beta)\mu\backslash Y|, \\
&= |[(X\beta\backslash Y) \cup (X\beta \cap Y)]\mu\backslash Y|, \\
&= |[(X\beta\backslash Y)\mu \cup (X\beta \cap Y)\mu]\backslash Y|, \\
&= |[(X\beta\backslash Y)\mu\backslash Y] \cup [(X\beta \cap Y)\mu\backslash Y]|, \\
&= |(X\beta\backslash Y)\mu\backslash Y| \leq |(X\beta\backslash Y)\mu| \leq |X\beta\backslash Y|.
\end{aligned}$$

Conversely, assume that the conditions hold and write

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix},$$

where $A_i \cap Y \neq \emptyset; B_j, C_k \subseteq X\backslash Y; \{a_i\} \subseteq Y, \{b_j\} \subseteq Y\backslash\{a_i\}$, $\{c_k\} \subseteq X\backslash Y$. By the assumption we can write

$$\beta = \begin{pmatrix} U_i & U_l & V_m & W_n & W_k \\ u_i & u_l & v_m & w_n & w_k \end{pmatrix},$$

where $U_i \cap Y \neq \emptyset \neq U_l \cap Y; V_m, W_n, W_k \subseteq X\backslash Y; \{u_i, u_l\} \subseteq Y, \{v_m\} \subseteq Y\backslash\{u_i, u_l\}$, $\{w_n, w_k\} \subseteq X\backslash Y$ and $|I| + |J| + |K| \leq |I| + |L| + |M| + |N| + |K|$. We consider in two cases:

Case 1 : $|J| \leq |L| + |M| + |N|$. Let $L \cup M \cup N = P\dot{\cup}Q$ where $|P| = |J|$. Then we can write $\{U_l\} \cup \{V_m\} \cup \{W_n\} = \{S_p\} \cup \{S_q\}$ and rewrite β as follows:

$$\beta = \begin{pmatrix} U_i & S_p & S_q & W_k \\ u_i & s_p & s_q & w_k \end{pmatrix}.$$

Since $|J| = |P|$, there is a bijection $\varphi : J \to P$. Now define

$$\lambda = \begin{pmatrix} A_i & B_j & C_k \\ x_i & y_{j\varphi} & z_k \end{pmatrix},$$

where $x_i \in U_i \cap Y, y_{j\varphi} \in S_{j\varphi}, z_k \in W_k$. So $\lambda \in S(X, Y)$. Choose $i_0 \in I$ and let $I' = I\backslash\{i_0\}$. Then define

$$\mu = \begin{pmatrix} u_{i'} & s_{j\varphi} & w_k & X\backslash\{u_{i'}, s_{j\varphi}, w_k\} \\ a_{i'} & b_j & c_k & a_{i_0} \end{pmatrix}.$$

So $\mu \in S(X, Y)$ and $\alpha = \lambda\beta\mu$.

Case 2 : $|J| > |L| + |M| + |N|$. Then $X\beta$ is infinite (for if $X\beta$ is finite, then $|X\alpha| = |I| + |J| + |K| > |I| + |L| + |M| + |N| + |K| = |X\beta|$ which is a contradiction). Hence $|J| \leq |I|$ or $|J| \leq |K|$ are infinite cardinals. If $|J| \leq |I|$ is an infinite cardinal, then write $I = P\dot{\cup}Q$ where $|P| = |I|, |Q| = |J|$. Thus we can write $\{U_i\} = \{S_p\} \cup \{S_q\}$ and rewrite β as follows:

$$\beta = \begin{pmatrix} S_p & S_q & U_l & V_m & W_n & W_k \\ s_p & s_q & u_l & v_m & w_n & w_k \end{pmatrix}.$$

Since $|I| = |P|$ and $|J| = |Q|$, there are bijections $\varphi : I \to P$ and $\psi : J \to Q$. Then define λ and μ as follows:

$$\lambda = \begin{pmatrix} A_i & B_j & C_k \\ x_{i\varphi} & y_{j\psi} & z_k \end{pmatrix},$$

where $x_{i\varphi} \in S_{i\varphi} \cap Y, y_{j\psi} \in S_{j\psi}, z_k \in W_k$,
and

$$\mu = \begin{pmatrix} s_{i'\varphi} & s_{j\psi} & w_k & X\backslash\{s_{i'\varphi}, s_{j\psi}, w_k\} \\ a_{i'} & b_j & c_k & a_{i_0} \end{pmatrix},$$

where $I' = I\backslash\{i_0\}$ for some fixed $i_0 \in I$. So, we see that $\lambda, \mu \in S(X, Y)$ and $\alpha = \lambda\beta\mu$. For the case $|J| \leq |K|$ is an infinite cardinal, we write $K = G\dot{\cup}H$

where $|G| = |J|, |H| = |K|$. Write $\{W_k\} = \{T_g\} \cup \{T_h\}$ and rewrite β as follows:

$$\beta = \begin{pmatrix} U_i & U_l & V_m & W_n & T_g & T_h \\ u_i & u_l & v_m & w_n & t_g & t_h \end{pmatrix}.$$

As above, we can define $\lambda, \mu \in S(X,Y)$ such that $\alpha = \lambda\beta\mu$. □

The following example shows that in general $\mathcal{D} \neq \mathcal{J}$ on $S(X,Y)$.

Example 2. Let $X = \mathbb{N}$ and Y the set of positive even integers. Then we define

$$\alpha = \begin{pmatrix} n \\ 2n \end{pmatrix}_{n\in\mathbb{N}} \text{ and } \beta = \begin{pmatrix} 2n & X\backslash Y \\ 4n & 2 \end{pmatrix}_{n\in\mathbb{N}}.$$

Hence $\alpha, \beta \in S(X,Y)$ and $|X\alpha| = \aleph_0 = |X\beta|, |Y\alpha| = \aleph_0 = |Y\beta|$, $|X\alpha\backslash Y| = 0 = |X\beta\backslash Y|$, so $\alpha\mathcal{J}\beta$. Since $|(X\alpha \cap Y)\backslash Y\alpha| = \aleph_0 \neq 1 = |(X\beta \cap Y)\backslash Y\beta|$, we have α and β are not $\mathcal{D}$-related on $S(X,Y)$.

Even Y is a finite proper subset of X, we still have $\mathcal{D} \neq \mathcal{J}$ on $S(X,Y)$.

Example 3. Let $X = \mathbb{N}$ and $Y = \{1,2,3,4\}$. Then we define

$$\alpha = \begin{pmatrix} \{1,2\} & \{3,4\} & n+4 \\ 3 & 1 & n+4 \end{pmatrix}_{n\in\mathbb{N}} \text{ and } \beta = \begin{pmatrix} \{1,2,4\} & 3 & \{5,6\} & n+6 \\ 1 & 2 & 3 & n+6 \end{pmatrix}_{n\in\mathbb{N}}.$$

Hence $\alpha, \beta \in S(X,Y)$ and $|X\alpha| = \aleph_0 = |X\beta|, |Y\alpha| = 2 = |Y\beta|$, $|X\alpha\backslash Y| = \aleph_0 = |X\beta\backslash Y|$, so $\alpha\mathcal{J}\beta$. Since $|X\alpha \cap Y| = 2$ but $|X\beta \cap Y| = 3$, we have α and β are not $\mathcal{D}$-related on $S(X,Y)$ by Corollary 3.

Theorem 6. *$\mathcal{D} = \mathcal{J}$ on $S(X,Y)$ if and only if X is a finite set or $X = Y$ or $|Y| = 1$.*

Proof. If X is a finite set, then by [3], Proposition 2.1.4 we have $\mathcal{D} = \mathcal{J}$. If $X = Y$, then $S(X,Y) = T(X)$ and thus $\mathcal{D} = \mathcal{J}$ by [1], Theorem 2.9(i). If $|Y| = 1$, then $S(X,Y) = E$. Let $\alpha, \beta \in S(X,Y)$ be such that $\alpha\mathcal{J}\beta$. So, $|X\alpha| = |X\beta|, |Y\alpha| = |Y\beta|$ and $|X\alpha\backslash Y| = |X\beta\backslash Y|$. Since $\alpha, \beta \in E$, we have $|(X\alpha \cap Y)\backslash Y\alpha| = |Y\alpha\backslash Y\alpha| = 0 = |Y\beta\backslash Y\beta| = |(X\beta \cap Y)\backslash Y\beta|$ and hence $\alpha\mathcal{D}\beta$. Thus $\mathcal{D} = \mathcal{J}$.

Conversely, assume that $\mathcal{D} = \mathcal{J}$ on $S(X,Y)$, and suppose on contrary that X is an infinite set, $Y \subsetneq X$ and $|Y| \geq 2$. Let a, b be two distinct elements in Y and $c \in X\backslash Y$. We consider in two cases:

Case 1 : Y is a finite set. Then $|X\backslash Y| = |X|$ and define $\alpha, \beta \in S(X,Y)$ as follows:

$$\alpha = \begin{pmatrix} Y & c & x \\ a & b & x \end{pmatrix}_{x\in X\backslash(Y\cup\{c\})} \text{ and } \beta = \begin{pmatrix} Y & x \\ a & x \end{pmatrix}_{x\in X\backslash Y}.$$

Thus $|X\alpha| = |X\backslash Y| = |X\beta|, |Y\alpha| = 1 = |Y\beta|$ and $|X\alpha\backslash Y| = |X\backslash Y| = |X\beta\backslash Y|$, we get $\alpha\mathcal{J}\beta$. But, $|X\alpha \cap Y| = 2 \neq 1 = |X\beta \cap Y|$, so α and β are not $\mathcal{D}$-related by Corollary 3 and this leads to a contradiction.

Case 2 : Y is an infinite set. Then define $\alpha, \beta \in S(X,Y)$ as follows:

$$\alpha = \begin{pmatrix} x & \{a,b\} & X\backslash Y \\ x & a & b \end{pmatrix}_{x \in Y\backslash\{a,b\}} \text{ and } \beta = \begin{pmatrix} x & (X\backslash Y)\cup\{a\} \\ x & a \end{pmatrix}_{x\in Y\backslash\{a\}}.$$

Since $|X\alpha| = |Y| = |X\beta|, |Y\alpha| = |Y| = |Y\beta|$ and $|X\alpha\backslash Y| = 0 = |X\beta\backslash Y|$, we get $\alpha\mathcal{J}\beta$. But, $|(X\alpha \cap Y)\backslash Y\alpha| = 1 \neq 0 = |(X\beta \cap Y)\backslash Y\beta|$, so α and β are not $\mathcal{D}$-related which is a contradiction. □

As a direct consequence of Theorem 6, we have the following corollary.

Corollary 4. *If X is an infinite set and Y is a finite subset of X, then $\mathcal{D} = \mathcal{J}$ on $S(X,Y)$ if and only if $|Y| = 1$.*

4. Ideals of $S(X,Y)$

Let p be any cardinal number and let

$$p' = \min\{q : q > p\}.$$

Note that p' always exists since the cardinals are well-ordered and when p is finite we have $p' = p+1 =$ the successor of p. As shown by Malcev [5], the ideals of $T(X)$ for any set X are precisely the sets:

$$T_r = \{\alpha \in T(X) : |X\alpha| < r\},$$

where $2 \leq r \leq |X|'$ (see also [1] vol 2, Theorem 10.59).

To describe ideals of $S(X,Y)$ for any set X and any nonempty subset Y of X, we let $|X| = a, |Y| = b$ and $|X\backslash Y| = c$. In addition, for each cardinals r,s,t such that $2 \leq r \leq a', 2 \leq s \leq b'$ and $1 \leq t \leq c'$, define

$$S(r,s,t) = \{\alpha \in S(X,Y) : |X\alpha| < r, |Y\alpha| < s \text{ and } |X\alpha\backslash Y| < t\}.$$

Then if $r = a'$ or $s = b'$ or $t = c'$, the set $S(r,s,t)$ can be deduced in a simple form:

$$\begin{aligned}
S(a',s,t) &= \{\alpha \in S(X,Y) : |Y\alpha| < s \text{ and } |X\alpha\backslash Y| < t\},\\
S(r,b',t) &= \{\alpha \in S(X,Y) : |X\alpha| < r \text{ and } |X\alpha\backslash Y| < t\},\\
S(r,s,c') &= \{\alpha \in S(X,Y) : |X\alpha| < r \text{ and } |Y\alpha| < s\},\\
S(a',b',t) &= \{\alpha \in S(X,Y) : |X\alpha\backslash Y| < t\},\\
S(r,b',c') &= \{\alpha \in S(X,Y) : |X\alpha| < r\},\\
S(a',s,c') &= \{\alpha \in S(X,Y) : |Y\alpha| < s\},
\end{aligned}$$

and $\quad S(a',b',c') = S(X,Y).$

We observe that: if $X = Y$, then $|X| = a = |Y|$ and $|X\backslash Y| = 0$, thus $S(r,r,1) = \{\alpha \in S(X,Y) : |X\alpha| < r\} = \{\alpha \in T(X) : |X\alpha| < r\}$ which is an ideal of $T(X)$.

Theorem 7. *The set $S(r,s,t)$ is an ideal of $S(X,Y)$.*

Proof. Let $\alpha \in S(r,s,t)$ and $\lambda, \mu \in S(X,Y)$. Then $|X\alpha| < r, |Y\alpha| < s$ and $|X\alpha \backslash Y| < t$. Thus by using the same proof as given in Theorem 5, we get $|X(\lambda\alpha\mu)| \leq |X\alpha| < r$, $|Y(\lambda\alpha\mu)| \leq |Y\alpha| < s$ and $|X(\lambda\alpha\mu)\backslash Y| \leq |X\alpha\backslash Y| < t$. Hence $\lambda\alpha\mu \in S(r,s,t)$. Therefore, $S(r,s,t)$ is an ideal of $S(X,Y)$. □

We note that if $r \leq u, s \leq v$ and $t \leq w$, then $S(r,s,t) \subseteq S(u,v,w)$. The following example shows that there is an ideal in $S(X,Y)$ which is not of the form $S(r,s,t)$ and the set of ideals of $S(X,Y)$ does not form a chain under the set inclusion.

Example 4. Let $X = \{1,2,3,4\}$ and $Y = \{1,2\}$. Then $|X| = 4, |Y| = 2$ and $|X\backslash Y| = 2$. Since $S(3,3,1)$ and $S(4,2,2)$ are ideals of $S(X,Y)$, we have $S(3,3,1) \cup S(4,2,2)$ is also an ideal of $S(X,Y)$. Suppose that $S(3,3,1) \cup S(4,2,2) = S(\ell,m,n)$ for some $2 \leq \ell \leq 5, 2 \leq m \leq 3$ and $1 \leq n \leq 3$. If $\ell < 4$ or $n < 2$, then there is $\alpha = \begin{pmatrix} \{1,2\} & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix} \in S(4,2,2)\backslash S(\ell,m,n)$, and if $m < 3$, then there is $\beta = \begin{pmatrix} 1 & \{2,3,4\} \\ 1 & 2 \end{pmatrix} \in S(3,3,1)\backslash S(\ell,m,n)$. Both cases contradict our supposition. So $\ell \geq 4, m \geq 3$ and $n \geq 2$. Consider $\delta = \begin{pmatrix} 1 & 2 & \{3,4\} \\ 1 & 2 & 3 \end{pmatrix} \in S(4,3,2)$, but $\delta \notin S(3,3,1) \cup S(4,2,2)$, so $S(3,3,1) \cup S(4,2,2) \neq S(r,s,t)$ for all $r \geq 4$, $s \geq 3$ and $t \geq 2$. Since $\alpha \in S(4,2,2)\backslash S(3,3,1)$ and $\beta \in S(3,3,1)\backslash S(4,2,2)$, we conclude that the set of ideals of $S(X,Y)$ does not form a chain.

To obtain ideals of $S(X,Y)$ we need the following notation. Let Z be a nonempty subset of $S(X,Y)$. Define

$$K(Z) = \{\alpha \in S(X,Y) : |X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta| \text{ and } |X\alpha\backslash Y| \leq |X\beta\backslash Y| \text{ for some } \beta \in Z\}.$$

Then we see that $Z \subseteq K(Z)$ and $Z_1 \subseteq Z_2$ implies $K(Z_1) \subseteq K(Z_2)$.

Theorem 8. *The ideals of $S(X,Y)$ are precisely the set $K(Z)$ for some nonempty subset Z of $S(X,Y)$.*

Proof. Let I be an ideal of $S(X,Y)$. We prove that $I = K(I)$. If $\alpha \in K(I)$, then $|X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta|$ and $|X\alpha\backslash Y| \leq |X\beta\backslash Y|$ for some $\beta \in I$ and thus by Theorem 5 we have $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in S(X,Y)$. Since $\beta \in I$ is an ideal of $S(X,Y)$, it follows that $\alpha = \lambda\beta\mu \in I$, and that $K(I) \subseteq I$. Usually, we have $I \subseteq K(I)$. Therefore, $I = K(I)$.

Conversely, we prove that $K(Z)$ is an ideal of $S(X,Y)$. Let $\alpha \in K(Z)$ and $\lambda, \mu \in S(X,Y)$. Then $|X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta|$ and $|X\alpha\backslash Y| \leq |X\beta\backslash Y|$ for some $\beta \in Z$. Like before, we have $|X(\lambda\alpha\mu)| \leq |X\alpha|$, $|Y(\lambda\alpha\mu)| \leq |Y\alpha|$ and $|X(\lambda\alpha\mu)\backslash Y| \leq |X\alpha\backslash Y|$. Thus $|X(\lambda\alpha\mu)| \leq |X\beta|$, $|Y(\lambda\alpha\mu)| \leq |Y\beta|$ and $|X(\lambda\alpha\mu)\backslash Y| \leq |X\beta\backslash Y|$. Hence $\lambda\alpha\mu \in K(Z)$ and therefore $K(Z)$ is an ideal of $S(X,Y)$. □

The following result was first proved by Malcev [5] in 1952.

Corollary 5. *The ideals of* $T(X)$ *are precisely the set* $T(r) = \{\alpha \in T(X) : |X\alpha| < r\}$*, where* $2 \le r \le |X|'$*.*

Proof. By taking $Y = X$ in Theorem 8, we see that the ideals of $S(X, X) = T(X)$ are precisely the set $K(Z)$ for some nonempty subset Z of $T(X)$. Let r be the least cardinal of $A = \{s : s > |X\beta| \text{ for all } \beta \in Z\}$ (A is nonempty since $|X|' \in A$). Then for each $\alpha \in K(Z)$, there is $\beta \in Z$ such that $|X\alpha| \le |X\beta| < r$ and thus $K(Z) \subseteq T(r)$. Conversely, suppose that $\alpha \notin K(Z)$, then $|X\alpha| > |X\beta|$ for all $\beta \in Z$. Thus $|X\alpha| \in A$ and hence $|X\alpha| \ge r$ since r is the least cardinal of A, that means $\alpha \notin T(r)$. So, $T(r) \subseteq K(Z)$ and therefore $K(Z) = T(r)$. □

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