

NEW CONDITIONS ON EXISTENCE AND GLOBAL ASYMPTOTIC STABILITY OF PERIODIC SOLUTIONS FOR BAM NEURAL NETWORKS WITH TIME-VARYING DELAYS

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ABSTRACT. In this paper, the problem on periodic solutions of the bidirectional associative memory neural networks with both periodic coefficients and periodic time-varying delays is discussed. By using degree theory, inequality technique and Lyapunov functional, we establish the existence, uniqueness, and global asymptotic stability of a periodic solution. The obtained results of stability are less restrictive than previously known criteria, and the hypotheses for the boundedness and monotonicity on the activation functions are removed.

1. Introduction

A class of two-layer interassociative network called bidirectional associative memory (BAM) neural network is an important model with the ability of information association, which is crucial for application in pattern recognition, solving optimization problems and automatic control engineering [16, 17]. In such applications, the dynamical characteristics of networks play an important role. As is well known, in both biological and man-made neural networks, time delays occur due to finite switching speed of the amplifiers and communication time. The delays are usually time-varying, and sometimes vary violently with time. They slow down the transmission rate and can influence the stability of designed neural networks by creating oscillatory or unstable phenomena [3]. So it is more in accordance with this fact to study the BAM neural networks with time-varying delays. The circuits diagram and connection pattern implementing for the delayed BAM neural networks can be found in [8]. In recent years, some useful results on the uniqueness and global stability of the equilibrium point for the delayed recurrent neural networks and delayed BAM neural

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networks have been given, for example, see [1, 2, 3], [8], [13], [15], [17, 18, 19] and references therein. It is well known that studies on neural dynamical system not only involve a discussion of stability properties, but also involve many dynamic behavior such as periodic oscillatory behavior, synchronization, dissipativity, bifurcation, and chaos [5, 8, 9]. In [9], the authors studied the global point dissipativity of neural networks with mixed time-varying delays. In [5], based on the invariant principle of functional differential equations, a simple adaptive feedback scheme is proposed for the synchronization of almost all kinds of coupled identical neural networks with or without time-varying delays. In many applications, the properties of periodic oscillatory solutions are of great interest, it has been found applications in learning theory, which is motivated by the fact that learning usually requires repetition. Hence, it is of prime importance to study periodic oscillatory solutions of neural networks. In addition, an equilibrium point can be viewed as a special periodic solution of neural networks with arbitrary periodic. In this sense, the analysis of periodic solutions of neural networks may be considered to be more general than that of equilibrium point. Recently, periodic solution for BAM neural networks with delays has been studied, for example, see [4, 6, 7, 8, 10, 11, 12, 20, 21] and references therein. In [5, 7, 8, 10, 12], some sufficient conditions ensuring the existence, uniqueness and global exponential stability of periodic solution were given for BAM neural networks with constants delays. In [4, 20, 21], under the hypothesis for the boundedness or monotonicity on the activation functions and the differentiability on the time-varying delays, the authors gave several sufficient conditions ensuring the existence and global exponential stability of periodic solution for BAM neural networks with time-varying delays. However, in some applications, one requires to use unbounded activation functions. For example, when neural networks are designed for solving optimization problems in the presence of constrains (linear, quadratic, or more general programming problems), unbounded activations modeled by diode-like exponential-type functions are needed to impose constraints satisfaction. The extension of the quoted results to the unbounded case is not straightforward. When considering the widely employed piecewise-linear neural networks, infinite intervals with zero slope are present in activations, it is of great interest to drop the assumptions of monotonicity. Therefore, it seems that for some purposes, nonmonotonic functions might be better candidates for neural activation in designing and implementing an artificial neural networks. Motivated by the above discussion, in [22], the authors were concerned with the following model:

$$(1.1) \quad \frac{du_i(t)}{dt} = -a_i(t)u_i(t) + \sum_{j=1}^m h_{ij}(t)f_j(\lambda_j v_j(t - \tau_{ij}(t))) + I_i(t), \quad i = 1, 2, \dots, n$$

$$\begin{aligned} \frac{dv_j(t)}{dt} &= -b_j(t)v_j(t) \\ &+ \sum_{i=1}^n w_{ji}(t)g_i(\mu_i u_i(t - \sigma_{ji}(t))) + J_j(t), \quad j = 1, 2, \dots, m, \end{aligned}$$

for $t > 0$, with initial value conditions

$$(1.2) \quad \begin{cases} u_i(s) = \phi_i(s), \quad i = 1, 2, \dots, n, \quad s \in [-\sigma, 0], \quad \sigma = \max_{1 \leq i \leq n, 1 \leq j \leq m, 0 \leq t \leq \omega} \{\sigma_{ji}(t)\} \\ v_j(s) = \psi_j(s), \quad j = 1, 2, \dots, m, \quad s \in [-\tau, 0], \quad \tau = \max_{1 \leq i \leq n, 1 \leq j \leq m, 0 \leq t \leq \omega} \{\tau_{ij}(t)\}. \end{cases}$$

in which, $u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T \in \mathbb{R}^n$, $v(t) = (v_1(t), v_2(t), \dots, v_m(t))^T \in \mathbb{R}^m$, $u_i(t)$ and $v_j(t)$ are the state of the i th neurons from the neural field F_U and the j th neurons from the neural field F_V at time t , respectively; f_j, g_i denote the activation functions of the j th neurons from F_V and the i th neurons from F_U at time t , respectively; $I_i(t)$ and $J_j(t)$ denote the external inputs on the i th neurons from F_U and the j th neurons from F_V , respectively; $\tau_{ij}(t)$ and $\sigma_{ji}(t)$ correspond to the transmission delays and $\tau_{ij}(t) > 0$ and $\sigma_{ji}(t) > 0$; $a_i(t) > 0$ and $b_j(t) > 0$ represent the rate with which the i th neuron from F_U and the j th neurons from F_V will reset their potential to the resting state in isolation when disconnected from the networks and external inputs, respectively; $h_{ij}(t)$ and $w_{ji}(t)$ denote the connection strengths; $\lambda_j > 0$ and $\mu_i > 0$ are constants, correspond to the neural gains associated with the neural activations [3, 13].

In [22], the authors gave some novel sufficient conditions ensuring the existence, uniqueness, and global exponential stability of periodic solution for the system (1.1) with both periodic coefficients and periodic time-varying delays by using analytic methods, inequality technique and M -matrix theory.

However, to the best of our knowledge, the results on asymptotic stability of periodic solution of the system (1.1) are scarce. So, in this paper, the objective of us is to establish new sufficient conditions ensuring the existence, uniqueness, and global asymptotic stability of periodic solution for the system (1.1) with both periodic coefficients and periodic time-varying delays by using coincidence degree theory, inequality technique and Lyapunov functional. Our results on the existence, uniqueness, and global stability of periodic solution for the system (1.1) with both periodic coefficients and periodic time-varying delays not only remove the hypotheses as required in [5, 20, 21] for boundedness and monotonicity on the activation functions and remove the restriction that the value of activation functions at zero point is zero in [12, 10], but also are more concise than those obtained in [4, 6, 7, 8, 10, 12, 20, 21, 22] and more easy to verified. In our results of stability of periodic solution, we only need the activation functions satisfy Lipschitz condition and the parameters satisfy two very simple inequalities, while in [22], the matrix of parameters is required to be a complicated M matrix, while in [12], one hand, the matrix of parameters is required to be a complicated M matrix, on the other hand, the parameters are

needed to satisfy four complicated inequalities, while in [10], the parameters are needed to satisfy four very complicated inequalities.

The paper is organized as follows. In Section 2, we establish the existence and uniqueness of a periodic solution for the system (1.1) by using coincidence degree theory. In Section 3, we establish the global asymptotic stability of a periodic solution for the system (1.1) by constructing a proper Lyapunov functional. Finally, an example is given to demonstrate the effectiveness of our results on global asymptotic stability.

2. Existence and uniqueness of a periodic solution

In this section, we first establish the existence of a periodic solution by applying coincidence degree theory. To establish the existence of at least a periodic solution by applying coincidence degree theory, we recall some basic tools in the frame work of Mawhin's coincidence degree [14] that will be used to investigate the existence of periodic solutions.

Let X, Z be Banach spaces, $L: \text{Dom } L \subset X \rightarrow Z$ be a linear mapping and $N: X \rightarrow Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \text{Ker } L = \text{codim } \text{Im } L < \infty$ and $\text{Im } L$ is closed in Z . If L is a Fredholm mapping of index zero, then there exist continuous projectors $P: X \rightarrow X$ and $Q: Z \rightarrow Z$ such that $\text{Im } P = \text{Ker } L$ and $\text{Im } L = \text{Ker } Q = \text{Im}(I - Q)$. It follows that $L|_{\text{Dom } L \cap \text{Ker } P}: (I - P)X \rightarrow \text{Im } L$ is invertible. We denote the inverse of the map $L|_{\text{Dom } L \cap \text{Ker } P}$ by K_p . If Ω is an open bounded subset of X , the mapping N will be called L -compact on $\bar{\Omega}$ if $(QN)(\bar{\Omega})$ is bounded and $K_p(I - Q)N: \bar{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J: \text{Im } Q \rightarrow \text{Ker } L$.

In the proof of our existence theorem, we will use the continuation theorem of Gaines and Mawhin ([14]).

Lemma 2.1 (Continuation Theorem). *Let L be a Fredholm mapping of index zero and let N be L -compact on $\bar{\Omega}$. Suppose*

- (a) $Lx \neq \lambda N(x), \forall \lambda \in (0, 1), x \in \partial\Omega$;
- (b) $QN(x) \neq 0, \forall x \in \text{Ker } L \cap \partial\Omega$;
- (c) $\deg(JQN, \Omega \cap \text{Ker } L, 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\text{Dom } L \cap \bar{\Omega}$.

For the sake of convenience, we introduce some notations as follows: $|\cdot|$ denotes the norm in \mathbb{R} , $\bar{f} = \max_{0 \leq t \leq \omega} |f(t)|$, $\underline{f} = \min_{0 \leq t \leq \omega} |f(t)|$, where $f(t)$ is a continuously periodic function with common period ω .

Our main result on the existence and uniqueness of a periodic solution for the system (1.1) is stated in the following theorem.

Theorem 2.1. *We assume that the following conditions hold:*

- (i) $a_i(t), b_j(t), h_{ij}(t), w_{ji}(t), \tau_{ij}(t), \sigma_{ji}(t), I_i(t), J_j(t)$ are continuously periodic functions on $t \in [0, +\infty)$ with common period $\omega > 0, i = 1, 2, \dots, n; j = 1, 2, \dots, m$;

(ii) There exist positive constants A_j ($j = 1, 2, \dots, m$) such that for $\forall x, y \in \mathbb{R}$,

$$|f_j(x) - f_j(y)| \leq A_j|x - y|;$$

(iii) There exist positive constants B_i ($i = 1, 2, \dots, n$) such that for $\forall x, y \in \mathbb{R}$,

$$|g_i(x) - g_i(y)| \leq B_i|x - y|;$$

(iv) There exist two positive constants $l_i > 1$ ($i = 1, 2$) with

$$a_i > \sum_{j=1}^m \overline{h_{ij}} A_j \lambda_j l_1 \text{ and } b_j > \sum_{i=1}^n \overline{w_{ji}} B_i \mu_i l_2$$

such that $0 < \frac{1}{1-\tau'_{ij}} < l_1$ and $0 < \frac{1}{1-\sigma'_{ji}} < l_2$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.

Then the system (1.1) has one unique ω -periodic solution.

Proof. Part 1. The proof of existence of a periodic solution. In order to apply Lemma 2.1 to the system (1.1). Let

$$X = \{u = (u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m)^T \in C(\mathbb{R}, \mathbb{R}^{m+n}) : u(t + \omega) = u(t)\}$$

and

$$Z = \{z \in C(\mathbb{R}, \mathbb{R}^{m+n}) : z(t + \omega) = z(t)\}.$$

Define

$$\|u\| = \max_{t \in [0, \omega]} \sum_{i=1}^n |u_i(t)| + \max_{t \in [0, \omega]} \sum_{j=1}^m |v_j(t)|, \quad u \in X \quad \text{or} \quad Z.$$

Equipped with the above norm $\|\cdot\|$, X and Z are Banach spaces.

Let for $u \in X$,

$$Nu = \begin{pmatrix} H_i(t) \\ K_j(t) \end{pmatrix} = \begin{pmatrix} -a_i(t)u_i(t) + \sum_{j=1}^m h_{ij}(t)f_j(\lambda_j v_j(t - \tau_{ij}(t)) + I_i(t)] \\ -b_j(t)v_j(t) + \sum_{i=1}^n w_{ji}(t)g_i(\mu_i u_i(t - \sigma_{ji}(t)) + J_j(t)] \end{pmatrix},$$

$$Lu = u' = \frac{du(t)}{dt}, \quad Pu = \frac{1}{\omega} \int_0^\omega u(t)dt, \quad u \in X, \quad Qz = \frac{1}{\omega} \int_0^\omega z(t)dt, \quad z \in Z.$$

Then it follows that $\text{Ker}L = \mathbb{R}^{m+n}$, $\text{Im}L = \{z \in Z : \int_0^\omega z(t)dt = 0\}$ is closed in Z , $\dim \text{Ker}L = m + n = \text{codim} \text{Im}L$ and P, Q are continuous projectors such that

$$\text{Im}P = \text{Ker}L, \quad \text{Ker}Q = \text{Im}L = \text{Im}(I - Q).$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \text{Im}L \rightarrow \text{Ker}P \cap \text{Dom}L$ is given by

$$K_p(z) = \int_0^t z(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^s z(t)dt ds.$$

Then

$$QN u = \begin{pmatrix} \frac{1}{\omega} \int_0^\omega H_1(s) ds \\ \frac{1}{\omega} \int_0^\omega H_2(s) ds \\ \vdots \\ \frac{1}{\omega} \int_0^\omega H_n(s) ds \\ \frac{1}{\omega} \int_0^\omega K_1(s) ds \\ \frac{1}{\omega} \int_0^\omega K_2(s) ds \\ \vdots \\ \frac{1}{\omega} \int_0^\omega K_m(s) ds \end{pmatrix}$$

and

$$K_p(I-Q)Nu = \begin{pmatrix} f_0^t H_1(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t H_1(s) ds dt + (\frac{1}{2} - \frac{t}{\omega}) \int_0^\omega H_1(s) ds \\ f_0^t H_2(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t H_2(s) ds dt + (\frac{1}{2} - \frac{t}{\omega}) \int_0^\omega H_2(s) ds \\ \vdots \\ f_0^t H_n(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t H_n(s) ds dt + (\frac{1}{2} - \frac{t}{\omega}) \int_0^\omega H_n(s) ds \\ f_0^t K_1(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t K_1(s) ds dt + (\frac{1}{2} - \frac{t}{\omega}) \int_0^\omega K_1(s) ds \\ \vdots \\ f_0^t K_m(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t K_m(s) ds dt + (\frac{1}{2} - \frac{t}{\omega}) \int_0^\omega K_m(s) ds \end{pmatrix}.$$

Obviously, QN and $K_p(I-Q)N$ are continuous. It is not difficult to show that $K_p(I-Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$ by using the Arzela-Ascoli theorem. Moreover, $QN(\bar{\Omega})$ is clearly bounded. Thus, N is L -compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

The conditions (i) and (ii) in Theorem 2.1 imply that $\forall x \in \mathbb{R}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$,

$$(2.1) \quad |f_j(x)| \leq A_j |x| + |f_j(0)|$$

and

$$(2.2) \quad |g_i(x)| \leq B_i |x| + |g_i(0)|.$$

Corresponding the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have for $i = 1, 2, \dots, n$; $j = 1, \dots, m$,

$$(2.3) \quad \begin{cases} \frac{du_i(t)}{dt} = \lambda H_i(t), \\ \frac{dv_j(t)}{dt} = \lambda K_j(t). \end{cases}$$

Assume that $u \in X$ is a solution of the system (2.3) for some $\lambda \in (0, 1)$. Multiplying the first equation of the system (2.3) by $u_i(t)$ and integrating over $[0, \omega]$, we have

$$(2.4) \quad \int_0^\omega u_i(t) \left\{ -a_i(t)u_i(t) + \sum_{j=1}^m h_{ij}(t)f_j(\lambda_j v_j(t - \tau_{ij}(t))) + I_i(t) \right\} dt = 0.$$

Multiplying the second equation of the system (2.3) by $v_j(t)$ and integrating over $[0, \omega]$, we have

$$(2.5) \quad \int_0^\omega v_j(t) \left\{ -b_j(t)v_j(t) + \sum_{i=1}^n w_{ji}(t)g_i(\mu_i u_i(t - \sigma_{ji}(t))) + J_j(t) \right\} dt = 0.$$

By conditions (2.1) and (2.2), from (2.4) and (2.5), we obtain

$$(2.6) \quad \underline{a}_i \int_0^\omega |u_i(t)|^2 dt \leq \int_0^\omega |u_i(t)| \left\{ \sum_{j=1}^m \overline{h}_{ij} \left(A_j \lambda_j |v_j(t - \tau_{ij}(t))| + |f_j(0)| \right) + \overline{I}_i \right\} dt$$

and

$$(2.7) \quad \underline{b}_j \int_0^\omega |v_j(t)|^2 dt \leq \int_0^\omega |v_j(t)| \left\{ \sum_{i=1}^n \overline{w}_{ji} \left(B_i \mu_i |u_i(t - \sigma_{ji}(t))| + |g_i(0)| \right) + \overline{J}_j \right\} dt.$$

Hence

$$\underline{a}_i \int_0^\omega |u_i(t)|^2 dt \leq \left(\int_0^\omega |u_i(t)|^2 dt \right)^{\frac{1}{2}} \left\{ \sum_{j=1}^m \overline{h}_{ij} \left[A_j \lambda_j \left(\int_0^\omega |v_j(t - \tau_{ij}(t))|^2 dt \right)^{\frac{1}{2}} + \sqrt{\omega} |f_j(0)| \right] + \sqrt{\omega} \overline{I}_i \right\}$$

and

$$\underline{b}_j \int_0^\omega |v_j(t)|^2 dt \leq \left(\int_0^\omega |v_j(t)|^2 dt \right)^{\frac{1}{2}} \left\{ \sum_{i=1}^n \overline{w}_{ji} \left[B_i \mu_i \left(\int_0^\omega |u_i(t - \sigma_{ji}(t))|^2 dt \right)^{\frac{1}{2}} + \sqrt{\omega} |g_i(0)| \right] + \sqrt{\omega} \overline{J}_j \right\}.$$

That is

$$(2.8) \quad \underline{a}_i \left(\int_0^\omega |u_i(t)|^2 dt \right)^{\frac{1}{2}} \leq \sum_{j=1}^m \overline{h}_{ij} \left[A_j \lambda_j \left(\int_0^\omega |v_j(t - \tau_{ij}(t))|^2 dt \right)^{\frac{1}{2}} + \sqrt{\omega} |f_j(0)| \right] + \sqrt{\omega} \overline{I}_i$$

and

$$(2.9) \quad \underline{b}_j \left(\int_0^\omega |v_j(t)|^2 dt \right)^{\frac{1}{2}} \leq \sum_{i=1}^n \overline{w}_{ji} \left[B_i \mu_i \left(\int_0^\omega |u_i(t - \sigma_{ji}(t))|^2 dt \right)^{\frac{1}{2}} + \sqrt{\omega} |g_i(0)| \right] + \sqrt{\omega} \overline{J}_j.$$

Denoting $s = t - \tau_{ij}(t) = g(t)$, $\sigma = t - \sigma_{ji}(t) = h(t)$, then

$$(2.10) \quad \int_0^\omega |v_j(t - \tau_{ij}(t))| dt = \int_0^\omega \frac{|v_j(s)|}{1 - \tau'_{ij}(g^{-1}(s))} ds,$$

$$(2.11) \quad \int_0^\omega |u_i(t - \sigma_{ji}(t))| dt = \int_0^\omega \frac{|u_i(\sigma)|}{1 - \sigma'_{ji}(h^{-1}(\sigma))} d\sigma,$$

$$(2.12) \quad \left(\int_0^\omega |v_j(t - \tau_{ij}(t))|^2 dt \right)^{\frac{1}{2}} = \left(\int_0^\omega \frac{|v_j(s)|^2}{1 - \tau'_{ij}(g^{-1}(s))} ds \right)^{\frac{1}{2}},$$

and

$$(2.13) \quad \left(\int_0^\omega |u_i(t - \sigma_{ji}(t))|^2 dt \right)^{\frac{1}{2}} = \left(\int_0^\omega \frac{|u_i(\sigma)|^2}{1 - \sigma'_{ji}(h^{-1}(\sigma))} d\sigma \right)^{\frac{1}{2}}.$$

Applying condition (iv), substituting (2.13) into (2.9) and substituting (2.12) into (2.8) give for $i = 1, \dots, n; j = 1, \dots, m$,

$$(2.14) \quad \underline{a}_i \left(\int_0^\omega |u_i(t)|^2 dt \right)^{\frac{1}{2}} \leq \sum_{j=1}^m \overline{h}_{ij} A_j \lambda_j \sqrt{l_1} \left(\int_0^\omega |v_j(t)|^2 dt \right)^{\frac{1}{2}} + \sqrt{\omega} \left(\sum_{j=1}^m \overline{h}_{ij} |f_j(0)| + \overline{I}_i \right)$$

and

$$(2.15) \quad \underline{b}_j \left(\int_0^\omega |y_j(t)|^2 dt \right)^{\frac{1}{2}} \leq \sum_{i=1}^n \overline{w}_{ji} B_i \mu_i \sqrt{l_2} \left(\int_0^\omega |u_i(t)|^2 dt \right)^{\frac{1}{2}} + \sqrt{\omega} \left(\sum_{i=1}^n \overline{w}_{ji} |g_i(0)| + \overline{J}_j \right).$$

Denoting for the sake of convenience

$$\max_{1 \leq i \leq n} \left\{ \left(\int_0^\omega |u_i(t)|^2 dt \right)^{\frac{1}{2}} \right\} = \left(\int_0^\omega |u_i(t)|^2 dt \right)^{\frac{1}{2}}$$

and

$$\max_{1 \leq j \leq m} \left\{ \left(\int_0^\omega |v_j(t)|^2 dt \right)^{\frac{1}{2}} \right\} = \left(\int_0^\omega |v_j(t)|^2 dt \right)^{\frac{1}{2}}.$$

For (2.14) and (2.15), we consider two possible cases:

$$(i) \quad \left(\int_0^\omega |v_j(t)|^2 dt \right)^{\frac{1}{2}} \leq \left(\int_0^\omega |u_i(t)|^2 dt \right)^{\frac{1}{2}};$$

$$(ii) \quad \left(\int_0^\omega |v_j(t)|^2 dt \right)^{\frac{1}{2}} > \left(\int_0^\omega |u_i(t)|^2 dt \right)^{\frac{1}{2}}.$$

(i) When $\left(\int_0^\omega |v_j(t)|^2 dt \right)^{\frac{1}{2}} \leq \left(\int_0^\omega |u_i(t)|^2 dt \right)^{\frac{1}{2}}$, from (2.14), we have

$$\left(\underline{a}_i - \sum_{j=1}^m \overline{h}_{ij} A_j \lambda_j \sqrt{l_1} \right) \left(\int_0^\omega |u_i(t)|^2 dt \right)^{\frac{1}{2}} \leq \sqrt{\omega} \left(\sum_{j=1}^m \overline{h}_{i0j} |f_j(0)| + \overline{I}_{i0} \right),$$

from which and condition (iv), it follows that

$$\left(\int_0^\omega |u_i(t)|^2 dt\right)^{\frac{1}{2}} \leq \max_{1 \leq i \leq n} \left\{ \frac{\sqrt{\omega} \left(\sum_{j=1}^m \overline{h_{ij}} |f_j(0)| + \overline{I_i}\right)}{\underline{a_i} - \sum_{j=1}^m \overline{h_{ij}} A_j \lambda_j \sqrt{l_1}} \right\} \stackrel{\text{def}}{=} d_1.$$

Therefore

$$(2.16) \quad \left(\int_0^\omega |v_j(t)|^2 dt\right)^{\frac{1}{2}} \leq \left(\int_0^\omega |u_i(t)|^2 dt\right)^{\frac{1}{2}} \leq d_1.$$

(ii) When $\left(\int_0^\omega |v_j(t)|^2 dt\right)^{\frac{1}{2}} > \left(\int_0^\omega |u_i(t)|^2 dt\right)^{\frac{1}{2}}$, from (2.15), we have

$$\left(\int_0^\omega |v_j(t)|^2 dt\right)^{\frac{1}{2}} \leq \max_{1 \leq j \leq m} \left\{ \frac{\sqrt{\omega} \left(\sum_{i=1}^n \overline{w_{ji}} |g_i(0)| + \overline{J_j}\right)}{\underline{b_j} - \sum_{i=1}^n \overline{w_{ji}} B_i \mu_i \sqrt{l_2}} \right\} \stackrel{\text{def}}{=} d_2.$$

Therefore

$$(2.17) \quad \left(\int_0^\omega |u_i(t)|^2 dt\right)^{\frac{1}{2}} \leq \left(\int_0^\omega |v_j(t)|^2 dt\right)^{\frac{1}{2}} \leq d_2.$$

Then from (2.16) and (2.17), we obtain for $i = 1, 2, \dots, n, j = 1, 2, \dots, m$,

$$(2.18) \quad \left(\int_0^\omega |u_i(t)|^2 dt\right)^{\frac{1}{2}} < \max\{d_1, d_2\} \stackrel{\text{def}}{=} d$$

and

$$(2.19) \quad \left(\int_0^\omega |v_j(t)|^2 dt\right)^{\frac{1}{2}} < \max\{d_1, d_2\} = d.$$

From the first equation of the system (2.3), (2.10), (2.18) and (2.19), it follows that

$$\begin{aligned} (2.20) \quad & \int_0^\omega |u'_i(t)| dt \\ & \leq \overline{a_i} \int_0^\omega |u_i(t)| dt + \sum_{j=1}^m \overline{h_{ij}} A_j \lambda_j \left(\int_0^\omega |v_j(t - \tau_{ij}(t))| dt + \omega |f_j(0)| \right) + \omega \overline{I_i} \\ & \leq \overline{a_i} \sqrt{\omega} \left(\int_0^\omega |u_i(t)|^2 dt \right)^{\frac{1}{2}} + \sum_{j=1}^m \overline{h_{ij}} A_j \lambda_j l_1 \sqrt{\omega} \left(\int_0^\omega |v_j(t)|^2 dt \right)^{\frac{1}{2}} \\ & \quad + \omega \left(\sum_{j=1}^m \overline{h_{ij}} |f_j(0)| + \overline{I_i} \right) \\ & < \overline{a_i} \sqrt{\omega} d + \sqrt{\omega} \sum_{j=1}^m \overline{h_{ij}} A_j \lambda_j l_1 \sqrt{\omega} d + \omega \left(\sum_{j=1}^m \overline{h_{ij}} |f_j(0)| + \overline{I_i} \right) \stackrel{\text{def}}{=} c_1. \end{aligned}$$

Similarly, from the second equation of the system (2.3), (2.11), (2.18) and (2.19), it follows that there exists a positive constant c_2 such that

$$(2.21) \quad \int_0^\omega |v'_j(t)| dt < c_2.$$

From (2.18) and (2.19), it follows that there exist points t_i and \bar{t}_j such that

$$(2.22) \quad |u_i(t_i)| < \frac{d}{\sqrt{\omega}}$$

and

$$(2.23) \quad |v_j(\bar{t}_j)| \leq \frac{d}{\sqrt{\omega}}.$$

Since for $\forall t \in [0, \omega]$,

$$|u_i(t)| \leq |u_i(t_i)| + \int_0^\omega |u'_i(t)| dt$$

and

$$|v_j(t)| \leq |v_j(\bar{t}_j)| + \int_0^\omega |v'_j(t)| dt,$$

then from (2.20)-(2.23), we have

$$|u_i(t)| \leq \frac{d}{\sqrt{\omega}} + c_1$$

and

$$|v_j(t)| \leq \frac{d}{\sqrt{\omega}} + c_2.$$

Obviously, $\frac{d}{\sqrt{\omega}}, c_1, c_2$ are all independent of λ . Now let

$$\Omega = \left\{ u = (u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_m)^T \in X : \right. \\ \left. \|u\| < n\left(\frac{d}{\sqrt{\omega}} + r_1 + c_1\right) + m\left(\frac{d}{\sqrt{\omega}} + r_2 + c_2\right) \right\},$$

where r_1, r_2 are two chosen positive constants such that the bound of Ω is larger. Then Ω are bounded open subsets of X . Hence Ω satisfies the requirement (a) in Lemma 2.1. Next we prove that (b) in Lemma 2.1 holds. If it is not true, then when $u \in \partial\Omega \cap \text{Ker}L = \partial\Omega \cap \mathbb{R}^{m+n}$, we have

$$\begin{aligned} & QNu \\ &= \left(\frac{1}{\omega} \int_0^\omega H_1(t) dt, \frac{1}{\omega} \int_0^\omega H_2(t) dt, \dots, \frac{1}{\omega} \int_0^\omega H_n(t) dt; \frac{1}{\omega} \int_0^\omega K_1(t) dt, \dots, \frac{1}{\omega} \int_0^\omega K_m(t) dt \right)^T \\ &= (0, \dots, 0)^T. \end{aligned}$$

Therefore there exist points $\xi_i (i = 1, 2, \dots, n)$ and $\eta_j (j = 1, 2, \dots, m)$ such that

$$\begin{cases} H_i(\xi_i) = 0, \\ K_j(\eta_j) = 0. \end{cases}$$

From this and following the arguments of (2.22) and (2.23), we have for $\forall i = 1, 2, \dots, n; j = 1, 2, \dots, m, t \in [0, \omega]$

$$|u_i(t)| < \frac{d}{\sqrt{\omega}}$$

and

$$|v_j(t)| < \frac{d}{\sqrt{\omega}}.$$

Hence

$$\|u\| < n \frac{d}{\sqrt{\omega}} + m \frac{d}{\sqrt{\omega}}.$$

Thus $u \in \Omega \cap \mathbb{R}^{m+n}$. This contradicts the fact that $u \in \partial\Omega \cap \mathbb{R}^{m+n}$. Hence this proves (b) in Lemma 2.1 holds. Finally, we show that (c) in Lemma 2.1 holds. We only need to prove that $\deg \{-JQN u, \Omega \cap \text{Ker} L, (0, 0)^T\} \neq (0, 0, \dots, 0)^T$. Now we show that

$$\begin{aligned} & \deg \{-JQN u, \Omega \cap \text{Ker} L, (0, 0, \dots, 0)^T\} \\ &= \deg \left\{ (a_1 u_1, a_2 u_2, \dots, a_n u_n; b_1 v_1, \dots, b_m v_m)^T, \Omega \cap \text{Ker} L, (0, \dots, 0)^T \right\}. \end{aligned}$$

To this end, we define a mapping $\phi : \text{Dom} L \times [0, 1] \rightarrow X$ by

$$\begin{aligned} & \phi(u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_m, \mu) \\ &= -\frac{\mu}{\omega} \left(\int_0^\omega H_1(t) dt, \int_0^\omega H_2(t) dt, \dots, \int_0^\omega H_n(t) dt, \int_0^\omega K_1(t) dt, \dots, \int_0^\omega K_m(t) dt \right) \\ & \quad + (1 - \mu)(a_1 u_1, a_2 u_2, \dots, a_n u_n; b_1 v_1, \dots, b_m v_m), \end{aligned}$$

where $\mu \in [0, 1]$ is a parameter. We show that when $u \in \partial\Omega \cap \text{Ker} L = \partial\Omega \cap \mathbb{R}^{m+n}$, $\phi(u_1, u_2, \dots, u_n; v_1, \dots, v_m, \mu) \neq (0, 0, \dots, 0)^T$. If it is not true, then when $u \in \partial\Omega \cap \text{Ker} L = \partial\Omega \cap \mathbb{R}^{m+n}$, $\phi(u_1, u_2, \dots, u_n; v_1, \dots, v_m, \mu) = (0, 0, \dots, 0)^T$. Thus constant vector u with $u \in \partial\Omega$ satisfies for $i = 1, 2, \dots, n, j = 1, 2, \dots, m,$

$$(2.24) \quad \frac{\mu}{\omega} \int_0^\omega \left\{ a_i(t) u_i - \sum_{j=1}^m h_{ij}(t) f_j(\lambda_j v_j) - I_i(t) \right\} dt + (1 - \mu) a_i u_i = 0,$$

$$(2.25) \quad \frac{\mu}{\omega} \int_0^\omega \left\{ b_j(t) v_j - \sum_{i=1}^n w_{ji}(t) g_i(\mu_i u_i) - J_j(t) \right\} dt + (1 - \mu) b_j v_j = 0.$$

We make the following claims:

Claim 1. $|u_i|^* < \frac{d}{\sqrt{\omega}} + c_1 + r_1$. Otherwise, $|u_i|^* \geq \frac{d}{\sqrt{\omega}} + c_1 + r_1$. We consider two possible cases: (a) $|v_j|^* \leq |u_i|^*$; (b) $|v_j|^* > |u_i|^*$, where $|v_j|^* = \max_{1 \leq j \leq m} \{|v_j|\}, |u_i|^* = \max_{1 \leq i \leq n} \{|u_i|\}$.

(a) When $|v_j|^* \leq |u_i|^*$, we have

$$\begin{aligned} & \frac{\mu}{\omega} \int_0^\omega u_i \left\{ a_i(t)u_i - \sum_{j=1}^m h_{ij}(t)f_j(\lambda_j v_j) - I_i(t) \right\} dt + (1-\mu)\underline{a}_i u_i^2 \\ & \geq \mu[\underline{a}_i u_i^2 - \sum_{j=1}^m \overline{h}_{ij} |u_i| (\lambda_j A_j |v_j| + |f_j(0)|) - \overline{I}_i |u_i|] + (1-\mu)\underline{a}_i u_i^2 \\ & \geq \underline{a}_i |u_i|^2 - \sum_{j=1}^m \overline{h}_{ij} (\lambda_j A_j |v_j| |u_i| + |u_i| |f_j(0)|) - \overline{I}_i |u_i|. \end{aligned}$$

Then

$$\begin{aligned} & \underline{a}_i (|u_i|^*)^2 - \sum_{j=1}^m \overline{h}_{ij} (A_j \lambda_j |u_i| |v_j| + |u_i| |f_j(0)|) - \overline{I}_i |u_i|^* \\ & > \underline{a}_i (|u_i|^*)^2 - \sum_{j=1}^m \overline{h}_{ij} (A_j \lambda_j |u_i|^* |u_i|^* + |u_i|^* |f_j(0)|) - \overline{I}_i |u_i|^* \\ & > |u_i|^* \left\{ (\underline{a}_i - \sum_{j=1}^m \overline{h}_{ij} A_j \lambda_j) \left(\frac{d}{\sqrt{\omega}} + c_1 + r_1 \right) - \sum_{j=1}^m \overline{h}_{ij} A_j \lambda_j |f_j(0)| - \overline{I}_i \right\} \\ & > |u_i|^* (c_1 + r_1) (\underline{a}_i - \sum_{j=1}^m \overline{h}_{ij} A_j \lambda_j) \\ & > 0, \end{aligned}$$

which contradicts (2.24).

(b) When $|v_j|^* > |u_i|^*$, we have

$$\begin{aligned} & \frac{\mu}{\omega} \int_0^\omega v_j \left\{ b_j(t)v_j - \sum_{i=1}^n w_{ji}(t)g_i(\mu_i u_i) - J_j(t) \right\} dt + (1-\mu)\underline{b}_j v_j^2 \\ & \geq \mu[\underline{b}_j v_j^2 - \sum_{i=1}^n \overline{w}_{ji} |v_j| (\mu_i B_i |u_i| + |g_i(0)|) - \overline{J}_j |v_j|] + (1-\mu)\underline{b}_j v_j^2 \\ & \geq \underline{b}_j |v_j|^2 - \sum_{i=1}^n \overline{w}_{ji} (\mu_i B_i |v_j| |u_i| + |v_j| |g_i(0)|) - \overline{J}_j |v_j|. \end{aligned}$$

Then

$$\begin{aligned} & \underline{b}_j (|v_j|^*)^2 - \sum_{i=1}^n \overline{w}_{ji} (B_i \mu_i |u_i| |v_j| + |v_j| |g_i(0)|) - \overline{J}_j |v_j|^* \\ & > \underline{b}_j (|v_j|^*)^2 - \sum_{i=1}^n \overline{w}_{ji} (B_i \mu_i |v_j|^* |v_j|^* + |v_j|^* |g_i(0)|) - \overline{J}_j |v_j|^* \\ & > |v_j|^* \left\{ (\underline{b}_j - \sum_{i=1}^n \overline{w}_{ji} B_i \mu_i) \left(\frac{d}{\sqrt{\omega}} + c_2 + r_2 \right) - \sum_{i=1}^n \overline{w}_{ji} B_i \mu_i |g_i(0)| - \overline{J}_j \right\} \end{aligned}$$

$$\begin{aligned}
 &> |v_j|^*(c_2 + r_2)(\underline{b}_j - \sum_{i=1}^n \overline{w_{ji}} B_i \mu_i) \\
 &> 0,
 \end{aligned}$$

which contradicts (2.25). From the arguments of (a) and (b), Claim 1 holds.

Claim 2. $|v_j|^* \leq \frac{d}{\sqrt{\omega}} + c_2 + r_2$. Otherwise, $|v_j|^* > \frac{d}{\sqrt{\omega}} + c_2 + r_2$. We consider two possible cases: (a) $|u_i|^* \leq |v_j|^*$; (b) $|u_i|^* > |v_j|^*$. This arguments of (a) and (b) are similar to this arguments of (b) and (a) in Claim 1, then the proof is omitted. Hence Claim 2 holds. Thus, $|u_i| < \frac{d}{\sqrt{\omega}} + c_1 + r_1, i = 1, 2, \dots, n$ and $|v_j| < \frac{d}{\sqrt{\omega}} + c_2 + r_2, j = 1, 2, \dots, m$. Then $u \in \Omega \cap \mathbb{R}^{m+n}$. This contradicts the fact $u \in \partial\Omega \cap \mathbb{R}^{m+n}$. According to topological degree theory and by taking $J = I$ since $\text{Ker}L = \text{Im}Q$, we obtain,

$$\begin{aligned}
 &\text{deg} \{ -JQN u, \Omega \cap \text{Ker}L, (0, 0)^T \} \\
 &= \text{deg} \{ \phi(u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_m, 1), \Omega \cap \text{Ker}L, (0, 0)^T \} \\
 &= \text{deg} \{ \phi(u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_m, 0), \Omega \cap \text{Ker}L, (0, 0)^T \} \\
 &= \text{deg} \{ (\underline{a}_1 u_1, \underline{a}_2 u_2, \dots, \underline{a}_n u_n; \underline{b}_1 v_1, \dots, \underline{b}_m v_m)^T, \Omega \cap \text{Ker}L, (0, \dots, 0)^T \} \\
 &\neq 0.
 \end{aligned}$$

So far, we have proved that Ω satisfies all the assumptions in Lemma 2.1. Therefore, the system (1.1) has at least one ω -periodic solutions.

Part 2. The proof of uniqueness of a periodic solution. Let

$$\begin{aligned}
 u &= (u_1(t), u_2(t), \dots, u_n(t); v_1(t), v_2(t), \dots, v_m(t))^T, \\
 u^* &= (u_1^*(t), u_2^*(t), \dots, u_n^*(t); v_1^*(t), v_2^*(t), \dots, v_m^*(t))^T
 \end{aligned}$$

be two ω periodic solutions of the system (1.1). Denote

$$\begin{aligned}
 \int_0^\omega (u_i(t) - u_i^*(t))^2 dt &= \max_{1 \leq i \leq n} \{ \int_0^\omega (u_i(t) - u_i^*(t))^2 dt \}, \\
 \int_0^\omega (v_j(t) - v_j^*(t))^2 dt &= \max_{1 \leq j \leq m} \{ \int_0^\omega (v_j(t) - v_j^*(t))^2 dt \}.
 \end{aligned}$$

From the system (1.1), we have

$$\begin{aligned}
 u_i'(t) - (u_i^*)'(t) &= -a_i(t)(u_i(t) - u_i^*(t)) \\
 &\quad + \sum_{j=1}^m h_{ij}(t) \{ f_j(\lambda_j v_j(t - \tau_{ij}(t))) - f_j(\lambda_j v_j^*(t - \tau_{ij}(t))) \}
 \end{aligned}$$

and

$$\begin{aligned}
 v_j'(t) - (v_j^*)'(t) &= -b_j(t)(v_j(t) - v_j^*(t)) \\
 &\quad + \sum_{i=1}^n w_{ji}(t) \{ g_i(\mu_i u_i(t - \sigma_{ji}(t))) - g_i(\mu_i u_i^*(t - \sigma_{ji}(t))) \}.
 \end{aligned}$$

Then from conditions (iv), (ii) and (iii), we have

$$\begin{aligned}
 (2.26) \quad & \underline{a}_i \int_0^\omega (u_i(t) - u_i^*(t))^2 dt \\
 & \leq \sum_{j=1}^m \overline{h}_{ij} \int_0^\omega \lambda_j A_j |u_i(t) - u_i^*(t)| |v_j(t - \tau_{ij}(t)) - v_j^*(t - \tau_{ij}(t))| dt \\
 & \leq \sum_{j=1}^m \overline{h}_{ij} \lambda_j A_j \left(\int_0^\omega |u_i(t) - u_i^*(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^\omega |v_j(t - \tau_{ij}(t)) - v_j^*(t - \tau_{ij}(t))|^2 dt \right)^{\frac{1}{2}} \\
 & \leq \sum_{j=1}^m \overline{h}_{ij} \lambda_j A_j \sqrt{l_1} \left(\int_0^\omega |u_i(t) - u_i^*(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^\omega |v_j(t) - v_j^*(t)|^2 dt \right)^{\frac{1}{2}}
 \end{aligned}$$

and

$$\begin{aligned}
 (2.27) \quad & \underline{b}_j \int_0^\omega (v_j(t) - v_j^*(t))^2 dt \\
 & \leq \sum_{i=1}^n \overline{w}_{ji} \mu_i B_i \sqrt{l_2} \left(\int_0^\omega |v_j(t) - v_j^*(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^\omega |u_i(t) - u_i^*(t)|^2 dt \right)^{\frac{1}{2}}.
 \end{aligned}$$

From (2.26) and (2.27), we have

$$\begin{aligned}
 (2.28) \quad & \underline{a}_i \left(\int_0^\omega (u_i(t) - u_i^*(t))^2 dt \right)^{\frac{1}{2}} \\
 & \leq \sum_{j=1}^m \overline{h}_{ij} \lambda_j A_j \sqrt{l_1} \left(\int_0^\omega |v_j(t) - v_j^*(t)|^2 dt \right)^{\frac{1}{2}}
 \end{aligned}$$

and

$$\begin{aligned}
 (2.29) \quad & \underline{b}_j \left(\int_0^\omega (v_j(t) - v_j^*(t))^2 dt \right)^{\frac{1}{2}} \\
 & \leq \sum_{i=1}^n \overline{w}_{ji} \mu_i B_i \sqrt{l_2} \left(\int_0^\omega |u_i(t) - u_i^*(t)|^2 dt \right)^{\frac{1}{2}}.
 \end{aligned}$$

Substituting (2.29) into (2.28) gives, for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$,

$$\left(\underline{a}_i - \sum_{j=1}^m \overline{h}_{ij} \lambda_j A_j \sqrt{l_1} \frac{\sum_{i=1}^n \overline{w}_{ji} \mu_i B_i \sqrt{l_2}}{\underline{b}_j} \right) \left(\int_0^\omega (u_i(t) - u_i^*(t))^2 dt \right)^{\frac{1}{2}} \leq 0.$$

Then for $i = 1, 2, \dots, n$,

$$\left(\int_0^\omega (u_i(t) - u_i^*(t))^2 dt \right)^{\frac{1}{2}} = 0.$$

Hence for $i = 1, 2, \dots, n$,

$$u_i(t) = u_i^*(t).$$

Similarly, substituting (2.28) into (2.29) gives for $j = 1, 2, \dots, m$,

$$v_j(t) = v_j^*(t).$$

Therefore the proof of uniqueness is complete. This completes the proof of Theorem 2.1. \square

3. Globally asymptotic stability of the unique periodic solution

In this section, by constructing a Lyapunov functional, we derive new sufficient conditions for the global asymptotic stability of the unique periodic solution of the system (1.1).

Theorem 3.1. *In addition to conditions (i), (ii), (iii) in Theorem 2.1, we assume further that the following conditions hold:*

(H₁) *There exists two positive constants $l_i \geq 1$ ($i = 1, 2$) with*

$$\underline{a}_i > \max\left\{\sum_{j=1}^m \overline{h}_{ij} A_j \lambda_j l_1, \sum_{j=1}^m \overline{w}_{ji} \mu_i B_i l_2\right\} \text{ and}$$

$$\underline{b}_j > \max\left\{\sum_{i=1}^n \overline{w}_{ji} B_i \mu_i l_2, \sum_{i=1}^n \overline{h}_{ij} \lambda_j A_j l_1\right\}$$

such that

$$0 < \frac{1}{1 - \tau'_{ij}} < l_1,$$

$$0 < \frac{1}{1 - \sigma'_{ji}} < l_2;$$

(H₂) *There exist constants σ_{ji} and τ_{ij} , $i = 1, 2, \dots, n; j = 1, 2, \dots, m$, such that*

$$0 < \sigma_{ij}(t) < \sigma_{ij}, 0 < \tau_{ji}(t) < \tau_{ji}.$$

Then the unique ω periodic solution of the system (1.1) is globally asymptotically stable.

Proof. By Theorem 2.1, in view of (H₁), system (1.1) has a unique ω periodic solution, say, $u^*(t) = (u_1^*(t), u_2^*(t), \dots, u_n^*(t); v_1^*(t), v_2^*(t), \dots, v_m^*(t))^T$. Suppose that $u(t) = (u_1(t), u_2(t), \dots, u_n(t); v_1(t), v_2(t), \dots, v_m(t))^T$ is any ω periodic solution of the system (1.1) with the initial conditions (1.2).

We define a Lyapunov functional as follows for $t > 0$, $i = 1, 2, \dots, n; j = 1, 2, \dots, m$,

$$(3.1) \quad V(u_i(t), v_j(t)) = \sum_{i=1}^n |u_i(t) - u_i^*(t)| + \sum_{j=1}^m |v_j(t) - v_j^*(t)|$$

$$+ \sum_{i=1}^n \sum_{j=1}^m \overline{h}_{ij} \lambda_j A_j \int_{t-\tau_{ij}(t)}^t \frac{|v_j(\sigma) - v_j^*(\sigma)|}{1 - \tau'_{ij}(g^{-1}(\sigma))} d\sigma$$

$$+ \sum_{i=1}^n \sum_{j=1}^m \overline{w_{ji}} \mu_i B_i \int_{t-\sigma_{ji}(t)}^t \frac{|u_i(\sigma) - u_i^*(\sigma)|}{1 - \sigma'_{ji}(h^{-1}(\sigma))} d\sigma,$$

where, $t - \tau_{ij}(t) = g(t)$, $t - \sigma_{ji}(t) = h(t)$.

Calculating the upper right derivative $D^+V(u_i(t), v_j(t))$ of $V(u_i(t), v_j(t))$ along the solutions of the system (1.1), we obtain from conditions (ii) and (iii),

(3.2)

$$\begin{aligned} & D^+V(u_i(t), v_j(t)) \\ &= \sum_{i=1}^n \left[-a_i(t)|u_i(t) - u_i^*(t)| + \sum_{j=1}^m \overline{h_{ij}} \lambda_j A_j |v_j(t - \tau_{ij}(t)) - v_j^*(t - \tau_{ij}(t))| \right] \\ &+ \sum_{j=1}^m \left[-b_j(t)|v_j(t) - v_j^*(t)| + \sum_{i=1}^n \overline{w_{ji}} \mu_i B_i |u_i(t - \sigma_{ji}(t)) - u_i^*(t - \sigma_{ji}(t))| \right] \\ &+ \sum_{i=1}^n \sum_{j=1}^m \overline{h_{ij}} \lambda_j A_j \left[\frac{|v_j(t) - v_j^*(t)|}{1 - \tau'_{ij}(g^{-1}(t))} - |v_j(t - \tau_{ij}(t)) - v_j^*(t - \tau_{ij}(t))| \right] \\ &+ \sum_{i=1}^n \sum_{j=1}^m \overline{w_{ji}} \mu_i B_i \left[\frac{|u_i(t) - u_i^*(t)|}{1 - \sigma'_{ji}(h^{-1}(t))} - |u_i(t - \sigma_{ji}(t)) - u_i^*(t - \sigma_{ji}(t))| \right] \\ &\leq - \sum_{i=1}^n (\underline{a}_i - \sum_{j=1}^m \overline{w_{ji}} \mu_i B_i l_2) |u_i(t) - u_i^*(t)| \\ &\quad - \sum_{j=1}^m (\underline{b}_j - \sum_{i=1}^n \overline{h_{ij}} \lambda_j A_j l_1) |v_j(t) - v_j^*(t)|. \end{aligned}$$

In view of condition (iv), it follows that $D^+V(u_i(t), v_j(t)) < 0$ for $u_i(t) \neq u_i^*(t), v_j(t) \neq v_j^*(t)$. Hence, by the Lyapunov-type theorem in functional differential equations we can conclude that the unique ω periodic solution of the system (1.1) is globally asymptotically stable. \square

4. An example

Example 1. Consider the following BAM neural networks with time-varying delays:

$$(4.1) \quad \begin{aligned} \frac{du_1(t)}{dt} &= -(9 + \sin t)u_1(t) + v_1[t - (1 + \frac{\sin t}{2})] + \sin t, \\ \frac{dv_1(t)}{dt} &= -(8 + \cos t)v_1(t) + u_1[t - (1 + \frac{\sin t}{3})] + \cos t. \end{aligned}$$

In Theorem 3.1, $A_1 = 1$, $B_1 = 1$, $l_1 = 2$, $l_2 = 3$, $\tau_{11} = \frac{3}{2}$, $\sigma_{11} = \frac{4}{3}$, $\overline{h_{11}} = 1$, $\overline{w_{11}} = 1$, $\lambda_1 = \mu_1 = 1$, $\underline{a}_1 = \underline{b}_1 = 2$.

It is easy to prove that all conditions in Theorem 3.1 are satisfied. By Theorem 3.1, the system (4.1) has a unique ω periodic solution which is globally asymptotically stable.

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